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SOLVING TIME DOMAIN HELMHOLTZ WAVE EQUATION WITH MOD-FDM

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Abstract—In this work, we present a marching-on in degree finite difference method (MOD-FDM) to solve the time domain Helmholtz wave equation. This formulation includes electric and magnetic current densities that are expressed in terms of the incident field for scattering problems for an open region to implement a plane wave excitation. The unknown time domain functional variations for the electric field are approximated by an orthogonal basis function set that is derived using the Laguerre polynomials. These temporal basis functions are also used to expand current densities. With the representation of the derivatives of the time domain variable in an analytic form, all the time derivatives of the field and current density can be handled analytically. By applying a temporal testing procedure, we get a matrix equation that is solved in a marching-on in degree technique as the degree of the temporal basis functions is increased. Numerical results computed using the proposed formulation are presented and compared with the solutions of the conventional time domain finite difference method (TD-FDM) and analytic solutions.

1. INTRODUCTION

The finite difference method in the time domain proposed by Yee has been extensively employed to analyze transient scattered fields from conducting and dielectric materials [1–7]. This technique is often called the finite-difference time-domain (FDTD) method [2, 3]. Recently, a marching-on in degree finite difference method (MOD-FDM) with the entire domain temporal orthogonal basis using the associated Laguerre functions was proposed to obtain an unconditionally stable solution [8]. This is in contrast to the conventional FDTD analysis. The MOD-FDM methodology has been successfully implemented in time domain integral equations [9–11], and time domain finite element method [12]. Based on this scheme [8], various works have been published [13–20]. However all the previous works were developed to solve the first-order Maxwell's curl equations.

In this paper, we present a new method to solve the second-order time domain Helmholtz wave equation in a lossy media with the MOD-FDM to obtain transient electromagnetic scattering responses [21, 22]. The formulation uses the volume electric and magnetic current densities for scattering problems in an open region to implement a plane wave excitation [23]. These current densities are expressed in term of the incident field. The time domain unknown coefficient of the electric field is approximated by a set of orthogonal basis functions that are derived from the Laguerre polynomials [24]. The Laguerre polynomials are defined only over the temporal interval from zero to infinity, and hence, are considered to be more suited for the transient problems, as they naturally enforce causality. The temporal basis functions used in this work are completely convergent to zero as time increases. Therefore this basis may never cause any late time instabilities in the solution as is often the case in solving time domain problems. So, the transient response spanned by these basis functions is also convergent to zero as time progresses. These temporal basis functions are used to expand the various current densities. Since all these temporal basis functions have analytical derivatives, in the time domain formulation, all the time derivatives of the fields and current densities can be handled analytically. Use of this temporal expansion function characterizing the time variable also decouples the space-time continuum in an analytic fashion. By applying a temporal testing procedure, we get a matrix equation that is solved using a marching-on in degree as the degree of temporal basis functions is increased.

This paper is organized as follows. In the next section we present the conventional finite difference scheme using the Yee algorithm for the Helmholtz wave equation briefly, and then derive the MOD-FDM formulation with a plane wave incidence for scattering problems in an open region. In Section 3, numerical results computed using the proposed formulation are presented and compared with the solutions of the conventional time domain finite difference methods (TD-FDM) and compared with analytic solutions. Finally, in Section 4, we present some conclusions drawn from this work.

2. FORMULATION

In the time domain, the general Maxwell's equations in a lossy dielectric media can be written as

