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MONTE CARLO INTEGRATION TECHNIQUE FOR THE ANALYSIS OF ELECTROMAGNETIC SCATTERING FROM CONDUCTING SURFACES

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Abstract—A new numerical method is proposed for the analysis of electromagnetic scattering from conducting surfaces. The method involves Monte Carlo integration technique in the Method of Moments solution of the Electric Field Integral Equation for determining the unknown induced current distribution on the surface of the scatterers. The unknown current distribution is represented in terms of a modified entire domain polynomial basis functions satisfying the appropriate edge conditions and symmetry conditions of the problem. This leads to very small order of the Method of Moments matrix as compared to the conventional sub-domain basis functions. The accuracy and the effectiveness of the method are demonstrated in three cases of scattering from conducting circular disks and results are compared with the solutions using conventional sub-domain basis functions. While the sub domain analysis is incapable of handling large domain problems, the proposed method overcomes this limitation. It is also observed that the proposed method is superior to conventional sub-domain method in dealing with singularity problem of the integral equation easily and efficiently.

1. INTRODUCTION

Over several decades, electromagnetic (EM) scattering problems from various metallic objects have attracted the researchers with a strong interest in the radar cross sections (RCS's) of large dynamic range. Scattering from various metallic surfaces such as square, cylindrical, circular, spherical [2–7] are commonly used as test cases in computational electromagnetics, because analytical solutions for scattered fields can be derived for these geometries. The scattering from conducting circular disk surfaces has been extensively studied in the literature. Many numerical techniques have been proposed for the solution of electromagnetic scattering from large conducting circular disks. These methods can be divided into various categories. First category is based on the Physical Optics (PO) [8] approximation which is an approximate technique and is accurate for predicting the far-field pattern near the main beam. The second category which is more accurate than PO technique is based on the Physical Theory of Diffraction (PTD) [9], which takes care of the singularity at the edges. Further modifications in PTD are proposed by Michaeli [10, 11]. The third category presenting similar accuracy as PTD belongs to Geometrical Theory of Diffraction (GTD) [12, 13]. Several modifications to this technique have been introduced by other researchers such as Uniform Geometrical Theory of Diffraction (UGTD) by Kouyoumijian and Pathak [14], Uniform Asymptotic Technique by Lee & Deschamps [15], and High-Order Geometrical Theory of Diffraction by Bechtel [16], Ryan & Peters [17] and by Marsland et al. [18]. The fourth category is the Method of Moments (MoM) [1] and last category its Hybrid variants such as Hybrid Asymptotic Moment Method by Kaye, Murthy and Thiele [19, 20]. Another hybrid technique similar to MoM which combines the eigenfunction expansion technique, the mode matching technique, the least squares fitting method, and the dyadic Green's function technique is proposed by Li et al. [21].

The Method of Moments (MoM) is the most commonly used technique to solve the Electric Field Integral Equation (EFIE) for problems involving electromagnetic scattering from metallic objects. This involves expansion of the unknown current on the surface of the scatterer in terms of known basis functions. This involves two approaches, the sub domain (SD) and the entire domain (ED) approaches, based on the two kinds of basis functions employed for the expansion. The entire domain basis functions extend over the whole structure, whereas the sub domain basis functions are defined to exist over a section of the structure and have a zero value over the rest of its portion. The SD approach is limited to electrically small and moderately large structures, as the number of sub domains required to model large structures accurately becomes very large. This results in the moment matrix of a large size increasing the computational memory requirement. The entire domain basis functions, on the other hand, require a very few number of expansion terms as the structure under investigation does not require to be divided into a number of sub domains [22–24].

Another key issue in the MoM solution is the evaluation of singular integrals which can be performed by both numerical and semi-analytical techniques [25]. In most of the cases, commonly used approach is to apply singularity subtraction (extraction) technique [26, 27]. In this method singular terms are subtracted from the kernel and integrated analytically. The remaining function is then integrated numerically. Though this method is well suited for lower order basis functions such as constant and linear basis functions, like the Rao-Wilton-Glisson basis functions [28], for the higher order basis functions it is found to be unsuitable.

