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SIMPLIFIED FORMULATION OF DYADIC GREEN'S FUNCTIONS AND THEIR DUALITY RELATIONS FOR GENERAL ANISOTROPIC MEDIA

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Abstract—A simplified method to obtain the complete set of the dyadic Green's functions (DGFs) for general anisotropic media is presented. The method is based on the k-domain representation of the fields in terms of wave matrices. The Fourier transformed Green's functions are calculated through the inverses of wave matrices. The inverses of the wave matrices, which lead to the final form of DGF, are obtained using dyadic decomposition technique. This facilitates the inverse operation significantly and gives DGFs clear vector representation, which helps their physical interpretation. The dyadic decomposition of the wave matrices has been presented for uniaxially anisotropic, biaxially anisotropic and gyrotropic media. The method of deriving DGF using the technique given in this paper is applied on a uniaxially anisotropic medium and verified with the existing results. It is shown that the knowledge of the inverse of one type of wave matrix is adequate to find the complete set of the dyadic Green's functions for a general anisotropic medium using the method presented. The duality relations of dyadic Green's functions are also developed. It is shown that once the dyadic Green's functions for one of the dual media are obtained, the DGFs for the other dual medium can be found by application of the duality relations shown in this paper.

1. INTRODUCTION

In a variety of applications, such as geophysical prospecting, remote sensing, wave propagation, microstrip circuits and antennas, it is necessary to compute the electromagnetic fields in a medium under consideration. For a given set of sources, the fields may easily be found if the Green's function of the medium is available. There have been numerous publications on the Green's functions because of their widespread use in electromagnetic problems [1–9].

Various methods have been developed to obtain dyadic Green's functions (DGFs) for anisotropic media. The most commonly used methods employed to find DGFs are the Fourier transform method with differential formulation [10], the method of eigenfunction expansion [11], and the matrix formulation [12]. In [10], DGFs are obtained for uniaxial and biaxial layered, planar structures excited by a surface electric current density when the medium has diagonalized permittivity tensor. In [11], DGFs for the multilayered anisotropic media are found using eigenfunction expansion formulation. The formulations are constructed based on the principle of scattering superposition. In [12], the DGFs for the anisotropic medium are derived with matrix exponential function approach based on Cayley-Hamilton theorem. This approach resembles the method similar to local transition matrix formulation, which was employed by Morgan et al. [13].

In this paper, the derivation of the complete set of the dyadic Greens function using a simplified method for a general unbounded homogeneous anisotropic medium is given. The analysis is carried out in the k-domain using wave matrix approach. In this domain, the set of equations, which relates the DGFs to the inverses of the wave matrices, are obtained with their solutions using dyadic decomposition technique. This approach simplifies the derivation of the DGF to the problem of finding the inverse of a wave matrix. The dyadic decomposition of the wave matrices has been presented for three different anisotropic media. The method given here is applied on a uniaxially anisotropic medium. The results are verified with the existing results. The method introduced here shows a simple and practical way of finding the complete set of DGFs for anisotropic media with the knowledge of the inverse of one type of wave matrix. Furthermore, the duality relations of DGFs are derived and used to obtain the DGFs of the dual media. The DGF formulation given in this paper is especially useful for radiation and scattering problems in the presence of an anisotropic medium of any type.

2. FORMULATION AND SOLUTIONS

Maxwell's equations for a general anisotropic medium with relative permittivity and permeability tensors, $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$, in the presence of impressed magnetic current density $\overline{M}$ and the electric current density $\overline{J}$ can be written as

$$\nabla \times \overline{E} = i\omega\mu_0 \overline{\overline{\mu}} \cdot \overline{H} - \overline{M} \tag{1}$$

$$\nabla \times \overline{H} = -i\omega\varepsilon_0 \overline{\overline{\varepsilon}} \cdot \overline{E} + \overline{J} \tag{2}$$

The linearity of Maxwell's equations implies linear dependence of $\overline{E}$ and $\overline{H}$ on the excitations $\overline{J}$ and $\overline{M}$. Then $\overline{E}$ and $\overline{H}$ can be represented as

$$\overline{E}(\overline{r}) = \int_{V'} \overline{\overline{G}}_{ee}(\overline{r},\overline{r}') \cdot \overline{J}(\overline{r}') d^3\overline{r}' + \int_{V'} \overline{\overline{G}}_{em}(\overline{r},\overline{r}') \cdot \overline{M}(\overline{r}') d^3\overline{r}' \tag{3}$$

$$\overline{H}(\overline{r}) = \int_{V'} \overline{\overline{G}}_{me}(\overline{r},\overline{r}') \cdot \overline{J}(\overline{r}') d^3\overline{r}' + \int_{V'} \overline{\overline{G}}_{mm}(\overline{r},\overline{r}') \cdot \overline{M}(\overline{r}') d^3\overline{r}' \tag{4}$$

or in matrix form

$$\begin{bmatrix} \overline{E}(\overline{r}) \\ \overline{H}(\overline{r}) \end{bmatrix} = \int_{V'} \begin{bmatrix} \overline{\overline{G}}_{ee}(\overline{r},\overline{r}') & \overline{\overline{G}}_{em}(\overline{r},\overline{r}') \\ \overline{\overline{G}}_{me}(\overline{r},\overline{r}') & \overline{\overline{G}}_{mm}(\overline{r},\overline{r}') \end{bmatrix} \cdot \begin{bmatrix} \overline{J}(\overline{r}') \\ \overline{M}(\overline{r}') \end{bmatrix} d^3\overline{r}' \tag{5}$$

The dyadic Green's functions $\overline{\overline{G}}_{ee}(\overline{r},\overline{r}'), \overline{\overline{G}}_{mm}(\overline{r},\overline{r}')$ are called electric type and magnetic type and $\overline{\overline{G}}_{me}(\overline{r},\overline{r}'), \overline{\overline{G}}_{em}(\overline{r},\overline{r}')$ are called magnetic-electric type and electric-magnetic type DGFs. Equation (5) can be simplified as

$$\begin{bmatrix} \overline{E}(\overline{r}) \\ \overline{H}(\overline{r}) \end{bmatrix} = \int_{V'} \overline{\overline{G}}(\overline{r},\overline{r}') \cdot \begin{bmatrix} \overline{J}(\overline{r}') \\ \overline{M}(\overline{r}') \end{bmatrix} d^3\overline{r}' \tag{6}$$

where

$$\overline{\overline{G}}(\overline{r},\overline{r}') = \begin{bmatrix} \overline{\overline{G}}_{ee}(\overline{r},\overline{r}') & \overline{\overline{G}}_{em}(\overline{r},\overline{r}') \\ \overline{\overline{G}}_{me}(\overline{r},\overline{r}') & \overline{\overline{G}}_{mm}(\overline{r},\overline{r}') \end{bmatrix} \tag{7}$$

In (6), the integral is taken over all space, i.e.,

$$\int d^3\overline{r}' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx' dy' dz' \tag{8}$$

The Fourier transform pair of the field vectors can be expressed as

$$\overline{F}(\overline{r}) = \frac{1}{(2\pi)^3} \int \overline{F}(\overline{k}) e^{i\overline{k}\cdot\overline{r}} d^3\overline{k} \tag{9a}$$

$$\overline{F}(\overline{k}) = \int \overline{F}(\overline{r}) e^{-i\overline{k}\cdot\overline{r}} d^3\overline{r} \tag{9b}$$

where $\overline{F} = \overline{E}$ for an electric field vector and $\overline{F} = \overline{H}$ for a magnetic field vector. Here the integration over $\overline{k}$ is three dimensional as the integration over $\overline{r}$, i.e., $d^3\overline{k} = dk_x dk_y dk_z$.