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \varepsilon(\boldsymbol{r}) \frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t} + \sigma(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{J}(\boldsymbol{r},t) \tag{1}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\mu(\boldsymbol{r}) \frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t} - \rho(\boldsymbol{r}) \boldsymbol{H}(\boldsymbol{r},t) - \boldsymbol{M}(\boldsymbol{r},t) \tag{2}$$

where ε, μ, σ and ρ are the permittivity, the permeability, the electric conductivity, and magnetic conductivity, respectively [2,3]. These parameters in the first place are considered independent of time. $\boldsymbol{E}$ is the electric field and $\boldsymbol{H}$ is the magnetic field. $\boldsymbol{J}$ and $\boldsymbol{M}$ are the volume electric and magnetic current densities, respectively, that are included to implement a plane wave excitation. From the two curl Equations (1) and (2), we obtain a vector Helmholtz wave equation for the electric field in a divergence free region without any net charge as [21,22]

$$\begin{aligned}&\nabla^2 \boldsymbol{E}(\boldsymbol{r},t) - \mu(\boldsymbol{r})\varepsilon(\boldsymbol{r}) \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r},t)}{\partial t^2} - [\mu(\boldsymbol{r})\sigma(\boldsymbol{r}) + \varepsilon(\boldsymbol{r})\rho(\boldsymbol{r})] \frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t} \\ &-\sigma(\boldsymbol{r})\rho(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r},t) = \mu(\boldsymbol{r}) \frac{\partial \boldsymbol{J}(\boldsymbol{r},t)}{\partial t} + \rho(\boldsymbol{r})\boldsymbol{J}(\boldsymbol{r},t) + \nabla \times \boldsymbol{M}(\boldsymbol{r},t).\end{aligned} \tag{3}$$

For simplicity, we consider the one-dimensional problem with the field component E_y propagating along the x-direction. Then the wave Equation (3) becomes

$$\begin{aligned}&\frac{\partial^2 E_y(x,t)}{\partial x^2} - \mu(x)\varepsilon(x) \frac{\partial^2 E_y(x,t)}{\partial t^2} - [\mu(x)\sigma(x) + \varepsilon(x)\rho(x)] \frac{\partial E_y(x,t)}{\partial t} \\ &-\sigma(x)\rho(x)E_y(x,t) = \mu(x) \frac{\partial J_y(x,t)}{\partial t} + \rho(x)J_y(x,t) - \frac{\partial M_z(x,t)}{\partial x}.\end{aligned} \tag{4}$$

Following Yee's notation and discretizing the space and the time with the cell size Δx and time step Δt, respectively [1], we can express (4) at $x = x_i = i\Delta x$ and $t = t_n = n\Delta t$ as

$$\begin{aligned}&\frac{\partial^2}{\partial x^2} E_y\big|_i^n - \mu_i\varepsilon_i \frac{\partial^2}{\partial t^2} E_y\big|_i^n - (\mu_i\sigma_i + \varepsilon_i\rho_i) \frac{\partial}{\partial t} E_y\big|_i^n - \sigma_i\rho_i E_y\big|_i^n \\ &= \mu_i \frac{\partial}{\partial t} J_y\big|_i^n + \rho_i J_y\big|_i^n - \frac{\partial}{\partial x} M_z\big|_i^n\end{aligned} \tag{5}$$

where $E_y\big|_i^n = E_y(x_i, t_n)$. Applying the second-order central finite difference approximation to (5) yields

$$\frac{E_y\big|_{i+1}^n - 2\,E_y\big|_i^n + E_y\big|_{i-1}^n}{(\Delta x)^2} - \mu_i\varepsilon_i \frac{E_y\big|_i^{n+1} - 2\,E_y\big|_i^n + E_y\big|_i^{n-1}}{(\Delta t)^2}$$

$$
\begin{aligned}
&-\left(\mu_i\sigma_i+\varepsilon_i\rho_i\right)\frac{E_y|_i^{n+1}-E_y|_i^{n-1}}{2\Delta t}-\sigma_i\rho_i\,E_y|_i^n\\
&=\ \mu_i\frac{J_y|_i^{n+1/2}-J_y|_i^{n-1/2}}{\Delta t}+\rho_i\,J_y|_i^n-\frac{M_z|_{i+1/2}^n-M_z|_{i-1/2}^n}{\Delta x}. \qquad (6)
\end{aligned}
$$

