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SOLITON PARAMETER DYNAMICS IN A NON-KERR LAW MEDIA

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Abstract—The adiabatic parameter dynamics of non-Kerr law optical solitons is obtained in this paper by the aid of soliton perturbation theory. The various kinds of perturbation terms that arise exhaustively in the context of optical solitons are considered in this paper. The new conserved quantity is also used to obtain the adiabatic dynamics of the soliton phase in all cases of non-Kerr laws studied in this paper. The non-Kerr law nonlinearities that are considered in this paper are power law, parabolic law as well as the dual-power law.

1. INTRODUCTION

The theory of optical solitons is a very important area of research for the last couple of decades [1–45]. The theoretical possibility of existence of optical solitons in a dielectric dispersive fiber was first predicted by Hasegawa and Tappert [9, 10]. A couple of years later Mollenauer et al. [29] successfully performed the famous experiment to verify this prediction. Important characteristic properties of these solitons are that they posses a localized waveform which remains intact upon interaction with another soliton. Because of their remarkable robustness, they attracted enormous interest in optical and telecommunication community. At present optical solitons are regarded as the natural data bits for transmission and processing of information in future, and an important alternative for the next generation of ultra high speed optical communication systems.

The fundamental mechanism of soliton formation namely the balanced interplay of linear group velocity dispersion (GVD) and nonlinearity induced self-phase modulation (SPM) is well understood. In the pico second regime, the nonlinear evolution equation that takes into account this interplay of GVD and SPM and which describes the dynamics of soliton is the well known nonlinear Schrödinger's equation (NLSE). The NLSE, which is the ideal equation in an ideal Kerr media, is in its original form found to be completely integrable by the method of Inverse Scattering Transform (IST) [1–4] and profound success has been achieved in the development of soliton theory in the framework of the NLSE model.

However, communication grade optical fibers or as a matter of fact any optical transmitting medium does posses finite attenuation coefficient, thus optical loss is inevitable and the pulse is often detoriated by this loss. Therefore, optical amplifiers have to be employed to compensate for this loss. When the gain bandwidth of the amplifier is comparable to the spectral width of the ultrashort optical pulse, the frequency and intensity dependent gain must be considered. Another hindrance to the stable propagation in a practical system is the noise induced Gordon-Hauss timing jitter [11]. An important aspect that has not been addressed with proper perspective is the fact that due to its nonsaturable nature, Kerr nonlinearity is inadequate to describe the soliton dynamics in the ultrahigh bit rate transmission. For example, when transmission bit rate is very high, for soliton formation the peak power of the incident field accordingly become very large. On the other hand higher order nonlinearities may become significant even at moderate intensities in certain materials such as semiconductor doped glass fibers. Under circumstances, as mentioned above, non-Kerr law nonlinearities come into play changing essentially the physical features of optical soliton propagation [4]. Therefore when very high bit rate transmission or transmission through materials with higher nonlinear coefficients are considered, it is necessary to take into account higher order nonlinearities. This problem can be addressed by incorporating the non-Kerr law nonlinearity in the NLSE.

It has been realized that the Gordon-Haus timing jitter can be reduced by introducing bandpass filtering. Stabilization of soliton propagation with the aid of nonlinear gain or under combined operation of gain and saturable absorption was recommended by Kodama et al. [11]. Thus, in order to model these features in the soliton dynamics, from a practical standpoint, the NLSE should be modified by incorporating additional terms. Thus, the concept of control of soliton propagation described by the NLSE with non-Kerr law nonlinearity is new and important developments in the application of solitons for optical communication systems. Because the NLSE with non-Kerr law is not integrable, perturbation methods or numerical techniques have to be incorporated.