Similarly, the Fourier transform pairs for DGF $\overline{\overline{G}}(\overline{r}, \overline{r}')$ can be written as

$$\overline{\overline{G}}(\overline{r}, \overline{r}') = \frac{1}{(2\pi)^3} \int \overline{\overline{G}}(\overline{k}) e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3\overline{k} \tag{10}$$

$$\overline{\overline{G}}(\overline{k}) = \int \overline{\overline{G}}(\overline{r}, \overline{r}') e^{-i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3(\overline{r}-\overline{r}') \tag{11}$$

Translational invariance assumed in the use of $(\overline{r}-\overline{r}')$ comes from the unbounded nature of the problem. Substitution of (9a), (10) into (6) gives

$$\frac{1}{(2\pi)^3} \int \begin{bmatrix} \overline{E}(\overline{k}) \\ \overline{H}(\overline{k}) \end{bmatrix} e^{i\overline{k}\cdot\overline{r}} d^3\overline{k} = \int \left[\frac{1}{(2\pi)^3} \int \overline{\overline{G}}(\overline{k}) e^{i\overline{k}\cdot(\overline{r}-\overline{r}')} d^3\overline{k} \right] \begin{bmatrix} \overline{J}(\overline{r}') \\ \overline{M}(\overline{r}') \end{bmatrix} d^3\overline{r}' \tag{12}$$

where $\overline{\overline{G}}(\overline{k})$ is given by

$$\overline{\overline{G}}(\overline{k}) = \begin{bmatrix} \overline{\overline{G}}_{ee}(\overline{k}) & \overline{\overline{G}}_{em}(\overline{k}) \\ \overline{\overline{G}}_{me}(\overline{k}) & \overline{\overline{G}}_{mm}(\overline{k}) \end{bmatrix} \tag{13}$$

When we change the order of integration in (12), we obtain

$$\begin{aligned} \int \begin{bmatrix} \overline{E}(\overline{k}) \\ \overline{H}(\overline{k}) \end{bmatrix} e^{i\overline{k}\cdot\overline{r}} d^3\overline{k} &= \int \overline{\overline{G}}(\overline{k}) e^{i\overline{k}\cdot\overline{r}} \int \begin{bmatrix} \overline{J}(\overline{r}') \\ \overline{M}(\overline{r}') \end{bmatrix} e^{-i\overline{k}\cdot\overline{r}'} d^3\overline{r}' d^3\overline{k} \\ &= \int \overline{\overline{G}}(\overline{k}) \begin{bmatrix} \overline{J}(\overline{k}) \\ \overline{M}(\overline{k}) \end{bmatrix} e^{i\overline{k}\cdot\overline{r}} d^3\overline{k} \end{aligned} \tag{14}$$

Using (14), we can relate the field vectors to dyadic Green's functions in the k-domain as

$$\begin{bmatrix} \overline{E}(\overline{k}) \\ \overline{H}(\overline{k}) \end{bmatrix} = \begin{bmatrix} \overline{\overline{G}}_{ee}(\overline{k}) & \overline{\overline{G}}_{em}(\overline{k}) \\ \overline{\overline{G}}_{me}(\overline{k}) & \overline{\overline{G}}_{mm}(\overline{k}) \end{bmatrix} \begin{bmatrix} \overline{J}(\overline{k}) \\ \overline{M}(\overline{k}) \end{bmatrix} \tag{15}$$

Now, assuming the solutions for the fields in the form of $e^{i\overline{k}\cdot\overline{r}}$ and using

$$\overline{k} \times \overline{E} = \overline{\overline{k}}\,\overline{E} \tag{16}$$

where

$$\overline{\overline{k}} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{17}$$

Maxwell's equations given by (1) and (2) can be transformed into the k-domain and we can derive the matrix equations for $\overline{E}(\overline{k})$ and $\overline{H}(\overline{k})$ as

$$\left[\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\mu}}\right] \cdot \overline{H}(\overline{k}) = -i\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{J}(\overline{k}) - i\omega\varepsilon_0\overline{M}(\overline{k}) \tag{18}$$

$$\left[\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}\right] \cdot \overline{E}(\overline{k}) = -i\omega\mu_0\overline{J}(\overline{k}) + i\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{M}(\overline{k}) \tag{19}$$

where $k_0^2 = \omega^2\mu_0\varepsilon_0$. The derivation of equations (18) and (19) is given in Appendix A. We define

$$\overline{\overline{W}}_H = \left[\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\mu}}\right] \tag{20}$$

as a *magnetic wave matrix* and

$$\overline{\overline{W}}_E = \left[\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}\right] \tag{21}$$

as an *electric wave matrix*. Note that the wave matrices are "dual" of each other in the sense that one can be obtained from the other by replacing $\overline{\overline{\varepsilon}} \leftrightarrow \overline{\overline{\mu}}$.

