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FRACTIONAL DUALITY IN HOMOGENEOUS BI-ISOTROPIC MEDIUM

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Abstract—The fractional dual solutions of Maxwell equations in bi-isotropic medium are determined using the field decomposition approach. Both negative phase velocity and positive phase velocity propagation have been considered. The results are compared with the corresponding available results for isotropic and chiral medium. Time average power associated with fractional dual fields and corresponding source distribution are also studied.

1. INTRODUCTION

More than a decade before, Engheta initiated an effort to bring together the tools of fractional derivatives/fractional integrals and the theory of electromagnetism. His contribution developed an area in electromagnetics that is named fractional paradigm in electromagnetic theory [1–6]. Fractional derivative/integrals are mathematical operators involving differentiation/integration to arbitrary non-integer orders. These operators, possess interesting mathematical properties and have been studied in the field of fractional calculus [7]. In his study, he applied the tools of fractional calculus in various problems of electromagnetic fields and waves, and obtained interesting results. These results highlight certain notable features and promising potential applications of these operators in electromagnetic theory [1–6]. He investigated the notion of fractionalization of some other linear operators in electromagnetic theory, e.g., curl operator, kernel of integral transform. Fractionalization of such operators has led us to novel solutions, interpretable as "fractional solutions", for certain electromagnetic problems [8–11]. An interesting and useful work done by Engheta is fractionalization of curl operator [8]. Mathematical recipe to fractionalize a linear operator is available in [8, 12]. Some interesting works are reported in [13, 14]. Problem of implementation of fractional order electric potential had been addressed in [13]. Debnath collected recent applications of fractional calculus in science and engineering [14].

Engheta used the fractional curl operator to find the new set of solutions to Maxwell's equations by fractionalizing the principle of duality [8]. New set of solutions is named as fractional dual solutions to the Maxwell equations. In electromagnetics, principle of duality states that if $(\mathbf{E}, \eta\mathbf{H})$ is one set of solutions (original solutions) to Maxwell equations, then other set of solutions (dual to the original solutions) is $(\eta\mathbf{H}, -\mathbf{E})$, where η is the impedance of the medium. The solutions which may be regarded as intermediate step between the original and dual to the original solutions may be obtained using the following relations [8]

$$\mathbf{E}_{fd} = \frac{1}{(jk)^{\alpha}}(\nabla\times)^{\alpha}\mathbf{E}$$
$$\eta\mathbf{H}_{fd} = \frac{1}{(jk)^{\alpha}}(\nabla\times)^{\alpha}\eta\mathbf{H}$$

where $(\nabla\times)^{\alpha}$ means fractional curl operator and $k = \omega\sqrt{\mu\epsilon}$ is the wavenumber of the medium. It may be noted that fd means fractional dual solutions. It is obvious from above set of equations that for $\alpha = 0$,

$$\mathbf{E}_{fd} = \mathbf{E}, \qquad \eta\mathbf{H}_{fd} = \eta\mathbf{H}$$

and for $\alpha = 1$

$$\mathbf{E}_{fd} = \eta\mathbf{H}, \qquad \eta\mathbf{H}_{fd} = -\mathbf{E}$$

Which are two sets of solutions to Maxwell's equations. The solutions which may be regarded intermediate step between the above two sets of solutions may be obtained by varying parameter α between zero and one. Naqvi et al. [15] afterward extended the work [8] and discussed the behavior of fractional dual solutions in an unbounded chiral medium. Lakhtakia [16] derived theorem which shows that a dyadic operator which commutes with curl operator can be used to find new solutions of the Faraday and Ampere-Maxwell equations. Veliev and Engheta [17] utilized the fractional curl operator to a fixed solution and obtained the fractional fields that represent the solution of reflection problem from an anisotropic surface. Naqvi and Abbas studied the behavior of fractional curl operator for complex and higher orders [18]. Hussain and Naqvi [19] and Hussain et al. [20, 21] proposed the idea of fractional transmission line and fractional waveguides respectively. Naqvi and Abbas [22] extended the work [8] for metamaterials with negative permittivity and permeability, while Naqvi and Rizvi [23] determined the sources corresponding to fractional dual solutions. Recently, Naqvi et al., has modelled the transmission through chiral slab in terms of fractional curl operator [24]. Two recent work are reported in [25–27].