Solving for $E_y|_i^{n+1}$ and rearranging terms, we get an alternate equation with changing $n+1$ to n as

$$
\begin{aligned}
E_y|_i^n\ =\ &\frac{2\left(\Delta t\right)^2}{2\mu_i\varepsilon_i+\Delta t\left(\mu_i\sigma_i+\varepsilon_i\rho_i\right)}\left[\frac{1}{(\Delta x)^2}\,E_y|_{i-1}^{n-1}\right.\\
&+\left(\frac{\mu_i\varepsilon_i}{(\Delta t)^2}-\frac{2}{(\Delta x)^2}-\sigma_i\rho_i\right)E_y|_i^{n-1}+\frac{1}{(\Delta x)^2}\,E_y|_{i+1}^{n-1}\\
&-\frac{2\mu_i\varepsilon_i-\Delta t\left(\mu_i\sigma_i+\varepsilon_i\rho_i\right)}{2\left(\Delta t\right)^2}\,E_y|_i^{n-2}-\frac{\mu_i}{\Delta t}\left(J_y|_i^{n-1/2}-J_y|_i^{n-3/2}\right)\\
&\left.-\rho_i\,J_y|_i^{n-1}+\frac{1}{\Delta x}\left(M_z|_{i+1/2}^{n-1}-M_z|_{i-1/2}^{n-1}\right)\right]. \qquad (7)
\end{aligned}
$$

This is generally the equation used in a TD-FDM to obtain the electric field.

Now we derive a MOD-FDM formulation. To carry out the time derivatives analytically, we expand all the temporal quantities in terms of the associate Laguerre polynomials given by

$$\phi_p(st)=\mathrm{e}^{-st/2}L_p(st) \qquad (8)$$

where s is a time scale parameter which takes care of the units along the time axis [25], and L_p is the Laguerre polynomial with degree p [24]. This temporal basis functions are orthogonal as

$$\int_0^\infty \phi_p(st)\phi_q(st)d(st)=\delta_{pq} \qquad (9)$$

where δ_{pq} is Kronecker delta with value 1 when $p=q$ and 0 otherwise.

A continuous function $F(x,t)$ defined for any value of time $t\geq 0$ can be expanded by the associate Laguerre basis functions as

$$F(x,t)=\sum_{p=0}^{\infty}F_p(x)\phi_p(st) \qquad (10)$$

where F_p is the coefficient which can be obtained from

$$F_p(x)=\int_0^\infty F(x,t)\phi_p(st)d(st). \qquad (11)$$

The first and second derivatives of the function $F(x,t)$ can be written as [9]

$$\frac{d}{dt}F(x,t) = s\sum_{p=0}^{\infty}\left(\frac{1}{2}F_p(x)+\sum_{m=0}^{p-1}F_m(x)\right)\phi_p(st) \tag{12}$$

$$\frac{d^2}{dt^2}F(x,t) = s^2\sum_{p=0}^{\infty}\left(\frac{1}{4}F_p(x)+\sum_{m=0}^{p-1}(p-m)F_m(x)\right)\phi_p(st). \tag{13}$$

By expanding E_y, J_y, and M_z with (10), (12), and (13), and putting them in (4) we have

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\sum_{p=0}^{\infty}E_y^p(x)\phi_p(st)\\
&-\mu(x)\varepsilon(x)s^2\sum_{p=0}^{\infty}\left(\frac{1}{4}E_y^p(x)+\sum_{m=0}^{p-1}(p-m)\,E_y^m(x)\right)\phi_p(st)\\
&-[\mu(x)\sigma(x)+\varepsilon(x)\rho(x)]\,s\sum_{p=0}^{\infty}\left(\frac{1}{2}E_y^p(x)+\sum_{m=0}^{p-1}E_y^m(x)\right)\phi_p(st)\\
&-\sigma(x)\rho(x)\sum_{p=0}^{\infty}E_y^p(x)\phi_p(st)=\mu(x)s\sum_{p=0}^{\infty}\left(\frac{1}{2}J_y^p(x)+\sum_{m=0}^{p-1}J_y^m(x)\right)\phi_p(st)\\
&+\rho(x)\sum_{p=0}^{\infty}J_y^p(x)\phi_p(st)-\frac{\partial}{\partial x}\sum_{p=0}^{\infty}M_z^p(x)\phi_p(st)
\end{aligned} \tag{14}$$

where E_y^p, J_y^p and, M_z^p are the coefficients of the associate Laguerre polynomials basis for E_y, J_y, and M_z, respectively.