Equations (18) and (19) can be expressed using wave matrices as

$$\overline{\overline{W}}_H \cdot \overline{H}(\overline{k}) = -i\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{J}(\overline{k}) - i\omega\varepsilon_0\overline{M}(\overline{k}) \tag{22}$$

$$\overline{\overline{W}}_E \cdot \overline{E}(\overline{k}) = i\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{M}(\overline{k}) - i\omega\mu_0\overline{J}(\overline{k}) \tag{23}$$

We can represent (22) and (23) in matrix form as

$$\begin{bmatrix} \overline{\overline{W}}_E(\overline{k}) & 0 \\ 0 & \overline{\overline{W}}_H(\overline{k}) \end{bmatrix} \begin{bmatrix} \overline{E}(\overline{k}) \\ \overline{H}(\overline{k}) \end{bmatrix} = \begin{bmatrix} -i\omega\mu_0\overline{\overline{I}} & i\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1} \\ -i\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1} & -i\omega\varepsilon_0\overline{\overline{I}} \end{bmatrix} \begin{bmatrix} \overline{J}(\overline{k}) \\ \overline{M}(\overline{k}) \end{bmatrix} \tag{24}$$

Equation (24) can be modified as

$$\begin{bmatrix} \overline{E}(\overline{k}) \\ \overline{H}(\overline{k}) \end{bmatrix} = \begin{bmatrix} -i\omega\mu_0\overline{\overline{W}}_E^{-1} & i\overline{\overline{W}}_E^{-1}\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1} \\ -i\overline{\overline{W}}_H^{-1}\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1} & -i\omega\varepsilon_0\overline{\overline{W}}_H^{-1} \end{bmatrix} \begin{bmatrix} \overline{J}(\overline{k}) \\ \overline{M}(\overline{k}) \end{bmatrix} \tag{25}$$

Equation (25) relates the field vectors to the inverses of the wave matrices in the k-domain. When (15) and (25) are compared, we obtain the following relation

$$\begin{bmatrix} \overline{\overline{G}}_{ee}(\overline{k}) & \overline{\overline{G}}_{em}(\overline{k}) \\ \overline{\overline{G}}_{me}(\overline{k}) & \overline{\overline{G}}_{mm}(\overline{k}) \end{bmatrix} = \begin{bmatrix} -i\omega\mu_0\overline{\overline{W}}_E^{-1} & i\overline{\overline{W}}_E^{-1}\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1} \\ -i\overline{\overline{W}}_H^{-1}\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1} & -i\omega\varepsilon_0\overline{\overline{W}}_H^{-1} \end{bmatrix} \tag{26}$$

Equation (26) is the representation of DGFs in terms of the inverses of the wave matrices in the k-domain for a general anisotropic medium. More explicitly,

$$\overline{\overline{G}}_{ee}(\overline{k}) = -i\omega\mu_0\overline{\overline{W}}_E^{-1} \tag{27a}$$

$$\overline{\overline{G}}_{em}(\overline{k}) = i\overline{\overline{W}}_E\overline{\overline{k}}\overline{\overline{\mu}}^{-1} \tag{27b}$$

$$\overline{\overline{G}}_{me}(\overline{k}) = -i\overline{\overline{W}}_H\overline{\overline{k}}\overline{\overline{\varepsilon}}^{-1} \tag{27c}$$

$$\overline{\overline{G}}_{mm}(\overline{k}) = -i\omega\varepsilon_0\overline{\overline{W}}_H^{-1} \tag{27d}$$

The relation between the inverses of the wave matrices is derived as [14]

$$\overline{\overline{W}}_H^{-1} = \frac{1}{k_0^2}\left[\overline{\overline{\mu}}^{-1} - \overline{\overline{\mu}}^{-1}\overline{\overline{k}}\,\overline{\overline{W}}_E^{-1}\overline{\overline{k}}\overline{\overline{\mu}}^{-1}\right] \tag{28}$$

Equations (27)–(28) clearly show that the knowledge of the inverse of one type of wave matrix is sufficient to obtain the complete set of dyadic Green's functions in the k-domain. The final form of the dyadic Green's functions, which is valid everywhere but the source point, is obtained by substituting (27) into (10) as follows.

$$\overline{\overline{G}}(\overline{r},\overline{r}') = \frac{1}{(2\pi)^3}\int\begin{bmatrix} -i\omega\mu_0\overline{\overline{W}}_E^{-1} & i\overline{\overline{W}}_E^{-1}\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1} \\ -i\overline{\overline{W}}_H^{-1}\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1} & -i\omega\varepsilon_0\overline{\overline{W}}_H^{-1}\end{bmatrix} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')}d^3\overline{k} \tag{29}$$

In the following section, the method to obtain the inverse of the wave matrices using dyadic decomposition technique is discussed.

3. DYADIC REPRESENTATION

The problem of finding the complete set of DGFs for any type of anisotropic medium is simplified by equations (27)–(29) to finding the inverse of an electric wave matrix $\overline{\overline{W}}_E$ which is equal to

$$\overline{\overline{W}}_E^{-1} = \frac{adj\left(\overline{\overline{W}}_E\right)}{\left|\overline{\overline{W}}_E\right|} \tag{30}$$

$adj\left(\overline{\overline{W}}_E\right)$ is known as the adjoint and $\left|\overline{\overline{W}}_E\right|$ is known as the determinant of the electric wave matrix. The dispersion relation for anisotropic medium is found by using (19) in the source free region, i.e., $\overline{J} = 0$ and $\overline{M} = 0$, as

$$\overline{\overline{W}}_E \cdot \overline{E} = 0 \tag{31}$$

For the homogenous equation (31) to have a non-zero vector solution $\overline{E}$, the determinant of the electric wave matrix must vanish, i.e., $\left|\overline{\overline{W}}_E\right| = 0$, or

$$\left|\overline{\overline{W}}_E\right| = \left|\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\overline{\overline{\varepsilon}}\right| = 0 \tag{32}$$

As a result, once the Green's function in integral form given by equation (29) is obtained, the problem is reduced to finding the adjoint matrix $adj\left(\overline{\overline{W}}_E\right)$ and the determinant $\left|\overline{\overline{W}}_E\right|$ of the electric wave matrix. In the following sections, $adj\left(\overline{\overline{W}}_E\right)$ and $\left|\overline{\overline{W}}_E\right|$ for three different types of anisotropic media are derived and represented in dyadic forms. The dyadic representation of Green's functions is obtained by direct substitution of the results into (29).

3.1. Uniaxially Anisotropic Medium

A uniaxially anisotropic medium is defined by the following relative permittivity and permeability tensors

$$\overline{\overline{\varepsilon}} = \varepsilon_{11}\overline{\overline{I}} + (\varepsilon_{33} - \varepsilon_{11})\hat{p}\hat{p} \tag{33a}$$

$$\overline{\overline{\mu}} = \overline{\overline{I}} \tag{33b}$$

$\hat{p}$ shows the direction of the optic axis exists in the uniaxially anisotropic medium and equals $\hat{p} = \hat{z}$. When (33) is substituted into (21), the electric wave matrix in dyadic forms is found as

$$\overline{\overline{W}}_E = \left(k_0^2\varepsilon_{11} - k^2\right)\overline{\overline{I}} + \overline{k}\,\overline{k} + k_0^2(\varepsilon_{33} - \varepsilon_{11})\hat{p}\hat{p} \tag{34}$$

The dispersion relation for a uniaxially anisotropic medium is derived by substituting (33) into (32) as follows.

$$\left|\overline{\overline{W}}_E\right| = k_0^2\left(k^2 - k_0^2\varepsilon_{11}\right)\left[k_\rho^2\varepsilon_{11} + k_z^2\varepsilon_{33} - k_0^2\varepsilon_{11}\varepsilon_{33}\right] = 0 \tag{35}$$