The next very important issue in the MoM solution of the EFIE is the judicious choice of the basis functions, used for the expansion of the unknown surface current distribution. This problem is more critical in the entire domain solution, especially for large scale problems. The choice must incorporate the fact that the order of the resulting matrix should be small, thus consuming small computational resources but at the same time, must produce accurate solutions. Another significant requirement is the satisfaction of the edge conditions and the symmetry conditions by the basis functions as predicted from the geometry of the problem. Thus while selecting the basis functions judiciously; one must consider all the above factors carefully.

In the proposed work, all the above mentioned issues: to solve large scale problem efficiently, to handle singular integrals and to make a wise choice of the basis functions, are tackled simultaneously. It is proposed to introduce a new fast and efficient algorithm performing the numerical integration in MoM by Monte Carlo integration technique [29, 30]. For the field points close to or coinciding with the source point resulting into a singular integrand, the Monte Carlo integration is found to be most suitable. Both the issues of rendering the order of the resulting matrix of the MoM solution to a small value, and the basis functions for unknown current distribution satisfying the edge condition and the symmetry condition is achieved by a careful choice of polynomial basis functions and modifying them for satisfying the above conditions.

The application of Monte Carlo techniques in electromagnetic scattering problems has been limited to evaluating the statistical average of physical quantities over randomly oriented bodies. In [31], the bistatic radar scattering cross section of randomly oriented wires has been determined using the Monte Carlo methods, whereas the averaging fields in dense random media with dielectric spheroids with random positions and orientations have been evaluated in [32]. However, the use of Monte Carlo methods for integration of functions in the MoM solution of EFIE formulation of scattering problems has not been reported so far.

In this paper, Monte Carlo integration technique has been proposed to solve scattering problem for conducting circular disk in the MoM formulation of the problem. The entire domain basis functions which satisfy the symmetry and edge conditions are utilized in the MoM matrix solution. A good agreement in the comparison confirms the applicability of the method developed in this paper.

2. MONTE CARLO INTEGRATION TECHNIQUE

Monte Carlo integration methods are useful for obtaining solutions to problems involving integration which are too complicated to solve analytically or by other numerical methods. Standard numerical integration techniques do not work very well on high-dimensional domains, especially when the integrand is not smooth. Although the quadrature rules of integration typically work very well for one-dimensional integrals, problems occur when extending them to higher dimensions.

Monte Carlo integration methods have advantages over other numerical integration methods in a space of many dimensions. Their efficiencies relative to other methods increase when the dimension of the problem increases e.g., quadrature formula becomes very complex while Monte Carlo integration technique remains almost unchanged in more than one dimension. In addition to this, the convergence of the Monte Carlo integration is independent of dimensionality regardless of the smoothness of the integrand. Monte Carlo integration is simple since only two basic operations are required, namely sampling and point evaluation. It is also suited for large structures and highly complex problems for which definite integral formulation is not obvious and standard analytical techniques are ineffective. Sampling can be used even on domains that are not well-suited to numerical quadrature.

The idea of Monte Carlo integration is to evaluate the integral using random sampling as

$$I = \int_{\Omega} f(x)dx \tag{1}$$

where f is a function of vector $\boldsymbol{x}$, Ω is domain of integration. The Monte Carlo integration is popular for complex f and/or Ω. In its basic form, this is done by independently sampling N points $x_1, \ldots, x_N$ according to some convenient density function p, and then computing the estimate

$$F_N = \frac{\Omega}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \tag{2}$$

where $p(x_i)$ is the probability density function or *pdf.* Here the notation F_N is used rather than I to emphasize that the result is approximate, and that its properties depend on how many sample points are chosen. If $p(x_i)$ is the uniform probability density, then the integral is simply