In present work, Engheta's work [8] has been extended for homogenous bi-isotropic medium. The results obtained in present work have been compared with corresponding results obtained by Engheta for ordinary isotropic medium [8] and results obtained by Naqvi et al. for chiral medium [15]. We have also noted the effects of negative phase velocity (NPV) and positive phase velocity (PPV) on fractional dual solutions. "Negative phase velocity medium are defined as medium in which phase changes apposite to the direction of energy propagation [28–33]. It is of interest is to find a uniform procedure which may be utilized to find fractional dual fields in each case. Corresponding source distribution and power is also in interest.

2. FRACTIONAL DUAL SOLUTIONS

In bi-isotropic medium having constitutive parameters ϵ, μ, $\xi = (\chi - j\kappa)\sqrt{\mu_0\epsilon_0}$ and $\zeta = (\chi + j\kappa)\sqrt{\mu_0\epsilon_0}$, relations between four field vectors may be written as [34]

$$\mathbf{D} = \epsilon\mathbf{E} + \xi\mathbf{H} \tag{1a}$$

$$\mathbf{B} = \zeta\mathbf{E} + \mu\mathbf{H} \tag{1b}$$

where parameters ξ and ζ may be expressed in terms of Tellegen parameter χ and chirality parameter κ as

$$\begin{aligned} \xi &= (\chi - j\kappa)\sqrt{\epsilon_0\mu_0} \\ \zeta &= (\chi + j\kappa)\sqrt{\epsilon_0\mu_0} \\ \chi_r &= \frac{\chi}{n} = \sin\theta \end{aligned}$$

where n is the refractive index of the medium. It may be noted that $\theta = 0$ or $\chi = 0$ corresponds to chiral medium. In bi-isotropic materials, it is convenient not to work with electric and magnetic field vectors $\mathbf{E}$, $\mathbf{H}$, but with other field quantities, the wavefields, $\mathbf{E}_+$, $\mathbf{H}_+$ and $\mathbf{E}_-$, $\mathbf{H}_-$, which make up the total field as [34]

$$\mathbf{E} = \mathbf{E}_- + \mathbf{E}_+ \tag{2a}$$

$$\mathbf{H} = \mathbf{H}_- + \mathbf{H}_+ \tag{2b}$$

Each of the two wavefields sees the BI medium as an equivalent medium with respective medium parameters ϵ_+, μ_+ and ϵ_-, μ_-, that is

$$
\begin{aligned}
\epsilon_+ &= \epsilon(\cos\theta + \kappa_r)\exp(j\theta) \\
\epsilon_- &= \epsilon(\cos\theta - \kappa_r)\exp(-j\theta) \\
\mu_+ &= \mu(\cos\theta + \kappa_r)\exp(j\theta) \\
\mu_- &= \mu(\cos\theta - \kappa_r)\exp(-j\theta) \\
\eta_\pm &= \eta\exp(\mp j\theta) \\
k_\pm &= k(\cos\theta \pm \kappa_r) \\
\kappa_r &= \frac{\kappa}{n}
\end{aligned}
$$

NPV and PPV propagation in loss-less bi-isotropic medium may be described by the following conditions

Case 1:

$$\kappa_r < \cos\theta, \quad \epsilon > 0$$

Both wave fields travel with PPV.

Case 2:

$$\kappa_r < \cos\theta, \quad \epsilon < 0$$

Both wave fields travel with NPV.

Case 3:

$$-\kappa_r > \cos\theta$$

Wave field with wave number k_+ is of NPV and with k_- is of PPV type.

Case 4:

$$\kappa_r > \cos\theta$$

Wave field with wave number k_+ is of PPV and with k_- is of NPV type.