The adjoint matrix, $adj\left(\overline{\overline{W}}_E\right)$ is obtained in dyadic form using (34) as

$$\begin{aligned} adj\left(\overline{\overline{W}}_E\right) &= \left(k_0^2\varepsilon_{11} - k^2\right)\left[k_0^2\varepsilon_{11}\overline{\overline{I}} - \overline{k}\,\overline{k} - k_0^2(\varepsilon_{33} - \varepsilon_{11})\hat{p}\hat{p}\right] \\ &\quad + k_0^2(\varepsilon_{33} - \varepsilon_{11})\left(\overline{k} \times \hat{p}\right)\left(\overline{k} \times \hat{p}\right) \end{aligned} \tag{36}$$

3.2. Biaxially Anisotropic Medium

The relative permittivity and permeability tensors of a biaxially anisotropic medium can be expressed as

$$\overline{\overline{\varepsilon}} = \varepsilon_{22}\overline{\overline{I}}\left(\frac{\varepsilon_{33}-\varepsilon_{11}}{2}\right)(\hat{p}_1\hat{p}_2+\hat{p}_2\hat{p}_1) \tag{37a}$$

$$\overline{\overline{\mu}} = \overline{\overline{I}} \tag{37b}$$

$\hat{p}_1$ and $\hat{p}_2$ show the direction of two optic axes that exist in a biaxially anisotropic medium. They are given by equation (38) below.

$$\hat{p}_1 = \sqrt{\frac{\varepsilon_{33}-\varepsilon_{22}}{\varepsilon_{33}-\varepsilon_{11}}}\hat{z}+\sqrt{\frac{\varepsilon_{22}-\varepsilon_{11}}{\varepsilon_{33}-\varepsilon_{11}}}\hat{x} \tag{38a}$$

$$\hat{p}_2 = \sqrt{\frac{\varepsilon_{33}-\varepsilon_{22}}{\varepsilon_{33}-\varepsilon_{11}}}\hat{z}-\sqrt{\frac{\varepsilon_{22}-\varepsilon_{11}}{\varepsilon_{33}-\varepsilon_{11}}}\hat{x} \tag{38b}$$

We obtain the electric wave matrix in dyadic form by substituting (37) into (21) as

$$\overline{\overline{W}}_E = \overline{k}\,\overline{k}+\left[\left(\varepsilon_{22}k_0^2-k^2\right)\overline{\overline{I}}+k_0^2\left(\frac{\varepsilon_{33}-\varepsilon_{11}}{2}\right)(\hat{p}_1\hat{p}_2+\hat{p}_2\hat{p}_1)\right] \tag{39}$$

The dispersion relation for a biaxially anisotropic medium is derived by inserting (37) into (32) and given by

$$\begin{aligned}\left|\overline{\overline{W}}_E\right| &= \alpha\left[(\alpha+\beta\hat{p}_1\cdot\hat{p}_2)^2-\beta^2\right]\\ &\quad+\overline{k}\cdot\left[\alpha\left[(\alpha+2\beta\hat{p}_1\cdot\hat{p}_2)\overline{\overline{I}}-\beta(\hat{p}_1\hat{p}_2+\hat{p}_2\hat{p}_1)\right]\right.\\ &\quad\left.-\beta^2(\hat{p}_1\times\hat{p}_2)(\hat{p}_2\times\hat{p}_1)\right]\cdot\overline{k}=0\end{aligned} \tag{40}$$

where

$$\alpha = \varepsilon_{22}k_0^2-k^2 \tag{41}$$

$$\beta = k_0^2\left(\frac{\varepsilon_{33}-\varepsilon_{11}}{2}\right) \tag{42}$$

$adj\left(\overline{\overline{W}}_E\right)$ for a biaxially anisotropic medium is obtained using (39) and expressed in dyadic form as

$$\begin{aligned}adj\left(\overline{\overline{W}}_E\right) &= \alpha\left[(\alpha+2\beta\hat{p}_1\cdot\hat{p}_2)\overline{\overline{I}}-\beta(\hat{p}_1\cdot\hat{p}_2+\hat{p}_2\cdot\hat{p}_1)\right]\\ &\quad-\beta^2(\hat{p}_1\times\hat{p}_2)(\hat{p}_1\times\hat{p}_2)\\ &\quad+\left[(\alpha-(3\alpha+2\beta(\hat{p}_1\cdot\hat{p}_2)))\overline{\overline{I}}+\beta(\hat{p}_1\cdot\hat{p}_2+\hat{p}_2\cdot\hat{p}_1)\right]\cdot\left(\overline{k}\times\overline{\overline{I}}\right)\\ &\quad+\left[\left(\overline{k}\cdot\left[\alpha\overline{\overline{I}}+\beta(\hat{p}_1\cdot\hat{p}_2+\hat{p}_2\cdot\hat{p}_1)\right]\right)\times\overline{\overline{I}}\right]\cdot\left(\overline{k}\times\overline{\overline{I}}\right)\end{aligned} \tag{43}$$

3.3. Electrically Gyrotropic Medium

An electrically gyrotropic or a gyroelectric medium is a medium whose relative permittivity and permeability tensors are in the following dyadic forms

$$\overline{\overline{\varepsilon}} = \varepsilon_1 \left(\overline{\overline{I}} - \hat{b}_0\hat{b}_0\right) + i\varepsilon_2 \left(\hat{b}_0 \times \overline{\overline{I}}\right) + \varepsilon_3\hat{b}_0\hat{b}_0 \tag{44a}$$

$$\overline{\overline{\mu}} = \mu\overline{\overline{I}} \tag{44b}$$

where $\hat{b}_0$ shows the direction of the applied dc magnetic field. When $\overline{B}_0 \equiv \hat{b}_0 B_0 = \hat{z}B_0$, i.e., $\hat{b}_0 = \hat{z} = (0, 0, 1)$, $\mu = 1$ i.e., cold plasma, and (44) is substituted into (21), the electric wave matrix in dyadic form can be obtained as

$$\overline{\overline{W}}_E = \left(k_0^2\varepsilon_1 - k^2\right)\overline{\overline{I}} + \hat{b}_0\hat{b}_0 \left(k_0^2\varepsilon_3 - k_0^2\varepsilon_1\right) + ik_0^2\varepsilon_2 \left(\hat{b}_0 \times \overline{I}\right) + \overline{k}\,\overline{k} \tag{45}$$

Following the same steps outlined in the preceding sections, the dispersion relation for a gyroelectric medium is given by equation (46).

$$\left|\overline{\overline{W}}_E\right| = k_0^2\varepsilon_3 \left(k_z^2 - k_{zI}^2\right) \left(k_z^2 - k_{zII}^2\right) = 0 \tag{46}$$