$$I \cong \frac{\Omega}{N} \sum_{i=1}^{N} f(x_i). \tag{3}$$

In terms of handling the singularity problem in the integrand, Monte Carlo integration technique outperforms many of the existing techniques. It handles this problem very easily, effectively and efficiently even in situations where there is no analytical transformation available to remove the singularity without changing the form of the kernel of the integrand. There is also no need for any analytical preprocessing or employing any corrective technique. The singularity problem can be handled very efficiently by just preventing the randomly generated points falling in the neighborhood of the singular point. The singular domain can be marked as a small circular region about the point of singularity in two dimensional problems. This does not require any extra effort to handle the singularity problem as the required condition can be embedded directly in the Monte Carlo integration technique itself in a single statement of the computer program employed for the purpose in simulation.

3. FORMULATION OF THE CIRCULAR DISK PROBLEM

The proposed technique is applied to a standard scattering problem consisting of a circular metallic disk. The disk is assumed to be of negligible thickness and its radius considered as a. A plane electromagnetic wave of frequency f is incident upon it normally from the z direction with its electric field vector $\mathbf{E}$ is polarized along the y axis. The amplitude of $\mathbf{E}$, on the disk is, $1\,\mathrm{Vm}^{-1}$. The geometry of the problem is shown in Figure 1.

The incident wave induces current on the disk which is responsible for the scattered field in all directions. This current distribution is the unknown in the EFIE which is to be solved using the MoM technique.

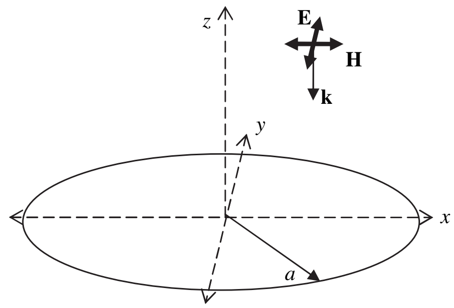

Figure 1. Circular metallic disk.

From Maxwell's equations we can obtain vector expression for the total electric field as

$$\mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s \tag{4}$$

where $\mathbf{E}^i$ is the incident field and $\mathbf{E}^s$ is the scattered field represented as:

$$\mathbf{E}^s = (1/j\omega\varepsilon_0\mu_0)[\nabla(\nabla.\mathbf{A}) + k^2\mathbf{A}]. \tag{5}$$

where $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\limits_{disc} \mathbf{J}(\mathbf{r}')\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$ is the vector potential and, $G(\mathbf{r},\mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ is the free space Green's function. Applying the boundary condition $\mathbf{n} \times \mathbf{E}_t = 0$ on the scatterer, the EFIE for the unknown current density $\mathbf{J}(\mathbf{r}')$ can be obtained as:

$$-\mathbf{E}^i = \mathbf{E}^s = (1/j\omega\varepsilon_0\mu_0)[\nabla(\nabla.\mathbf{A}) + k^2\mathbf{A}] \tag{6}$$

where $\mathbf{J}(\mathbf{r}')$ is included as given by (6).

For MoM solution of the integral equation, taking some known basis function $\mathbf{f}_n(\mathbf{r}')$, the unknown current on the conducting surface can be expanded as:

$$\mathbf{J}(\mathbf{r}') = \sum_{n=1}^{N} I_n\mathbf{f}_n(\mathbf{r}') \tag{7}$$

where I_n; $n = 1, 2, \ldots, N$; are unknown amplitudes of the basis functions and are to be determined. This expansion is applied over the same number of field points as the number of expansion terms, which transforms the integral equation into a set of simultaneous algebraic equations in the unknown coefficients, which can be written in the matrix form as:

$$[\mathbf{Z}_{mn}]\,[\mathbf{I}_n] = [-\mathbf{E}_n] \tag{8a}$$

or

$$[\mathbf{I}_n] = [\mathbf{Z}_{mn}]^{-1}\,[-\mathbf{E}_n]\,. \tag{8b}$$

The matrix $\mathbf{Z}$ is called the "moment" matrix. One typical moment matrix element $\mathbf{Z}_{mn}$ is given by

$$Z_{mn} = \frac{1}{j4\pi\omega\varepsilon_0}\int\limits_{disc} f_n(\mathbf{r}').\{\nabla(\nabla\cdot G(\mathbf{r}_m,\mathbf{r}')) + k^2 G(\mathbf{r}_m,\mathbf{r}')\}d\mathbf{r}' \tag{9}$$

Once the unknown coefficients are determined, current distribution over the entire region can be calculated using the expansion. The matrix elements are usually evaluated using numerical integration. It is evident from (9) that the matrix elements are singular at the points $\mathbf{r}' = \mathbf{r}_m$. Especially, the double differentiation of $G(\mathbf{r},\mathbf{r}')$ leads to highly singular behavior of the integrand in the vicinity of the points, $\mathbf{r}' = \mathbf{r}_m$. This integration cannot be performed analytically or numerically by ordinary quadrature rules and approximate methods have to be developed. This essentially makes the MoM solution an approximate one. Usually the function in the integrand is modified and approximated before performing the integration. However, a new numerical technique proposed in this paper does not require the integrand function to be modified. Rather, the singularity problem is removed due to the characteristic property of MCI i.e., integration based on random number generation. This technique is named the "local correction technique", where the uniformly distributed random points are avoided from falling in the vicinity of the observation points.

The problem under investigation is an entire domain MoM problem. Thus the basis functions are defined over the full domain of the problem. Also, due to the point matching technique, all the matrix elements that are evaluated; involve integration of singular integrands; unlike the sub domain formulation where usually only the diagonal elements of the moment matrix involve singular integration.

The geometry of the problem indicates many facts about the current distribution that will be induced on the surface of the disk. These facts are of much help in selecting the basis functions required for both x and y components of the unknown current, J_x and J_y. Both these components are functions of x and y coordinates. It is easy to observe that due to the edge effects, $J_x(\pm a, 0) = 0$ whereas

$J_x(0, \pm a)$ is divergent. Similarly, $J_y(0, \pm a) = 0$ whereas $J_y(\pm a, 0)$ diverges. Also, the symmetry of the problem suggests that J_x is anti symmetric about both x and y axes, i.e., $J_x(-x, y) = -J_x(x, y)$ and $J_x(x, -y) = -J_x(x, y)$. On the other hand, J_y is symmetric about both x and y axes. $J_y(-x, y) = J_y(x, y)$ and $J_y(x, -y) = J(x, y)$. Thus, starting with the double series of polynomials $x^i y^j$ and modifying the series from the knowledge of the current stated above [17], the basis function for the unknown current distribution is taken as:

$$J_x(x,y) \cong \sum_{\substack{i=1\\(2)}}^{n_{yx}} \left\{ \sum_{\substack{j=3\\(2)}}^{n_{xx}} c_{xij}(x^j - a^{j-1}x) \right\} \cdot \frac{y^i}{\sqrt{a^2 - y^2}} \tag{10a}$$

$$J_y(x,y) \cong \sum_{\substack{i=0\\(2)}}^{n_{xy}} \left\{ \sum_{\substack{j=2\\(2)}}^{n_{yy}} c_{yij}(y^j - a^j) \right\} \cdot \frac{x^i}{\sqrt{a^2 - x^2}} \tag{10b}$$

where $n_{xx} = n_{yy} = 8$ and $n_{xy} = n_{yx} = 7$. The subscript (2) below the summation sign means that the indices i and j are to be increased with a step size 2. Thus the total number of expansion terms for x-current is 12 and for y-current is 16, thus making the total number of unknown coefficients c_{xij} and c_{yij} is equal to 28. This leads to the formation of the moment matrix of order 28×28.