To eliminate the variable t and the infinite summation in (14), we test this equation in a Galerkin's methodology with $\phi_q(st)$. Due to the orthogonal property in (9) we have

$$\begin{aligned}
&\frac{d^2}{dx^2}E_y^q(x)-\mu(x)\varepsilon(x)s^2\left(\frac{1}{4}E_y^q(x)+\sum_{m=0}^{q-1}(q-m)\,E_y^m(x)\right)\\
&-[\mu(x)\sigma(x)+\varepsilon(x)\rho(x)]\,s\left(\frac{1}{2}E_y^q(x)+\sum_{m=0}^{q-1}E_y^m(x)\right)-\sigma(x)\rho(x)E_y^q(x)\\
&=\mu(x)s\left(\frac{1}{2}J_y^q(x)+\sum_{m=0}^{q-1}J_y^m(x)\right)+\rho(x)J_y^q(x)-\frac{d}{dx}M_z^q(x).
\end{aligned} \tag{15}$$

Using the finite difference in space to approximate the spatial derivatives is similar to the traditional FDTD method. By using second-order spatial difference in (15) at $x = i\Delta x$, we have

$$\frac{E_y|_{i+1}^q - 2\,E_y|_i^q + E_y|_{i-1}^q}{(\Delta x)^2} - \mu_i\varepsilon_i s^2\left(\frac{1}{4}\,E_y|_i^q + \sum_{m=0}^{q-1}(q-m)\,E_y|_i^m\right)$$
$$-(\mu_i\sigma_i + \varepsilon_i\rho_i)\,s\left(\frac{1}{2}\,E_y|_i^q + \sum_{m=0}^{q-1}E_y|_i^m\right) - \sigma_i\rho_i\,E_y|_i^q$$
$$= \mu_i s\left(\frac{1}{2}\,J_y|_i^q + \sum_{m=0}^{q-1}J_y|_i^m\right) + \rho_i\,J_y|_i^q - \frac{M_z|_{i+1/2}^q - M_z|_{i-1/2}^q}{\Delta x} \quad (16)$$

where $E_y|_i^q = E_y^q(x_i)$,

$$J_y|_i^q = \int_0^\infty J_y(x_i, t)\,\phi_q(st)d(st) \quad (17)$$
$$M_z|_{i\pm 1/2}^q = \int_0^\infty M_z\left(x_{i\pm 1/2}, t\right)\phi_q(st)d(st). \quad (18)$$

Rewriting (16) in a simple form, we have

$$E_y|_{i-1}^q + \alpha_{ii}\,E_y|_i^q + E_y|_{i+1}^q = \beta_i^q \quad (19)$$

where

$$\alpha_{ii} = -\left[2 + \mu_i\varepsilon_i\left(\frac{s\Delta x}{2}\right)^2 + \frac{(\mu_i\sigma_i + \varepsilon_i\rho_i)\,s(\Delta x)^2}{2} + \sigma_i\rho_i(\Delta x)^2\right] \quad (20)$$

$$\beta_i^q = \mu_i\varepsilon_i\,(s\Delta x)^2\sum_{m=0}^{q-1}(q-m)\,E_y|_i^m + (\mu_i\sigma_i + \varepsilon_i\rho_i)\,s(\Delta x)^2\sum_{m=0}^{q-1}E_y|_i^m$$
$$+(\Delta x)^2\left[\left(\frac{\mu_i s}{2} + \rho_i\right)J_y|_i^q + \mu_i s\sum_{m=0}^{q-1}J_y|_i^m\right] - \Delta x\left(M_z|_{i+1/2}^q - M_z|_{i-1/2}^q\right). \quad (21)$$