$adj\left(\overline{\overline{W}}_E\right)$ can be derived using (45) and expressed in dyadic form as

$$\begin{aligned} adj\overline{\overline{W}}_E = & \left(k_0^4 adj\overline{\overline{\varepsilon}} - k^2k_0^2\varepsilon_3\overline{\overline{I}}\right) + \hat{k}\hat{k}\left[k^2\left(k^2 - k_0^2\varepsilon_1\right)\right] \\ & + \hat{b}_0\hat{b}_0\left[k^2k_0^2(\varepsilon_3 - \varepsilon_1)\right] + (\hat{k} \times \hat{b}_0)(\hat{k} \times \hat{b}_0)\left[k^2k_0^2(\varepsilon_3 - \varepsilon_1)\right] \\ & + i\varepsilon_2k^2k_0^2\left[\hat{k}(\hat{k} \times \hat{b}_0) - (\hat{k} \times \hat{b}_0)\hat{k}\right] \end{aligned} \tag{47}$$

where

$$k^2 = k_\rho^2 + k_z^2 = k_x^2 + k_y^2 + k_z^2 \tag{48}$$

$$adj\overline{\overline{\varepsilon}} = \varepsilon_1\varepsilon_3\overline{\overline{I}} - i\varepsilon_2\varepsilon_3 \left(\hat{b}_0 \times \overline{\overline{I}}\right) + (\varepsilon_1 - \varepsilon_3)\hat{b}_0\hat{b}_0 \tag{49}$$

The results obtained in (45) and (47) check with the results given in [15]. It is to be noted that magnetically gyrotropic medium is dual of electrically gyrotropic medium with application of the following transformations

$$\overline{\overline{\varepsilon}} \rightarrow \overline{\overline{\mu}}, \qquad \mu \rightarrow \varepsilon, \qquad \mu_0 \rightarrow \varepsilon_0 \tag{50}$$

As a result, $adj\left(\overline{\overline{W}}_E\right)$ and $\left|\overline{\overline{W}}_E\right|$ for a magnetically gyrotropic medium can be obtained easily using the results for an electrically gyrotropic medium with the application of the duality relations given by (50).

4. VERIFICATION OF THE PROCEDURE

In this section, the method outlined in Section 2, using the results given in Section 3, is applied to obtain DGF for a uniaxially anisotropic medium. The final form of the DGF is compared with the results given in [16].

The integral form of the electric type DGF for a uniaxially anisotropic medium is given by (29) and equal to

$$\overline{\overline{G}}^{e}_{ee}(\overline{r},\overline{r}') = \frac{-i\omega\mu_0}{(2\pi)^3}\int \overline{\overline{W}}_E^{-1} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')}d^3\overline{k} \tag{51}$$

When (30) is substituted into (51), we obtain

$$\overline{\overline{G}}^{e}_{ee}(\overline{r},\overline{r}') = \frac{-i\omega\mu_0}{(2\pi)^3}\int \frac{adj\left(\overline{\overline{W}}_E\right)}{\left|\overline{\overline{W}}_E\right|} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')}d^3\overline{k} \tag{52}$$

Using the results given for a uniaxially anisotropic medium in Section 3.1, the dispersion relation for a uniaxially anisotropic medium can be found as

$$\left|\overline{\overline{W}}_E\right| = k_0^2\left(k^2 - k_0^2\varepsilon_{11}\right)\left[k_\rho^2\varepsilon_{11} + k_z^2\varepsilon_{33} - k_0^2\varepsilon_{11}\varepsilon_{33}\right] = 0 \tag{53a}$$

or

$$\left|\overline{\overline{W}}_E\right| = k_0^2\varepsilon_{33}\left(k_z^2 - \left(k_0^2\varepsilon_{11} - k_\rho^2\right)\right)\left[k_z^2 - \left(k_0^2\varepsilon_{11} - k_\rho^2\frac{\varepsilon_{11}}{\varepsilon_{33}}\right)\right] = 0 \tag{53b}$$

The wave numbers for a uniaxially anisotropic medium can be obtained using (53) as follows.

$$k_{zI}^2 = k_0^2\varepsilon_{11} - k_\rho^2 \tag{54a}$$

$$k_{zII}^2 = k_0^2\varepsilon_{11} - k_\rho^2\frac{\varepsilon_{11}}{\varepsilon_{33}} \tag{54b}$$

where

$$k^2 = k_\rho^2 + k_z^2 = k_x^2 + k_y^2 + k_z^2 \tag{55}$$

Equations (54) represent the two types of waves — type I wave or an ordinary wave, which is represented by k_{zI}, and type II wave or an extraordinary wave which is represented by k_{zII} exist in the uniaxially anisotropic medium. The wave numbers given by equation (54) check with the results given in [16] when optic axis is taken in the z-direction by setting tilt angle equal to zero, $\psi = 0$.

Equation (53b) can be put in the following form using (54)–(55)

$$\left|\overline{\overline{W}}_E\right| = k_0^2\varepsilon_{33}\left(k_z^2 - k_{zI}^2\right)\left(k_z^2 - k_{zII}^2\right) \tag{56}$$

$adj\left(\overline{\overline{W}}_E\right)$ for a uniaxially anisotropic medium is obtained in Section 3.1 and equal to

$$\begin{aligned} adj\left(\overline{\overline{W}}_E\right) &= \left(k_0^2\varepsilon_{11} - k^2\right)\left[k_0^2\varepsilon_{11}\overline{\overline{I}} - \overline{k}\,\overline{k} - k_0^2(\varepsilon_{33} - \varepsilon_{11})\hat{p}\hat{p}\right] \\ &\quad + k_0^2(\varepsilon_{33} - \varepsilon_{11})(\overline{k} \times \hat{p})(\overline{k} \times \hat{p}) \end{aligned} \tag{57}$$

$adj\left(\overline{\overline{W}}_E\right)$ can be put in the following matrix form

$$adj\left(\overline{\overline{W}}_E\right) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{58}$$

The matrix elements of $adj\left(\overline{\overline{W}}_E\right)$ are derived as

$$A_{11} = k^2k_x^2 - k_0^2\left(k_x^2 + k_z^2\right)\varepsilon_{33} - k_\rho^2k_0^2\varepsilon_{11} + k_0^4\varepsilon_{11}\varepsilon_{33} \tag{59a}$$

$$A_{12} = A_{21} = k^2k_xk_y - k_0^2k_xk_y\varepsilon_{33} \tag{59b}$$

$$A_{13} = A_{31} = k^2k_xk_z - k_0^2k_xk_z\varepsilon_{11} \tag{59c}$$

$$A_{22} = k^2\left(k_y^2 - k_0^2\varepsilon_{33}\right) + k_0^2\left(k_x^2[\varepsilon_{33} - \varepsilon_{11}] - k_y^2\varepsilon_{11}\right) + k_0^4\varepsilon_{11}\varepsilon_{33} \tag{59d}$$

$$A_{23} = A_{32} = k^2k_yk_z - k_0^2k_yk_z\varepsilon_{11} \tag{59e}$$

$$A_{33} = k^2\left(k_z^2 - k_0^2\varepsilon_{11}\right) - k_0^2k_z^2\varepsilon_{11} + k_0^4\varepsilon_{11}^2 \tag{59f}$$