Once the current distribution $\mathbf{J}(\mathbf{r}')$ over the scatterer is determined, the next step is to evaluate the Radar Cross Section (RCS) at far field distances. This evaluation involves the determination of the scattered fields at observation points. The scattered field, as given by (5), involves integration over the entire domain of the problem. In two dimensional problems, the conventionally used quadrature rules is time consuming. For two dimensional integration, the number of evaluations of the integrand performed is of the order of N^2 where N is the number of intervals into which the x and y axes are divided. On the other hand, the number of evaluations required in MCI is of the order of M, where M is the total number of uniformly distributed random points generated over the entire domain, independent of the dimensionality. It is shown in this paper that the MCI applied for the evaluation of RCS gives results which are in good agreement with those obtained from conventional sub domain MoM analysis.

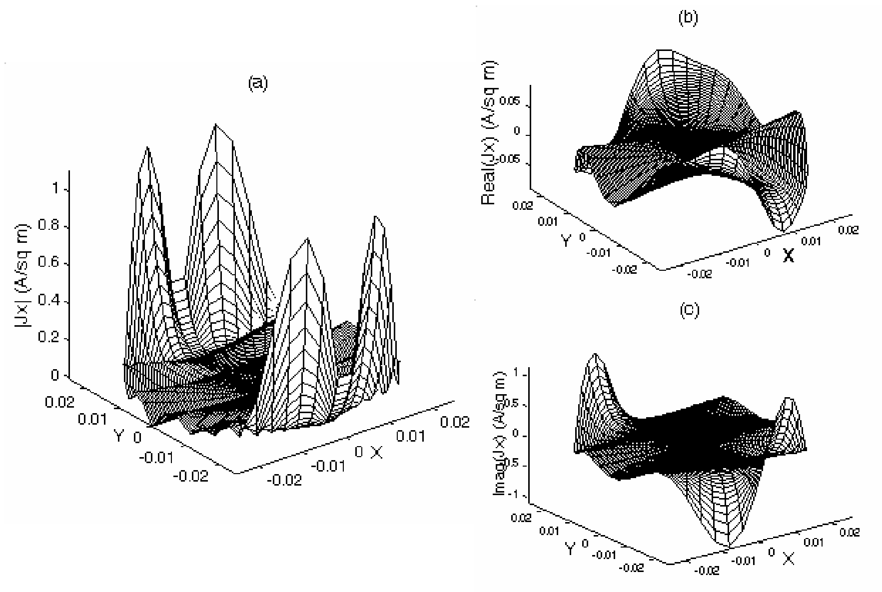

Figure 2. Solutions for the x component of current density induced over the circular disk scatterer with $a = 2.5\,\text{cm}$ and $f = 6\,\text{GHz}$. (a) Magnitude. (b) Real part. (c) Imaginary part.

4. RESULTS AND DISCUSSION

Simulations are performed for the EFIE formulation of the circular metallic disk scatterer using the MCI technique for different cases. To validate the choice of the basis function for the metallic circular disk scattering problem and to validate the implementation of the MCI in the MoM solution of the EFIE, the real part, imaginary part and the magnitude of J_x and J_y for one particular case have been plotted. Next, RCS is evaluated at far field distance due to the scattered field of the disks. For all the cases, RCS is evaluated in both the xz and the yz planes.