We can get a matrix equation form from (19)–(21) with a proper boundary condition as

$$[\alpha_{ij}]\left[E_y|_j^q\right] = [\beta_i^q], \quad q = 0, 1, 2\ldots. \quad (22)$$

Here we use the dispersion boundary condition derived with the associate Laguerre basis functions in [8]. By solving this matrix

Equation (22) recursively in a MOD manner and using (10), the electric field is expressed as

$$E_y\left(x_i, t_n\right) = \sum_{p=0}^{M-1} E_y|_i^p \, \phi_p\left(st_n\right) \tag{23}$$

where M is a finite number of temporal basis functions. Using the plane wave injector scheme, the corresponding electric and magnetic current densities are expressed in term of the incident electric field [23]. When a plane wave with y-polarization is incident to the x-direction, the current densities in (17) and (18) are given by [20]

$$J_y(x,t) = -\frac{E^{\text{inc}}(x,t)}{\eta \Delta x} \tag{24}$$

$$M_z(x,t) = -\frac{E^{\text{inc}}(x,t)}{\Delta x} \tag{25}$$

where η is the wave impedance of free space.

We can also construct a Helmholtz wave equation for the magnetic field using the two familiar curl Equations (1) and (2), and derive the matrix equation corresponding to (22) in a similar way. In this case, the matrix $[\alpha_{ij}]$ is same as in (22), but $[\beta_i^q]$ is slightly different due to the duality. Therefore we can obtain the coefficients from the magnetic field without an additional matrix inversion in this procedure when computing the electric field coefficients.

3. NUMERICAL EXAMPLES

The geometry to be analyzed here is a one-dimensional dielectric slab backed by a perfectly electric conductor (PEC). This slab has a relative permittivity 4, $\sigma = 0.05\,\text{S/m}$, and is 9 cm thick. The permeability is that of free space and the magnetic loss is assumed to be zero. The problem space consists of 310 cells with $\Delta x = 1.5\,\text{mm}$ and 250–310 cells for the slab. The PEC boundary condition is applied at the boundary $x = 46.5\,\text{cm}$. The incident field in (24) and (25) used in this work is the Gaussian pulse plane wave as defined in [26]

$$E^{\text{inc}}(x,t) = E_0 \frac{4}{T\sqrt{\pi}} \mathrm{e}^{-\gamma^2}, \quad \gamma = \frac{4}{T}\left(ct - ct_0 - x + x_s\right) \tag{26}$$

where T is the pulse width, c is the velocity of propagation in free space, t_0 is a time delay which represents the time at which the pulse peak at the origin, and x_s is the source position of the plane wave incidence. In the computation of the TD-FDM using (7), we set the time step size $\Delta t = \Delta x/2c$. In the numerical computation, the pulse width is $T/c = 400\,\text{ps}$, and $x_s/\Delta x = 50$. We set the number of Laguerre basis functions as $M = 500$ and the time scale parameter