2. MATHEMATICAL ANALYSIS

The dimensionless form of NLSE with non-Kerr law nonlinearity is given by

$$iq_t + \frac{1}{2}q_{xx} + F\left(|q|^2\right)q = 0. \tag{1}$$

where x represents the nondimensional distance along the fiber while, t represents time in dimensionless form. Equation (1) is a nonlinear partial differential equation that is not integrable, in general. The non-integrability is not necessarily related to the nonlinear term in it. Also, in (1), F is a real-valued algebraic function and it is necessary to have the smoothness of the complex function $F\left(|q|^2\right)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, the function $F\left(|q|^2\right)q$ is k times continuously differentiable, so that [45]

$$F\left(|q|^2\right)q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m);R^2\right) \tag{2}$$

The soliton solution of (1), although not integrable, is assumed to be given in the form [5]

$$q(x,t) = A(t)g\left[B(t)\left\{x-\bar{x}(t)\right\}\right]e^{i\phi(x,t)} \tag{3}$$

where

$$\phi(x,t) = -\kappa(t)x + \omega(t)t + \theta(t) \tag{4}$$

In (3), A represents the amplitude of the soliton, B is the inverse width of the soliton, $\bar{x}(t)$ is the center position and g represents the functional form of the soliton which depends on the type of nonlinearity that is being considered. In (4), κ is the frequency, ω is the wave number and θ is the phase. The soliton width and the amplitude are related as $B(t) = \Lambda(A(t))$ where the functional form of Λ depends on the type of nonlinearity in (1). Therefore, one can write

$$\frac{d\bar{x}}{dt} = v = -\kappa = \frac{\partial \phi}{\partial x} \tag{5}$$

Equation (1) has three integrals of motion [5] also known as the conserved quantities. They are the energy (E) also known as the wave power or L_2 norm, linear momentum (M) and the Hamiltonian (H) that are respectively given by

$$E = \int_{-\infty}^{\infty} |q|^2 \, dx = \frac{A^2}{B} I_{0,2,0,0,0,0,0,0} \tag{6}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (q^* q_x - q q_x^*) \, dx = -\kappa \frac{A^2}{B} I_{0,2,0,0,0,0,0,0} \tag{7}$$

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[\frac{1}{2} |q_x|^2 - \int_0^I F(\xi) d\xi \right] dx \\ &= \frac{A^2 B}{2} I_{0,0,2,0,0,0,0,0} + \frac{\kappa^2 A^2}{2B} I_{0,2,0,0,0,0,0,0} - \int_{-\infty}^{\infty}\int_0^I F(s) ds dx \end{aligned} \tag{8}$$

where the intensity I is given by $I = |q|^2$. Here, the integral

$$I_{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8} = \int_{-\infty}^{\infty} \tau^{n_1} g^{n_2}(\tau) \left(\frac{dg}{d\tau}\right)^{n_3} \left(\frac{d^2 g}{d\tau^2}\right)^{n_4} \left(\frac{d^3 g}{d\tau^3}\right)^{n_5} \left(\frac{d^4 g}{d\tau^4}\right)^{n_6} \left(\frac{d^5 g}{d\tau^5}\right)^{n_7} \left(\frac{d^6 g}{d\tau^6}\right)^{n_8} d\tau \tag{9}$$

is defined for non-negative integers n_j for $1 \leq j \leq 8$ with

$$\tau = B(t)\{x - \bar{x}(t)\} \tag{10}$$

Besides these conserved quantities, the conserved quantity that governs the conservation of phase is given by [5, 17]

$$\frac{i}{2}\int_{-\infty}^{\infty} (q q_t^* - q^* q_t) \, dx - \frac{i}{2}\frac{d}{dt}\int_{-\infty}^{\infty} x (q q_x^* - q^* q_x) \, dx = -\int_{-\infty}^{\infty} \left(\frac{3}{2}|q_x|^2 - c|q|^4\right) dx \tag{11}$$

where

$$c = \frac{2\omega_1(A,B)I_{0,2,0,0,0,0,0,0,0} - 3B^2 I_{0,0,2,0,0,0,0,0,0}}{2A^2 I_{0,4,0,0,0,0,0,0,0}} \tag{12}$$