In the first case, a circular metallic disk is considered with $f = 6\,\text{GHz}$ and $a = 5\,\text{cm}$, which corresponds to the diameter equal to the wavelength. The current distributions for this case are plotted in Figure 2 and Figure 3. It is clear from these Figures that both J_x and J_y components of the currents induced on the surface of the scatterer satisfy the edge and symmetry conditions simultaneously as predicted. This validates the choice of the basis functions given by (10) as well as the MCI technique in the MoM solution. The RCS calculated

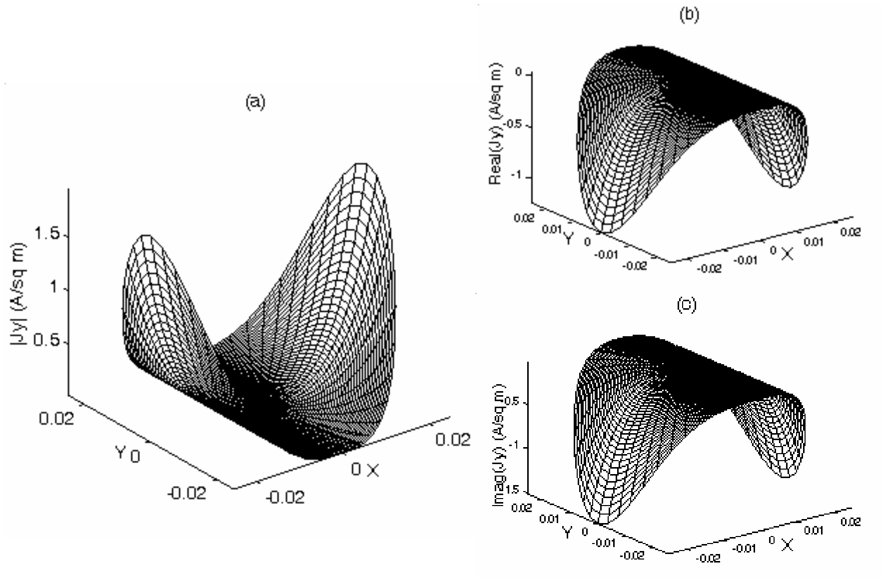

Figure 3. Solution for the y component of current density induced over the circular disk scatterer with $a = 2.5\,\text{cm}$ and $f = 6\,\text{GHz}$. (a) Magnitude. (b) Real part. (c) Imaginary part.

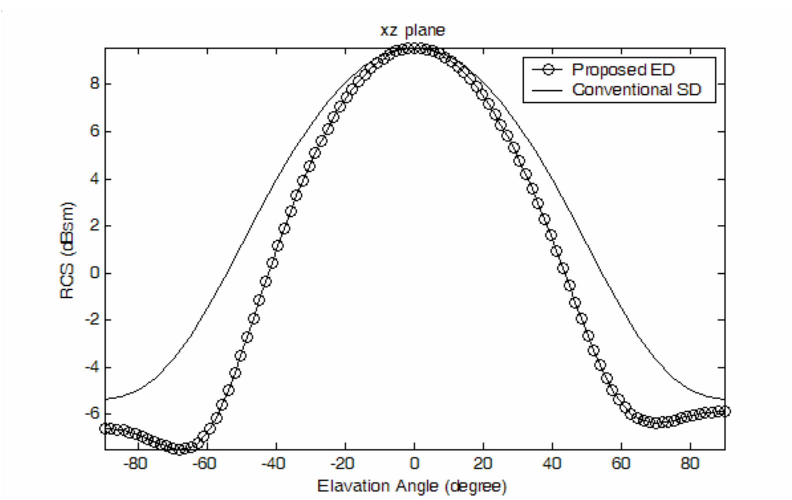

Figure 4. RCS on xz plane for the disk with $a = 2.5\,\text{cm}$ and $f = 6\,\text{GHz}$.