For homogeneous bi-isotropic medium we have the wavefields [34]

$$\mathbf{E}_\pm = \frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right] \tag{3a}$$

$$\eta\mathbf{H}_\pm = \frac{1}{2\cos\theta}\left[\exp(\pm j\theta)\eta\mathbf{H} \pm j\mathbf{E}\right] \tag{3b}$$

Equation (3) is known as Bohren decomposition [35]. Above equation may be written as

$$\begin{aligned}\mathbf{E}_\pm &= \frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right] \\ \eta_\pm\mathbf{H}_\pm &= \frac{1}{2\cos\theta}\left[\eta\mathbf{H} \pm j\mathbf{E}\exp(\mp j\theta)\right]\end{aligned}$$

It may be considered that wavefields $(\mathbf{E}_\pm, \eta_\pm\mathbf{H}_\pm)$ are propagating in an isotropic medium having intrinsic impedance $\eta_\pm$. Above relations are related through the principle of duality. Duality principle states that if $(\mathbf{E}_\pm, \eta_\pm\mathbf{H}_\pm)$ is one set of solutions to Maxwell equations then $(\eta_\pm\mathbf{H}_\pm, -\mathbf{E}_\pm)$ is the dual set of solutions. As wavefields are solutions of Maxwell equations therefore, using the Maxwell equations or application of duality transformation yields the following relations, for

Case 1:

$$\begin{aligned}\mathbf{OE}_\pm &= \exp\left(\pm j\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right]\right] \\ &= \exp\left(\pm j\frac{\pi}{2}\right)\mathbf{E}_\pm = \eta_\pm\mathbf{H}_\pm \\ \mathbf{O}\eta_\pm\mathbf{H}_\pm &= -\exp\left(\mp j\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\eta\mathbf{H} \pm j\exp(\mp j\theta)\mathbf{E}\right]\right] \\ &= \exp\left(\pm j\frac{\pi}{2}\right)\eta_\pm\mathbf{H}_\pm = -\mathbf{E}_\pm\end{aligned}$$

above equations may be written in an appropriate form as

$$\mathbf{OE}_\pm = \exp\left(\pm j\frac{\pi}{2}\right)\mathbf{E}_\pm \tag{4a}$$

$$\mathbf{O}\eta_\pm\mathbf{H}_\pm = \exp\left(\pm j\frac{\pi}{2}\right)\eta_\pm\mathbf{H}_\pm \tag{4b}$$

Similarly for

Case 2:

$$\mathbf{OE}_\pm = \exp\left(\mp j\frac{\pi}{2}\right)\mathbf{E}_\pm \tag{4c}$$

$$\mathbf{O}\eta_\pm\mathbf{H}_\pm = \exp\left(\mp j\frac{\pi}{2}\right)\eta_\pm\mathbf{H}_\pm \tag{4d}$$

Case 3:

$$\mathbf{OE}_\pm = \exp\left(-j\frac{\pi}{2}\right)\mathbf{E}_\pm \tag{4e}$$

$$\mathbf{O}\eta_\pm\mathbf{H}_\pm = \exp\left(-j\frac{\pi}{2}\right)\eta_\pm\mathbf{H}_\pm \tag{4f}$$

Case 4:

$$\mathbf{O}\mathbf{E}_\pm = \exp\left(j\frac{\pi}{2}\right)\mathbf{E}_\pm \tag{4g}$$

$$\mathbf{O}\eta_\pm\mathbf{H}_\pm = \exp\left(j\frac{\pi}{2}\right)\eta_\pm\mathbf{H}_\pm \tag{4h}$$

In above equations vector operator $\mathbf{O}$ may be termed as duality operator and is given by

$$\mathbf{O} = \frac{1}{jk_\pm}\nabla\times$$

It may be noted that if wavefields are proportional to its curl then this type of vector fields are known as Beltrami fields [30]. In Equations (4), $\mathbf{E}_\pm$ and $\eta_\pm\mathbf{H}_\pm$ may be considered as eigenvectors of the duality operator $\mathbf{O}$ and its corresponding eigenvalues are $\exp\left(\pm j\frac{\pi}{2}\right)$ when both wavefields are of PPV type, and $\exp\left(\mp j\frac{\pi}{2}\right)$ when both wavefields are of NPV type etc.