when the wave number $\omega(A, B)$ can be written as

$$\omega(A,B) = \omega_1(A,B) - \frac{\kappa^2}{2} \tag{13}$$

and the function ω_1 is known only when the type of nonlinearity in (1) is known. This conservation law provides the law of variation of the soliton phase θ that is defined in (4). For the soliton given by (3), the parameters are now defined as follows [5]

$$\kappa(t) = \frac{i}{2}\frac{\int_{-\infty}^{\infty}(qq_x^* - q^*q_x)\,dx}{\int_{-\infty}^{\infty}|q|^2\,dx} = \frac{i}{2E}\int_{-\infty}^{\infty}(qq_x^* - q^*q_x)\,dx \tag{14}$$

$$\bar{x}(t) = \frac{\int_{-\infty}^{\infty}x\,|q|^2\,dx}{\int_{-\infty}^{\infty}|q|^2\,dx} = \frac{1}{E}\int_{-\infty}^{\infty}x\,|q|^2\,dx \tag{15}$$

Thus, the energy conservation, and the fact that $B(t) = \Lambda(A(t))$, implies

$$\frac{dA}{dt} = \frac{dB}{dt} = 0 \tag{16}$$

Again the momentum conservation and relation (14) gives

$$\frac{d\kappa}{dt} = 0 \tag{17}$$

Finally the conservation law of the phase given by (11) yields

$$\frac{d\theta}{dt} = 0 \tag{18}$$

Thus the relations given by (16)–(18) imply that the soliton amplitude, width, frequency and the phase all stay constant and do not vary with time during propagation down the fiber. Finally, differentiating both sides of (15) with respect to time, yields

$$\frac{d\bar{x}}{dt} = v \tag{19}$$

which is also seen in (5).

2.1. Perturbation Terms

The NLSE along with its perturbation terms that will be studied in this paper is given by

$$iq_t + \frac{1}{2}q_{xx} + F\left(|q|^2\right)q = i\epsilon R[q, q^*] \tag{20}$$

Here R is a spatio-differential or integro-differential operator while the perturbation parameter ϵ with $0 < \epsilon \ll 1$ is called the relative width of the spectrum that arises due to quasi-monochromaticity [4, 11]. In presence of the perturbation terms, the adiabatic dynamics of the soliton parameters are given by [4]

$$\frac{dE}{dt} = \epsilon \int_{\infty}^{\infty} (q^* R + qR^*)\, dx \tag{21}$$

$$\frac{dM}{dt} = i\epsilon \int_{\infty}^{\infty} (q_x^* R - q_x R^*)\, dx \tag{22}$$

$$\frac{d\kappa}{dt} = \frac{\epsilon}{E}\left[i\int_{-\infty}^{\infty} (q_x^* R - q_x R^*)\, dx + \kappa \int_{-\infty}^{\infty} (q^* R + qR^*)\, dx\right] \tag{23}$$

$$v = \frac{d\bar{x}}{dt} = -\kappa + \frac{\epsilon}{E}\int_{-\infty}^{\infty} x\,(q^* R + qR^*)\, dx \tag{24}$$

$$\frac{d\theta}{dt} = \frac{i\epsilon}{2}\int_{-\infty}^{\infty} \left[(qR^* - q^* R) + x\,(q_x R^* - q_x^* R)\, dx\right] dx \tag{25}$$

Equations (21)–(25) can be rewritten in the following alternative form

$$\frac{dE}{dt} = \frac{2\epsilon A}{B}\int_{-\infty}^{\infty} g(\tau)\Re\left[Re^{-i\phi}\right] d\tau \tag{26}$$

$$\frac{dM}{dt} = \frac{2\epsilon A}{B}\int_{-\infty}^{\infty} \left\{B\frac{dg}{d\tau}\Im\left[Re^{-i\phi}\right] - \kappa g(\tau)\Re\left[Re^{-i\phi}\right]\right\} d\tau \tag{27}$$