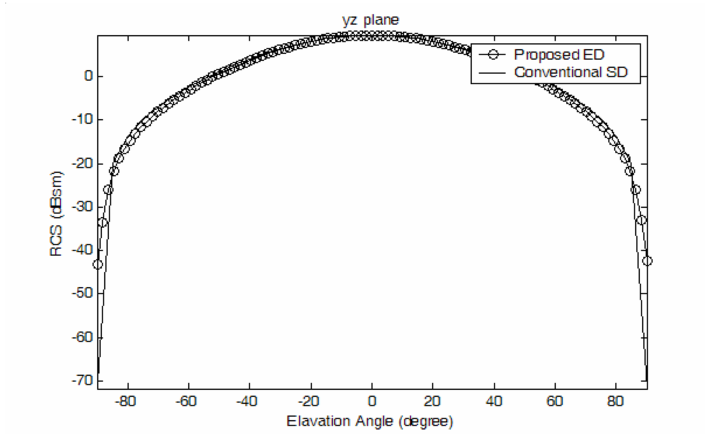

Figure 5. RCS on the yz plane for the disk with $a = 2.5\,\mathrm{cm}$ and $f = 6\,\mathrm{GHz}$.

with the proposed entire domain (ED) basis functions in this paper, is plotted and compared with that of the conventional sub domain (SD) basis functions, with respect to the elevation angle in xz and yz planes respectively. The results are plotted in Figure 4 and Figure 5 for this case.

In the second case $f = 10\,\mathrm{GHz}$ and $a = 2\,\mathrm{inch}$. Figure 6 and Figure 7 show the calculated RCS versus elevation angle for proposed ED and conventional SD methods in xz and yz planes respectively for this case.

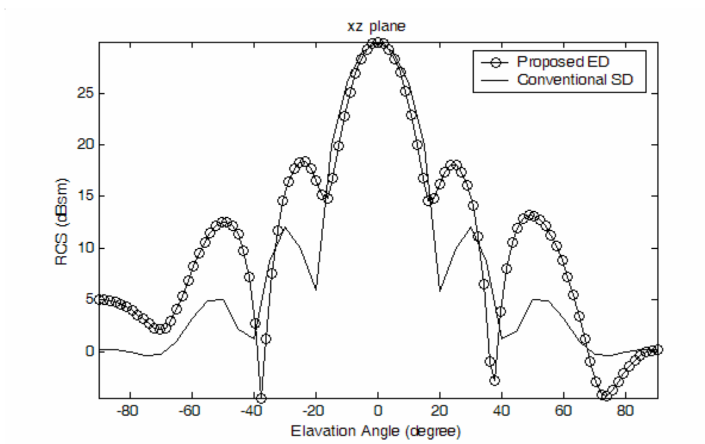

Figure 6. RCS on the xz plane for disk with $a = 2\,\mathrm{inch}$ and $f = 10\,\mathrm{GHz}$.

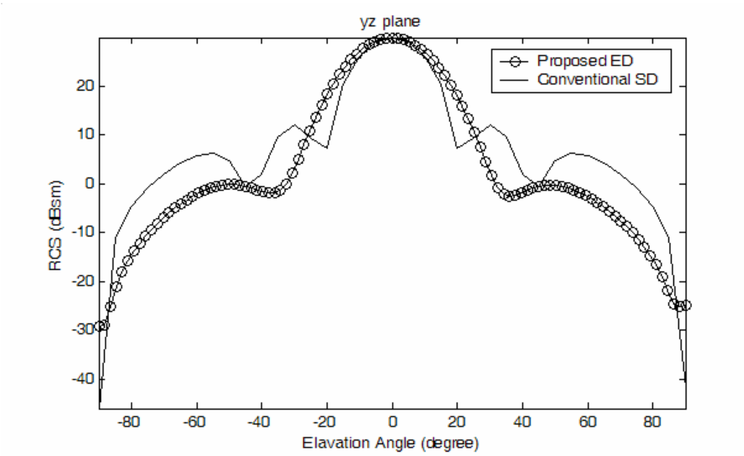

Figure 7. RCS on the yz plane for disk with $a = 2$ inch and $f = 10$ GHz.