Fractional dual of the wavefields may be obtained by fractionalizing the duality operator and fractionalization of linear operator means fractionalization of corresponding eigenvalues [8]. That is

Case 1:

$$\mathbf{E}_{fd\pm} = \mathbf{O}^\alpha\mathbf{E}_\pm = \exp\left(\pm j\alpha\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right]\right]$$

$$\eta_\pm\mathbf{H}_{fd\pm} = \mathbf{O}^\alpha\eta_\pm\mathbf{H}_\pm = \exp\left(\pm j\alpha\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\eta\mathbf{H} \pm j\exp(\mp j\theta)\mathbf{E}\right]\right]$$

or

$$\mathbf{E}_{fd\pm} = \exp\left(\pm j\alpha\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right]\right] \tag{5a}$$

$$\mathbf{H}_{fd\pm} = \frac{1}{\eta}\exp\left(\pm j\alpha\frac{\pi}{2} \pm j\theta\right)\left[\frac{1}{2\cos\theta}\left[\eta\mathbf{H} \pm j\exp(\mp j\theta)\mathbf{E}\right]\right] \tag{5b}$$

Case 2:

$$\mathbf{E}_{fd\pm} = \exp\left(\mp j\alpha\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right]\right] \tag{5c}$$

$$\mathbf{H}_{fd\pm} = \frac{1}{\eta}\exp\left(\mp j\alpha\frac{\pi}{2} \pm j\theta\right)\left[\frac{1}{2\cos\theta}\left[\eta\mathbf{H} \pm j\exp(\mp j\theta)\mathbf{E}\right]\right] \tag{5d}$$

Case 3:

$$\mathbf{E}_{fd\pm} = \exp\left(-j\alpha\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right]\right] \quad (5e)$$

$$\mathbf{H}_{fd\pm} = \frac{1}{\eta}\exp\left(-j\alpha\frac{\pi}{2} \pm j\theta\right)\left[\frac{1}{2\cos\theta}\left[\eta\mathbf{H} \pm j\exp(\mp j\theta)\mathbf{E}\right]\right] \quad (5f)$$

Case 4:

$$\mathbf{E}_{fd\pm} = \exp\left(j\alpha\frac{\pi}{2}\right)\left[\frac{1}{2\cos\theta}\left[\exp(\mp j\theta)\mathbf{E} \mp j\eta\mathbf{H}\right]\right] \quad (5g)$$

$$\mathbf{H}_{fd\pm} = \frac{1}{\eta}\exp\left(j\alpha\frac{\pi}{2} \pm j\theta\right)\left[\frac{1}{2\cos\theta}\left[\eta\mathbf{H} \pm j\exp(\mp j\theta)\mathbf{E}\right]\right] \quad (5h)$$

where α is the fractional parameter. Total fractional dual field may be obtained as

$$\mathbf{E}_{fd} = \mathbf{E}_{fd-} + \mathbf{E}_{fd+}$$
$$\mathbf{H}_{fd} = \mathbf{H}_{fd-} + \mathbf{H}_{fd+}$$

Substituting (5) in above expressions yields the following results

Case 1:

$$\mathbf{E}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{E}\cos\left(\alpha\frac{\pi}{2} - \theta\right) + \eta\mathbf{H}\sin\left(\alpha\frac{\pi}{2}\right)\right] \quad (6a)$$

$$\mathbf{H}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{H}\cos\left(\theta + \alpha\frac{\pi}{2}\right) - \frac{\mathbf{E}}{\eta}\sin\left(\alpha\frac{\pi}{2}\right)\right] \quad (6b)$$