$$\begin{aligned}\frac{d\kappa}{dt} &= \frac{2\epsilon\kappa A}{BE}\int_{-\infty}^{\infty} g(\tau)\Re\left[Re^{-i\phi}\right] d\tau \\ &\quad + \frac{2\epsilon A}{BE}\int_{-\infty}^{\infty} \left\{\kappa g(\tau)\Re\left[Re^{-i\phi}\right] + B\frac{dg}{d\tau}\Im\left[Re^{-i\phi}\right]\right\} d\tau\end{aligned} \tag{28}$$

$$v = \frac{d\bar{x}}{dt} = -\kappa + \frac{2\epsilon A}{BE}\int_{-\infty}^{\infty} x g(\tau)\Re\left[Re^{-i\phi}\right] d\tau \tag{29}$$

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{\epsilon A}{B}\int_{-\infty}^{\infty} g(\tau)\Im\left[Re^{-i\phi}\right] d\tau \\ &\quad \frac{\epsilon A}{B^2}\int_{-\infty}^{\infty} \tau\left\{B\frac{dg}{d\tau}\Im\left[Re^{-i\phi}\right] - \kappa g(\tau)\Re\left[Re^{-i\phi}\right]\right\} d\tau\end{aligned} \tag{30}$$

Equations (21)–(25) gives the adiabatic variation of soliton parameters while equations (26)–(30) gives the adiabatic dynamics of the same soliton parameters in amplitude-phase format.

In this paper, the following perturbation terms that are considered, are all exhaustively studied in the context of fiber optics and optical solitons.

$$\begin{aligned} R = & \ \delta |q|^{2m} q + \alpha q_x + \beta q_{xx} - \gamma q_{xxx} + \lambda \left(|q|^2 q\right)_x + \sigma \left(|q|^2\right)_x q \\ & + \rho |q_x|^2 q - i\xi \left(q^2 q_x^*\right)_x - i\eta q_x^2 q^* - i\zeta q^* \left(q^2\right)_{xx} - i\mu \left(|q|^2\right)_x q \\ & - i\chi q_{xxxx} - i\psi q_{xxxxxx} + (\sigma_1 q + \sigma_2 q_x) \int_{-\infty}^{x} |q|^2 ds \end{aligned} \tag{31}$$

In (31), δ is the coefficient of nonlinear damping or amplification [5] depending on its sign and m could be 0, 1, 2. For $m = 0$, δ is the linear amplification or attenuation according to δ being positive or negative. For $m = 1$, δ represents the two-photon absorption (or a nonlinear gain if $\delta > 0$). If $m = 2$, δ gives a higher order correction (saturation or loss) to the nonlinear amplification-absorption. Also, β is the bandpass filtering term [11]. In (31), λ is the self-steepening coefficient for short pulses [11] (typically $\leq$ 100 femto seconds), ν is the higher order dispersion coefficient [11]. Here μ is the coefficient of Raman scattering [4, 11] and α is the frequency separation between the soliton carrier and the frequency at the peak of EDFA gain [6]. Moreover, ρ represents the coefficient of nonlinear dissipation induced by Raman scattering [11]. The coefficients of ξ, η and ζ arise due to quasi-solitons [44]. The integro-differential perturbation terms with σ_1 and σ_2 are due to saturable amplifiers [11].

The coefficients of the higher order dispersion terms are respectively given by γ, χ and ψ. It is known that the NLSE, as given by (1), does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and so higher order dispersion terms come in. If the group velocity dispersion is close to zero, one needs to consider the third and higher order dispersion for performance enhancement along trans-oceanic and trans-continental distances. Also, for short pulse widths where changes in group velocity dispersion, within the spectral bandwidth of the signal, cannot be neglected, one needs to take into account the presence of higher order dispersion terms. This reasoning leads to the inclusion of the fourth and sixth order dispersion terms that are respectively given by the coefficients of χ and ψ.