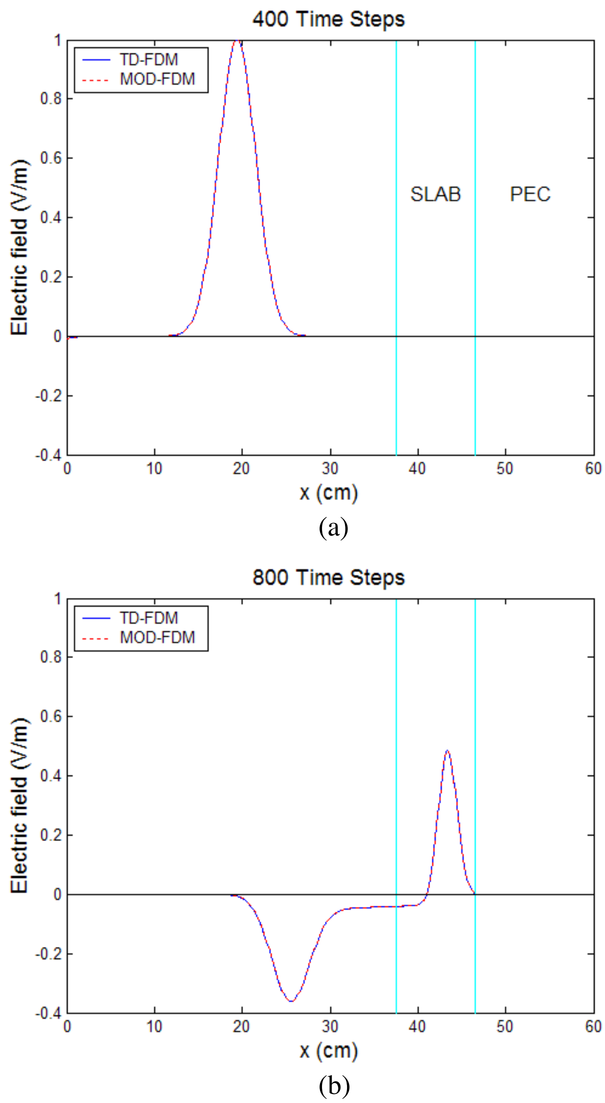


Figure 1. Electric field along x-position with the incidence of Gaussian pulse plane wave. (a) 400 time steps, (b) 800 time steps.

is set to $s = 1.2 \times 10^{10}$. We compare the electric field computed by the proposed MOD-FDM with the solution obtained using the TD-FDM and from the analytic solution. The analytic solution is obtained by the inverse discrete Fourier transform of the frequency domain solution as described in [27].

For the first example, Fig. 1 shows the electric field along the x-direction for an incident Gaussian pulse at $n = 400$ and 800 time steps of the TD-FDM computation. We set $E_0 = T\sqrt{\pi}/4$. The agreement between the conventional TD-FDM and the proposed MOD-FDM is very good. Fig. 2 shows the transient electric field at $x = 30\,\mathrm{cm}$ computed by the MOD-FDM and TD-FDM, and inverse Fourier transform of the analytic solution. All the three solutions agree well, as is evident from the figure.

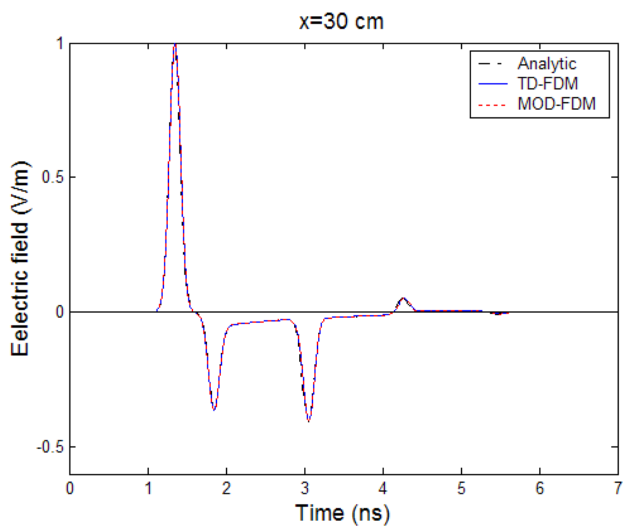
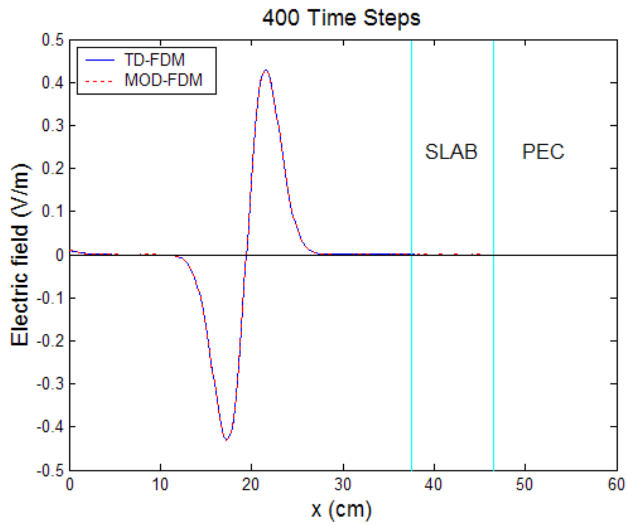
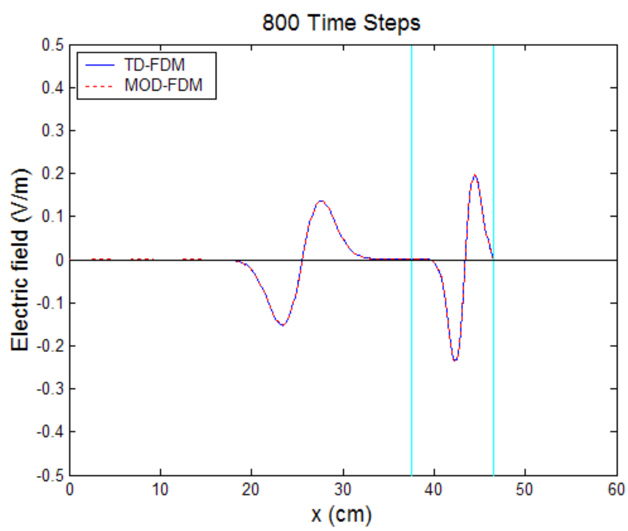