Case 2:

$$\mathbf{E}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{E}\cos\left(\alpha\frac{\pi}{2} + \theta\right) - \eta\mathbf{H}\sin\left(\alpha\frac{\pi}{2}\right)\right] \quad (6c)$$

$$\mathbf{H}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{H}\cos\left(\theta - \alpha\frac{\pi}{2}\right) + \frac{\mathbf{E}}{\eta}\sin\left(\alpha\frac{\pi}{2}\right)\right] \quad (6d)$$

Case 3:

$$\mathbf{E}_{fd} = \exp\left(-j\alpha\frac{\pi}{2}\right)\mathbf{E} \quad (6e)$$

$$\mathbf{H}_{fd} = \exp\left(-j\alpha\frac{\pi}{2}\right)\mathbf{H} \quad (6f)$$

Case 4:

$$\mathbf{E}_{fd} = \exp\left(j\alpha\frac{\pi}{2}\right)\mathbf{E} \quad (6g)$$

$$\mathbf{H}_{fd} = \exp\left(j\alpha\frac{\pi}{2}\right)\mathbf{H} \quad (6h)$$

3. PLANE WAVE PROPAGATION IN UNBOUNDED BI-ISOTROPIC MEDIUM

A plane wave polarized along $\mathbf{u}_x$ at $z = 0$ and propagating along positive z-axis in bi-isotropic medium may be written as sum of two plane waves [34]

$$\mathbf{E}(z) = E\exp(-jkz\cos\theta)\left[\mathbf{u}_x\cos(\kappa_r kz) - \mathbf{u}_y\sin(\kappa_r kz)\right]$$

The vector in square brackets is a unit vector which has been rotated from position $\mathbf{u}_x$ at $z = 0$, by angle $\phi = -\kappa_r kz$ in the right direction of propagation, while propagating through the bi-isotropic medium. Above expression may be rewritten with the help of rotation dyadic $R(\cdot)$ as

$$\mathbf{E}(z) = \exp(-jkz\cos\theta)R(-\kappa_r kz)\cdot\mathbf{E}(0) \tag{7a}$$

The corresponding magnetic field may be written as

$$\eta\mathbf{H} = R(\theta + \pi/2).\mathbf{E}(z) \tag{7b}$$

where

$$\begin{aligned} R(\cdot) &= I_t\cos(\cdot) + J\sin(\cdot) \\ I &= \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y + \mathbf{u}_z\mathbf{u}_z \\ I_t &= \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y \\ J &= \mathbf{u}_z \times \mathrm{I} \end{aligned}$$

Substituting (7) in (6a) and (6b) yields the following

$$\begin{aligned} \mathbf{E}_{fd} &= \frac{1}{\cos\theta}\left[I_t\cos\left(\alpha\frac{\pi}{2} - \theta\right) + R(\theta + \pi/2)\sin\left(\alpha\frac{\pi}{2}\right)\right]\cdot\mathbf{E}(z) \\ \eta\mathbf{H}_{fd} &= \frac{1}{\cos\theta}\left[R(\theta + \pi/2)\cos\left(\theta + \alpha\frac{\pi}{2}\right) - I_t\sin\left(\alpha\frac{\pi}{2}\right)\right]\cdot\mathbf{E}(z) \end{aligned}$$

Hence for bi-isotropic medium, where $\theta \neq 0$, we have

$$\begin{aligned} \mathbf{E}_{fd} &= \left[I_t\cos\left(\alpha\frac{\pi}{2}\right) + J\sin\left(\alpha\frac{\pi}{2}\right)\right]\cdot\mathbf{E}(z) = R(\alpha\pi/2)\cdot\mathbf{E}(z) \\ \eta\mathbf{H}_{fd} &= \left[J\cos\left(\alpha\frac{\pi}{2} + \theta\right) - I_t\sin\left(\alpha\frac{\pi}{2} + \theta\right)\right]\cdot\mathbf{E}(z) \\ &= R(\theta + \alpha\pi/2 + \pi/2)\cdot\mathbf{E}(z) \end{aligned}$$

In terms of rotation dyadic $R(\cdot)$, we have the expressions for bi-isotropic medium, for