For the perturbation terms given by (31), the adiabatic parameter dynamics is given by

$$\frac{dE}{dt} = \frac{\epsilon A^2}{B^2}\left[2\delta A^{2m} B I_{0,2m+2,0,0,0,0,0,0,0} - 2\beta B\left(B^2 I_{0,0,2,0,0,0,0,0,0} + \kappa^2 I_{0,2,0,0,0,0,0,0,0}\right) - 2\rho A^2 B\left(B^2 I_{0,2,2,0,0,0,0,0,0} + \kappa^2 I_{0,4,0,0,0,0,0,0,0}\right) + 2\sigma_1 A^2 \int_{-\infty}^{\infty} g^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau + \sigma_2 A^2\left\{B^2\int_{-\infty}^{\infty}\left(\frac{dg}{d\tau}\right)^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau + \kappa^2 \int_{-\infty}^{\infty} g^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau\right\}\right] \tag{32}$$

$$\frac{dM}{dt} = -\frac{\epsilon\kappa A^2}{B^2}\left[2\delta A^{2m} B I_{0,2m+2,0,0,0,0,0,0,0} - 2\beta B\left(B^2 I_{0,0,2,0,0,0,0,0,0} + \kappa^2 I_{0,2,0,0,0,0,0,0,0}\right) - 2\rho A^2 B\left(B^2 I_{0,2,2,0,0,0,0,0,0} + \kappa^2 I_{0,4,0,0,0,0,0,0,0}\right) + 2\sigma_1 A^2 \int_{-\infty}^{\infty} g^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau + \sigma_2 A^2\left\{B^2\int_{-\infty}^{\infty}\left(\frac{dg}{d\tau}\right)^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau + \kappa^2 \int_{-\infty}^{\infty} g^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau\right\}\right] \tag{33}$$

$$\frac{d\kappa}{dt} = \frac{\epsilon}{B I_{0,2,0,0,0,0,0,0,0}}\left[2\beta\kappa B^3\left(I_{0,1,0,1,0,0,0,0,0} - I_{0,0,2,0,0,0,0,0,0}\right) - 4\mu A^2 B^3 I_{0,2,2,0,0,0,0,0,0} - \sigma_2\kappa A^2\left\{B^2\int_{-\infty}^{\infty}\left(\frac{dg}{d\tau}\right)^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau + \kappa^2 \int_{-\infty}^{\infty} g^2\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right) d\tau\right\}\right] \tag{34}$$

$$\frac{d\theta}{dt} = -\frac{\epsilon A^2}{4B^3}\Big[2\kappa A^2 B^2(\sigma + 3\lambda) I_{0,4,0,0,0,0,0,0,0} + 4\gamma\kappa B^2\left(2B^2 I_{0,0,2,0,0,0,0,0,0} + \kappa^2 I_{0,2,0,0,0,0,0,0,0}\right) + \xi A^2 B^2\left(4B^2 I_{0,2,2,0,0,0,0,0,0} - 5\kappa^2 I_{0,4,0,0,0,0,0,0,0} + 4B^2 I_{1,2,1,1,0,0,0,0} + 8B^2 I_{1,1,3,0,0,0,0,0,0}\right) + \eta A^2 B^2\left(4B^2 I_{0,2,2,0,0,0,0,0,0} - 5\kappa^2 I_{0,4,0,0,0,0,0,0,0} + 4B^2 I_{1,1,3,0,0,0,0,0,0}\right) + 4\zeta A^2 B^2\left(3\kappa^2 I_{0,4,0,0,0,0,0,0,0} - 4B^2 I_{0,2,2,0,0,0,0,0,0} - 2B^2 I_{1,2,1,1,0,0,0,0} - 2B^2 I_{1,1,3,0,0,0,0,0,0}\right) + \chi B^2\Big(6B^4 I_{0,0,0,2,0,0,0,0}$$