When (56)–(58) are substituted into (52), we obtain electric type DGF as

$$\begin{aligned} \overline{\overline{G}}_{ee}(\overline{r}, \overline{r}') &= \frac{-i\omega\mu_0}{(2\pi)^3}\int\frac{1}{k_0^2\varepsilon_{33}\left(k_z^2 - k_{zI}^2\right)\left(k_z^2 - k_{zII}^2\right)} \\ &\quad \times \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} e^{i\overline{k}\cdot(\overline{r}-\overline{r}')}dk_xdk_ydk_z \end{aligned} \tag{60}$$

We perform the integration over k_z to obtain the two dimensional form of the DGF that is widely used in radiation and scattering problems. This form of the DGF is useful especially for the layered structures when the stratification of the layers is in the z-direction. The poles of the integrand occur at the zeros of $\left|\overline{\overline{W}}_E\right|$ denoted by $k_z = \pm k_{zI}$

and $k_z = \pm k_{zII}$. We assume the medium to be slightly lossy, i.e., $\mathrm{Im}k_z \ll \mathrm{Re}k_z$, $\mathrm{Im}k_z > 0$. This guarantees that the radiation condition is satisfied at $z = \pm\infty$. When we perform the contour integration over k_z, we obtain the following result for $z > z'$

$$\begin{aligned}\overline{\overline{G}}_{ee}(\overline{r},\overline{r}') &= \frac{\omega\mu_0}{4\pi k_0^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_x dk_y \left\{ \frac{adk\overline{\overline{W}}_{EI}}{2k_{zI}\left(k_{zI}^2 - k_{zII}^2\right)\varepsilon_{33}} e^{i\overline{k}_I\cdot(\overline{r}-\overline{r}')} \right. \\ &\left. + \frac{adj\overline{\overline{W}}_{EII}}{2k_{zII}\left(k_{zII}^2 - k_{zI}^2\right)\varepsilon_{33}} e^{i\overline{k}_{II}\cdot(\overline{r}-\overline{r}')} \right], \quad z > z' \end{aligned} \tag{61}$$

In (61), $adj\overline{\overline{W}}_{EI}$ in matrix form can be obtained using (58)–(59) as

$$adj\overline{\overline{W}}_{EI} = adj\overline{\overline{W}}_E\Big|_{k_z = k_{zI}} \tag{62}$$

So,

$$adj\overline{\overline{W}}_{EI} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{63}$$

The matrix elements are

$$B_{11} = k_0^2 k_y^2(\varepsilon_{33} - \varepsilon_{11}) \tag{64a}$$

$$B_{12} = B_{21} = k_0^2 k_x^2 k_y^2(\varepsilon_{11} - \varepsilon_{33}) \tag{64b}$$

$$B_{13} = B_{31} = 0 \tag{64c}$$

$$B_{22} = k_0^2 k_x^2(\varepsilon_{33} - \varepsilon_{11}) \tag{64d}$$

$$B_{23} = B_{32} = 0 \tag{64e}$$

$$B_{33} = 0 \tag{64f}$$

Following the same procedure, $adj\overline{\overline{W}}_{EII}$ in matrix form can be obtained as

$$adj\overline{\overline{W}}_{EII} = adj\overline{\overline{W}}_E\Big|_{k_z = k_{zII}} \tag{65}$$

So,

$$adj\overline{\overline{W}}_{EII} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{66}$$

The matrix elements for $adj\overline{\overline{W}}_{EII}$ are

$$C_{11} = k_x^2 k_{zII}^2 \left(\frac{\varepsilon_{11} - \varepsilon_{33}}{\varepsilon_{11}}\right) \tag{67a}$$

$$C_{12} = C_{21} = k_x k_y k_{zII}^2 \left(\frac{\varepsilon_{11} - \varepsilon_{33}}{\varepsilon_{11}}\right) \tag{67b}$$

$$C_{13} = C_{31} = k_x k_{zII} \left(\frac{\varepsilon_{11} - \varepsilon_{33}}{\varepsilon_{11}}\right) \left(-k_\rho^2 \frac{\varepsilon_{11}}{\varepsilon_{33}}\right) \tag{67c}$$

$$C_{22} = k_y^2 k_{zII}^2 \left(\frac{\varepsilon_{11} - \varepsilon_{33}}{\varepsilon_{11}}\right) \tag{67d}$$

$$C_{23} = C_{32} = k_y k_{zII} \left(\frac{\varepsilon_{11} - \varepsilon_{33}}{\varepsilon_{11}}\right) \left(-k_\rho^2 \frac{\varepsilon_{11}}{\varepsilon_{33}}\right) \tag{67e}$$

$$C_{33} = \left(\frac{\varepsilon_{11} - \varepsilon_{33}}{\varepsilon_{11}}\right) \left(k_\rho^2 \frac{\varepsilon_{11}}{\varepsilon_{33}}\right)^2 \tag{67f}$$

The final form of the DGF given in (61) and the coefficients given in (64) and (67) check with the results given in [16] when the optic axis is in the z-direction, i.e., tilt angle is zero $\psi = 0°$.

5. DUALITY OF DYADIC GREEN'S FUNCTIONS

The duality of dyadic Green's functions is best explained on gyrotropic medium. Gyrotropic medium becomes electrically gyrotropic if it is defined by the following relative permittivity and permeability tensors

$$\overline{\overline{\varepsilon}} = \varepsilon_1 \left(\overline{\overline{I}} - \hat{b}_0 \hat{b}_0\right) + i\varepsilon_2 \left(\hat{b}_0 \times \overline{\overline{I}}\right) + \varepsilon_3 \hat{b}_0 \hat{b}_0 \tag{68a}$$

$$\overline{\overline{\mu}} = \mu \overline{\overline{I}} \tag{68b}$$

whereas gyrotropic medium becomes magnetically gyrotropic if it is defined by

$$\overline{\overline{\mu}} = \mu_1 \left(\overline{\overline{I}} - \hat{b}_0 \hat{b}_0\right) + i\mu_2 \left(\hat{b}_0 \times \overline{\overline{I}}\right) + \mu_3 \hat{b}_0 \hat{b}_0 \tag{69a}$$

$$\overline{\overline{\varepsilon}} = \varepsilon \overline{\overline{I}} \tag{69b}$$