Case 1:

$$\mathbf{E}_{fd} = R(\alpha\pi/2) \cdot \mathbf{E}(z) \tag{8a}$$
$$\eta\mathbf{H}_{fd} = R(\theta + \alpha\pi/2 + \pi/2) \cdot \mathbf{E}(z) \tag{8b}$$

Similarly for

Case 2:

$$\mathbf{E}_{fd} = R(-\alpha\pi/2) \cdot \mathbf{E}(z) \tag{8c}$$
$$\eta\mathbf{H}_{fd} = R(\theta - \alpha\pi/2 + \pi/2) \cdot \mathbf{E}(z) \tag{8d}$$

Case 3:

$$\mathbf{E}_{fd} = \exp\left(-j\alpha\frac{\pi}{2}\right)\mathbf{E}(z) \tag{8e}$$
$$\eta\mathbf{H}_{fd} = \exp\left(-j\alpha\frac{\pi}{2}\right)R(\theta + \pi/2) \cdot \mathbf{E}(z) \tag{8f}$$

Case 4:

$$\mathbf{E}_{fd} = \exp\left(j\alpha\frac{\pi}{2}\right)\mathbf{E}(z) \tag{8g}$$
$$\eta\mathbf{H}_{fd} = \exp\left(j\alpha\frac{\pi}{2}\right)R(\theta + \pi/2) \cdot \mathbf{E}(z) \tag{8h}$$

It may be noted from above equations that, in case 1 and case 2, changing the parameter α rotates the plane of polarization. For case 3 and case 4, there is no rotation in the plane of polarization. Secondly, when we changes the value of α, plane of polarization in case 2 rotates opposite to that of case 1. Thirdly, in case 3 and 4, parameter α only effects the phase of the field. It is also obvious that direction of $\eta\mathbf{H}_{fd}$ makes an angle $\theta + \pi/2$ with that of $\mathbf{E}_{fd}$ as for the case of original set of solutions $(\mathbf{E}, \eta\mathbf{H})$. That is, for all cases

$$\eta\mathbf{H}_{fd} = R(\theta + \pi/2) \cdot \mathbf{E}_{fd}(z) \tag{9}$$

For chiral medium $\theta = 0$, Equations (8a) and (8b) simplifies to

$$\begin{aligned}\mathbf{E}_{fd} &= \left[I_t\cos\left(\alpha\frac{\pi}{2}\right) + R(\pi/2)\sin\left(\alpha\frac{\pi}{2}\right)\right] \cdot \mathbf{E}(z)\\ &= R(\alpha\pi/2) \cdot \mathbf{E}(z) \qquad (10a)\\ \eta\mathbf{H}_{fd} &= \left[R(\pi/2)\cos\left(\alpha\frac{\pi}{2}\right) - I_t\sin\left(\alpha\frac{\pi}{2}\right)\right] \cdot \mathbf{E}(z)\\ &= R(\alpha\pi/2 + \pi/2) \cdot \mathbf{E}(z) \qquad (10b)\end{aligned}$$

In terms of rotation dyadic, we have

$$\mathbf{E}_{fd} = R(\alpha\pi/2) \cdot \mathbf{E}(z) \tag{11a}$$
$$\eta\mathbf{H}_{fd} = R(\alpha\pi/2 + \pi/2) \cdot \mathbf{E}(z) \tag{11b}$$
$$\eta\mathbf{H}_{fd} = R(\pi/2) \cdot \mathbf{E}_{fd}(z) \tag{11c}$$

It is obvious that direction of $\eta\mathbf{H}_{fd}$ is orthogonal to $\mathbf{E}_{fd}$. For ordinary isotropic material, $\theta = \kappa_r = 0$, set of expressions given in (11) hold. It is obvious from above set of equations that in case 1 and 2, fractional curl operator effects only the rotation dyadic $R(\cdot)$ and in case 3 and 4 it effects only the phase of the field. Equation (8) is the general expression and fractional dual solution for chiral and ordinary medium may be derived very easily.