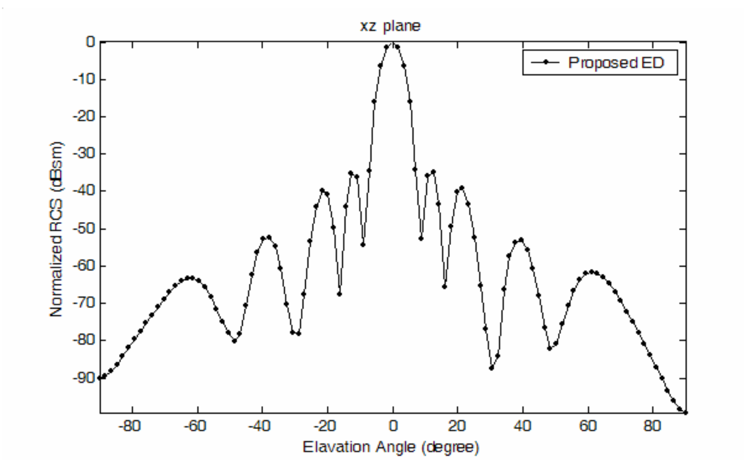

Figure 8. Normalized RCS in the xz plane for the disk with $a = 4.5$ inch and $f = 10$ GHz.

In the third and final case, $f = 10$ GHz and $a = 4.5$ inch. RCS versus elevation angle for proposed ED analysis in xz and yz planes respectively for this case are plotted in Figure 8 and Figure 9. It is a known fact that conventional sub domain methods cannot model electrically large structures, which is the situation in this case as the diameter of the disk is more than 7 times the wavelength. Thus RCS in this case is calculated and presented using only the proposed technique.

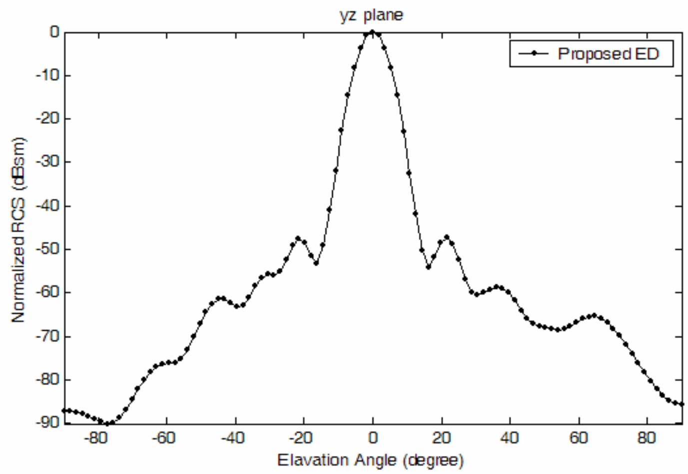

Figure 9. Normalized RCS in the yz plane for the disk with $a = 4.5$ inch and $f = 10$ GHz.

It is clear from Figure 4 through Figure 8 that RCS calculated with the proposed technique and MCI implementation in evaluation of scattered fields, is in good agreement with the conventional SD results.

5. CONCLUSIONS

In this paper, a new technique is proposed for the Method of Moments solution of the Electric Field Integral Equation. The method involves Monte Carlo Integration technique in evaluation of the moment matrix elements. This technique is not only capable of reducing the computational burden, but also overcomes the problem of singularity in the integral equation. Especially in case of two and higher dimensional problems, MCI technique is superior to the conventional quadrature methods for integration. This is because of the fact that this technique is independent of the dimensionality of the problem, involving only uniform generation of random points inside the domain of integration. In addition to this, one major advantage of the MCI technique is that it removes the singularity problem arising in integration of singular functions without any analytical modification or approximation to the integrand. The technique is applied to the problem of scattering from circular metallic disks. The basis functions employed for the problems are entire domain modified polynomial basis functions. Current distribution induced over the scatterer has been evaluated and plotted.

RCS for three different sizes of disks have been plotted and compared with conventional sub domain MoM results wherever possible. The results obtained are found to be in good agreement with each other.

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