We rewrite Eqs. (3) and (4) as

$$\overline{E}(\overline{r}) = \int_{V'} \overline{\overline{G}}_{ee}^{e,m}(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}') d^3\overline{r}' + \int_{V'} \overline{\overline{G}}_{em}^{e,m}(\overline{r}, \overline{r}') \cdot \overline{M}(\overline{r}') d^3\overline{r}' \tag{70}$$

$$\overline{H}(\overline{r}) = \int_{V'} \overline{\overline{G}}_{me}^{e,m}(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}') d^3\overline{r}' + \int_{V'} \overline{\overline{G}}_{mm}^{e,m}(\overline{r}, \overline{r}') \cdot \overline{M}(\overline{r}') d^3\overline{r}' \tag{71}$$

The dyadic Green's functions $\overline{\overline{G}}_{ee}^{e,m}(\overline{r}, \overline{r}'), \overline{\overline{G}}_{mm}^{e,m}(\overline{r}, \overline{r}')$ are called electric type and magnetic type, respectively and $\overline{\overline{G}}_{me}^{e,m}(\overline{r}, \overline{r}'), \overline{\overline{G}}_{em}^{e,m}(\overline{r}, \overline{r}')$ are called magnetic-electric type and electric-magnetic type DGFs, respectively. The superscript of the DGF refers to the type of the gyrotropic medium and the first and the second subscripts show the type of the dyadic Green's function. The subscript 'e' refers to an electric type and 'm' refers to a magnetic type DGF. The superscript 'e' stands for an electrically gyrotropic medium. The superscript 'm' stands for a magnetically gyrotropic medium.

To derive the first order dyadic differential equations for an electrically gyrotropic medium, we substitute (70), (71), and (68) into (1) and (2) and use

$$\overline{J}(\overline{r}) = \int_{V'} \delta(\overline{r}-\overline{r}')\overline{\overline{I}}\cdot\overline{J}(\overline{r}')d^3\overline{r}' \tag{72}$$

$$\overline{M}(\overline{r}) = \int_{V'} \delta(\overline{r}-\overline{r}')\overline{\overline{I}}\cdot\overline{M}(\overline{r}')d^3\overline{r}' \tag{73}$$

Note that in this case, the superscript of the DGF is "e" representing an electrically gyrotropic medium. We obtain

$$\nabla\times\overline{\overline{G}}^e_{ee}(\overline{r},\overline{r}') = i\omega\mu_0\mu\overline{\overline{G}}^e_{me}(\overline{r},\overline{r}') \tag{74}$$

$$\nabla\times\overline{\overline{G}}^e_{me}(\overline{r},\overline{r}') = -i\omega\varepsilon_0\overline{\overline{\varepsilon}}\cdot\overline{\overline{G}}^e_{ee}(\overline{r},\overline{r}') + \delta(\overline{r}-\overline{r}')\overline{\overline{I}} \tag{75}$$

$$\nabla\times\overline{\overline{G}}^e_{em}(\overline{r},\overline{r}') = i\omega\mu_0\mu\overline{\overline{G}}^e_{mm}(\overline{r},\overline{r}') - \delta(\overline{r}-\overline{r}')\overline{\overline{I}} \tag{76}$$

$$\nabla\times\overline{\overline{G}}^e_{mm}(\overline{r},\overline{r}') = -i\omega\varepsilon_0\overline{\overline{\varepsilon}}\cdot\overline{\overline{G}}^e_{em}(\overline{r},\overline{r}') \tag{77}$$

When equations (74)–(77) are decoupled they lead to the second order dyadic differential equations as follows.

$$\left[\nabla\times\nabla\times\overline{\overline{I}} - k_0^2\mu\overline{\overline{\varepsilon}}\right]\cdot\overline{\overline{G}}^e_{ee}(\overline{r},\overline{r}') = i\omega\mu_0\mu\overline{\overline{I}}\delta(\overline{r}-\overline{r}') \tag{78}$$

$$\left[\nabla\times\overline{\overline{\varepsilon}}^{-1}\cdot\nabla\times\overline{\overline{I}} - k_0^2\mu\overline{\overline{I}}\right]\cdot\overline{\overline{G}}^e_{mm}(\overline{r},\overline{r}') = i\omega\varepsilon_0\overline{\overline{I}}\delta(\overline{r}-\overline{r}') \tag{79}$$

$$\left[\nabla\times\nabla\times\overline{\overline{I}} - k_0^2\mu\overline{\overline{\varepsilon}}\right]\cdot\overline{\overline{G}}^e_{em}(\overline{r},\overline{r}') = -\nabla\times\overline{\overline{I}}\delta(\overline{r}-\overline{r}') \tag{80}$$

$$\left[\nabla\times\overline{\overline{\varepsilon}}^{-1}\cdot\nabla\times\overline{\overline{I}} - k_0^2\mu\overline{\overline{I}}\right]\cdot\overline{\overline{G}}^e_{me}(\overline{r},\overline{r}') = \nabla\times\overline{\overline{\varepsilon}}^{-1}\delta(\overline{r}-\overline{r}') \tag{81}$$

where

$$k_0^2 = \omega^2\mu_0\varepsilon_0$$

We repeat the same procedure to find the first order dyadic differential equations for a magnetically gyrotropic medium by substituting (70), (71) and (69) into (1) and (2). Note that in this case, the superscript of the DGF is "m" representing a magnetically gyrotropic medium. We obtain

$$\nabla\times\overline{\overline{G}}^m_{ee}(\overline{r},\overline{r}') = i\omega\mu_0\overline{\overline{\mu}}\cdot\overline{\overline{G}}^m_{me}(\overline{r},\overline{r}') \tag{82}$$

$$\nabla\times\overline{\overline{G}}^m_{me}(\overline{r},\overline{r}') = -i\omega\varepsilon_0\varepsilon\overline{\overline{G}}^m_{ee}(\overline{r},\overline{r}')+\delta(\overline{r}-\overline{r}')\overline{\overline{I}} \tag{83}$$

$$\nabla\times\overline{\overline{G}}^m_{em}(\overline{r},\overline{r}') = i\omega\mu_0\overline{\overline{\mu}}\cdot\overline{\overline{G}}^m_{mm}(\overline{r},\overline{r}')-\delta(\overline{r}-\overline{r}')\overline{\overline{I}} \tag{84}$$

$$\nabla\times\overline{\overline{G}}^m_{mm}(\overline{r},\overline{r}') = -i\omega\varepsilon_0\varepsilon\overline{\overline{G}}^m_{em}(\overline{r},\overline{r}') \tag{85}$$

When equations (82)–(85) are decoupled they lead to the second order dyadic differential equations as follows.