4. CORRESPONDING SOURCES AND TIME AVERAGE POWER

The expression for corresponding sources are obtained as [16, 23]

Case 1:

$$\mathbf{J}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{J}\cos\left(\alpha\frac{\pi}{2} - \theta\right) + \eta\mathbf{M}\sin\left(\alpha\frac{\pi}{2}\right)\right] \tag{12a}$$
$$\mathbf{M}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{M}\cos\left(\theta + \alpha\frac{\pi}{2}\right) - \frac{\mathbf{J}}{\eta}\sin\left(\alpha\frac{\pi}{2}\right)\right] \tag{12b}$$

Case 2:

$$\mathbf{J}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{J}\cos\left(\alpha\frac{\pi}{2} + \theta\right) - \eta\mathbf{M}\sin\left(\alpha\frac{\pi}{2}\right)\right] \tag{12c}$$
$$\mathbf{M}_{fd} = \frac{1}{\cos\theta}\left[\mathbf{M}\cos\left(\theta - \alpha\frac{\pi}{2}\right) + \frac{\mathbf{J}}{\eta}\sin\left(\alpha\frac{\pi}{2}\right)\right] \tag{12d}$$

Case 3:

$$\mathbf{J}_{fd} = \exp\left(-j\alpha\frac{\pi}{2}\right)\mathbf{J} \tag{12e}$$
$$\mathbf{M}_{fd} = \exp\left(-j\alpha\frac{\pi}{2}\right)\mathbf{M} \tag{12f}$$

Case 4:

$$\mathbf{J}_{fd} = \exp\left(j\alpha\frac{\pi}{2}\right)\mathbf{J} \tag{12g}$$
$$\mathbf{M}_{fd} = \exp\left(j\alpha\frac{\pi}{2}\right)\mathbf{M} \tag{12h}$$

where **J** and **M** are electric and magnetic current densities respectively corresponding to the original fields. It may be noted from above equations that, in case 1 and 2, the fractional sources are combination of ordinary electric and magnetic sources but not in case 3 and 4.

Time average power associated with fractional dual fields is given by [36–38], for all cases

$$\begin{aligned}\frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_{fd}\times\mathbf{H}_{fd}^{*}\right\} &= \frac{1}{2}\mathrm{Re}\left[\left\{\mathbf{E}_{fd-}\times\mathbf{H}_{fd-}^{*}\right\}+\left\{\mathbf{E}_{fd-}\times\mathbf{H}_{fd+}^{*}\right\}\right.\\ &\quad\left.+\left\{\mathbf{E}_{fd+}\times\mathbf{H}_{fd-}^{*}\right\}+\left\{\mathbf{E}_{fd+}\times\mathbf{H}_{fd+}^{*}\right\}\right]\\ &= \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}\times\mathbf{H}^{*}\right\}\end{aligned}$$

This shows that time average power associated with the fractional dual fields is same as time average power associated with the original set of fields.

5. CONCLUSIONS

Fractional dual fields for bi-isotropic medium are derived and written in a form involving rotation dyadic. We have used field decomposition approach. Discussion have been divided into four cases. In case 1, both wave fields travel with PPV, in case 2, both wave fields travel with NPV, in case 3, wave field with wave number k_+ travels with NPV while wave field with wave number k_- travels with PPV. In case 4, wave field with wave number k_+ travels with PPV while wave field with wave number k_- travels with NPV. It is noted that in case 1 and 2, fractional parameter appears in a rotation dyadic which relates original solution to the fractional dual solution. This means that effect of fractional parameter is to rotate the plane of polarization. Rotation of the plane of polarization in case 1 and case 2 is opposite of each other. For case 3 and 4, the fractional parameter effects only the phase of the field and does not changes the plane of polarization of the field. We have derived the corresponding fractional sources for all these cases. It is noted that for case 1 and 2, the fractional sources are constructed from superposition of ordinary electric and magnetic sources while in case 3 and 4 the fractional dual sources are obtained from electric source in case 3 while from magnetic source from case 4.

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