Figure 2. Transient electric field at $x = 30\,\mathrm{cm}$ versus time with the incidence of Gaussian pulse plane wave.

In the next example, Fig. 3 shows the electric field as a function of the x-coordinates at temporal locations of $n = 400$ and 800 time steps when the derivative of Gaussian pulse with $E_0 = T^2\sqrt{\pi}/(32c)$ is incident on the slab. Agreement between the solutions obtained using the proposed MOD-FDM and the conventional TD-FDM is excellent. Fig. 4 shows the transient electric field at $x = 30\,\mathrm{cm}$ for an incident field proportional to the derivative form of the Gaussian pulse. We can see that the agreement between the solutions obtained by the TD-FDM and MOD-FDM, and using the analytic solution is very good.



(a)



(b)

Figure 3. Electric field along x-position with the incidence of the derivative of Gaussian pulse plane wave. (a) 400 time steps, (b) 800 time steps.

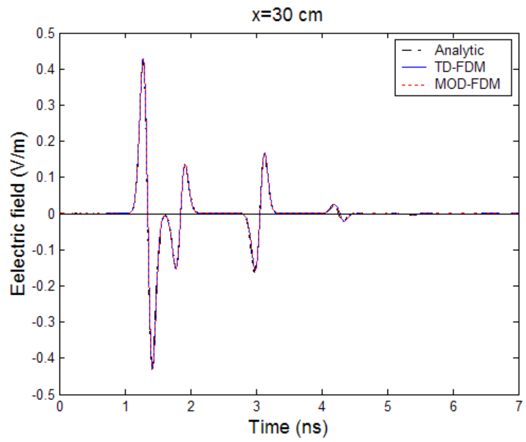


Figure 4. Transient electric field at $x = 30\,\text{cm}$ versus time with the incidence of the derivative of Gaussian pulse plane wave.

4. CONCLUSION

We have proposed a marching-on in degree finite difference method to solve the time domain Helmholtz wave equation. This formulation uses the electric and the magnetic current densities to excite a plane wave source in an open region. The time domain unknown coefficients for the electric field are approximated by a set of orthogonal basis functions that are derived from the Laguerre polynomials. With the representation of the derivatives of the time domain coefficients in an analytic form, all the time derivatives of the field and current density can be handled analytically. By applying a temporal testing procedure, we get a matrix equation that is solved using a marching-on in degree technique as the degree of the temporal basis functions is increased. The agreement between the solutions obtained using the proposed method and the traditional time domain finite difference method, and the inverse Fourier transform of the frequency domain analytic solution is excellent as a function of both the spatial and the temporal variables. The proposed formulation can be extended to two- and three-dimensional scattering problems directly.

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