$$\left[\nabla\times\overline{\overline{\mu}}^{-1}\cdot\nabla\times\overline{\overline{I}}-k_0^2\varepsilon\overline{\overline{I}}\right]\cdot\overline{\overline{G}}^m_{ee}(\overline{r},\overline{r}') = i\omega\mu_0\overline{\overline{I}}\delta(\overline{r}-\overline{r}') \tag{86}$$

$$\left[\nabla\times\nabla\times\overline{\overline{I}}-k_0^2\varepsilon\overline{\overline{\mu}}\right]\cdot\overline{\overline{G}}^m_{mm}(\overline{r},\overline{r}') = i\omega\varepsilon_0\varepsilon\delta(\overline{r}-\overline{r}') \tag{87}$$

$$\left[\nabla\times\overline{\overline{\mu}}^{-1}\cdot\nabla\times\overline{\overline{I}}-k_0^2\varepsilon\overline{\overline{I}}\right]\cdot\overline{\overline{G}}^m_{em}(\overline{r},\overline{r}') = -\nabla\times\overline{\overline{\mu}}^{-1}\delta(\overline{r}-\overline{r}') \tag{88}$$

$$\left[\nabla\times\nabla\times\overline{\overline{I}}-k_0^2\varepsilon\overline{\overline{\mu}}\right]\cdot\overline{\overline{G}}^m_{me}(\overline{r},\overline{r}') = \nabla\times\overline{\overline{I}}\delta(\overline{r}-\overline{r}') \tag{89}$$

When both sets of (78)–(81) and (86)–(89) are compared, it is seen that application of the duality transformation

$$\overline{\overline{\varepsilon}}\to\overline{\overline{\mu}},\qquad \mu\to\varepsilon,\qquad \mu_0\to\varepsilon_0 \tag{90}$$

on the problem of electrically gyrotropic medium transforms dyadic Green's functions into their dual ones for a magnetically gyrotropic medium as follows.

$$\overline{\overline{G}}^e_{ee}(\overline{r},\overline{r}')\to\overline{\overline{G}}^m_{mm}(\overline{r},\overline{r}') \tag{91}$$

$$\overline{\overline{G}}^e_{me}(\overline{r},\overline{r}')\to-\overline{\overline{G}}^m_{em}(\overline{r},\overline{r}') \tag{92}$$

$$\overline{\overline{G}}^e_{mm}(\overline{r},\overline{r}')\to\overline{\overline{G}}^m_{ee}(\overline{r},\overline{r}') \tag{93}$$

$$\overline{\overline{G}}^e_{em}(\overline{r},\overline{r}')\to-\overline{\overline{G}}^m_{me}(\overline{r},\overline{r}') \tag{94}$$

Thus, if the dyadic Green's functions are obtained for the given type of gyrotropic medium, the dyadic Green's functions for the dual medium can be obtained by applying the duality transformation introduced by (90) and (91)–(94).

6. CONCLUSION

In this paper, the complete set of the dyadic Green's functions for a general anisotropic medium is obtained by using a simplified method. The inverse operation, which is crucial in DGF formulation, is accomplished using the dyadic decomposition technique. The dyadic decomposition for wave matrices has been presented for several anisotropic media. The method is applied on a uniaxially anisotropic medium and verified with the existing results. It is shown that the knowledge of the inverse of one type of wave matrix is adequate to obtain the complete set of the DGFs for a general anisotropic medium. The duality relations of DGFs have also been developed and we have shown that once the DGFs for one of the dual media are obtained, DGFs for the other dual medium can be obtained by a simple application of the duality relations shown in this paper. The results presented here are generic and can be used in antenna and scattering problems in the presence of any type of anisotropic medium.

APPENDIX A.

Maxwell equations for a general anisotropic medium in the presence of an impressed magnetic current density $\overline{M}$ and an electric current density $\overline{J}$ can be written as

$$\nabla \times \overline{E} = i\omega\mu_0 \overline{\overline{\mu}} \cdot \overline{H} - \overline{M} \tag{A1}$$

$$\nabla \times \overline{H} = -i\omega\varepsilon_0 \overline{\overline{\varepsilon}} \cdot \overline{E} + \overline{J} \tag{A2}$$

First, we transform Maxwells equations into those in the k-domain and express them as

$$i\overline{k} \times \overline{E} = i\omega\mu_0 \overline{\overline{\mu}} \cdot \overline{H} - \overline{M} \tag{A3}$$

$$i\overline{k} \times \overline{H} = -i\omega\varepsilon_0 \overline{\overline{\varepsilon}} \cdot \overline{E} + \overline{J} \tag{A4}$$

Equations (A3) and (A4) can be written as

$$\overline{E}(\overline{k}) = -\frac{1}{i\omega\varepsilon_0} \overline{\overline{\varepsilon}}^{-1} \left[i\overline{k} \times \overline{H}(\overline{k}) - \overline{J}(\overline{k}) \right] \tag{A5}$$

$$\overline{H}(\overline{k}) = \frac{1}{i\omega\mu_0} \overline{\overline{\mu}}^{-1} \left[i\overline{k} \times \overline{E}(\overline{k}) + \overline{M}(\overline{k}) \right] \tag{A6}$$

Substitution of (A5) into (A3) gives

$$\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{\overline{k}}\overline{H}(\overline{k}) + i\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\overline{J}(\overline{k}) = -k_0^2 \overline{\overline{\mu}}\,\overline{\overline{H}}(\overline{k}) - i\omega\varepsilon_0 \overline{M}(\overline{k})$$

or

$$\boxed{\left[\overline{\overline{k}}\,\overline{\overline{\varepsilon}}^{-1}\,\overline{\overline{k}} + k_0^2 \overline{\overline{\mu}}\right] \cdot \overline{H}(\overline{k}) = -i\overline{k}\,\overline{\overline{\varepsilon}}^{-1}\overline{J}(\overline{k}) - i\omega\varepsilon_0 \overline{M}(\overline{k})} \tag{A7}$$

where $\overline{\overline{k}}$ is defined as

$$\overline{\overline{k}} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{A8}$$

and we have made the use of equation (16).

If we follow the same procedure by substituting (A6) into (A4), we get

$$-\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{\overline{k}}\overline{E}(\overline{k}) + i\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{M}(\overline{k}) = k_0^2\,\overline{\overline{\varepsilon}}\,\overline{E}(\overline{k}) + i\omega\mu_0\,\overline{J}(\overline{k})$$

or

$$\boxed{\left[\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{\overline{k}} + k_0^2\,\overline{\overline{\varepsilon}}\right]\cdot\overline{E}(\overline{k}) = -i\omega\mu_0\overline{J}(\overline{k}) + i\overline{\overline{k}}\,\overline{\overline{\mu}}^{-1}\overline{M}(\overline{k})} \tag{A9}$$

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