$$
\begin{aligned}
&+6\kappa^2 A^2 I_{0,4,0,0,0,0,0,0,0} + 29\kappa^2 B^2 I_{0,0,0,2,0,0,0,0,0}\Big) \\
&-2\psi\kappa B^2 \Big(3B^6 I_{0,0,0,0,2,0,0,0} + 27\kappa^4 B^2 I_{0,0,2,0,0,0,0,0,0} \\
&+27\kappa^2 B^4 I_{0,0,0,2,0,0,0,0} - 18\kappa^3 B^2 I_{0,0,2,0,0,0,0,0,0} + 3\kappa^4 I_{0,2,0,0,0,0,0,0,0}\Big) \\
&-4\sigma_1\kappa A^2 \int_{-\infty}^{\infty} \tau g^2(\tau)\left(\int_{-\infty}^{\tau} g^2(s)ds\right)d\tau\Big]
\end{aligned}
\tag{35}
$$

while the velocity change is given by

$$
\begin{aligned}
v \;=\; \frac{d\bar{x}}{dt} &= -\kappa + \frac{\epsilon}{I_{0,2,0,0,0,0,0,0,0}}\left[2\alpha I_{1,1,1,0,0,0,0,0,0}\right. \\
&+2A^2\left(3\lambda + 2\theta + 2\xi\kappa - 2\eta\kappa - 8\rho\kappa\right) I_{1,3,1,0,1,0,0,0} \\
&-4\psi\kappa\left(3B^4 I_{1,1,0,0,0,0,1,0} - 10B^2\kappa^2 I_{1,1,0,0,1,0,0,0} - \kappa^4 I_{1,1,1,0,0,0,0,0,0}\right) \\
&-2\gamma\left(B^2 I_{1,1,0,0,1,0,0,0} - 3\kappa^2 I_{1,1,1,0,0,0,0,0,0}\right) - 8\chi\kappa\left(B^2 I_{1,1,0,0,1,0,0,0}\right. \\
&\left.-\kappa^2 I_{1,1,1,0,0,0,0,0,0}\right) + \frac{2A^2}{B^3}\left\{\sigma_1 \int_{-\infty}^{\infty} \tau^2 g^2 \left(\int_{-\infty}^{\tau} g^2 d\tau_1\right)d\tau\right. \\
&\left.\left.+\sigma_2 B \int_{-\infty}^{\infty} \tau g \frac{dg}{d\tau}\left(\int_{-\infty}^{\tau} g^2 d\tau_1\right)d\tau\right\}\right]
\end{aligned}
\tag{36}
$$

These relations will now be applied to four laws of nonlinearity for $F(s)$ in the following sections. They are Kerr law, power law, parabolic law and dual-power law. The explicit adiabatic parameter dynamics will now be obtained in these four cases of nonlinearity.

3. KERR LAW

The Kerr law of nonlinearity originates from the fact that a light wave in an optical fiber faces nonlinear responses. Even though the nonlinear responses are extremely weak, their effects appear in various ways over long distance of propagation that is measured in terms of light wavelength. The origin of nonlinear response is related to the non-harmonic motion of bound electrons under the influence of an applied field. As a result the Fourier amplitude of the induced polarization from the electric dipoles is not linear in the electric field, but involves higher terms in electric field amplitude [1–11].

For Kerr law, $F(s) = s$. Thus, the NLSE modifies to

$$
iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0 \tag{37}
$$

This case, as mentioned before, is integrable by the IST [1, 2]. The form of the soliton is given by

$$q(x,t) = \frac{A}{\cosh\left[B(x-\bar{x}(t))\right]} e^{i(-\kappa x+\omega t+\theta)} \tag{38}$$

where

$$\kappa = -v \tag{39}$$

and

$$\omega = \frac{B^2-\kappa^2}{2} \tag{40}$$

while

$$A = B \tag{41}$$

In the Kerr law case, however, there are infinitely many integrals of motion. The first three integrals of motion that matches with those of the NLSE are respectively

$$E = \int_{-\infty}^{\infty} |q|^2 dx = 2A \tag{42}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (qq_x^* - q^* q_x) dx = -2\kappa A \tag{43}$$

$$H = \frac{1}{2}\int_{-\infty}^{\infty} (|q_x|^2 - |q|^4) dx = \frac{2}{3} A\left(3\kappa^2 - A^2\right) \tag{44}$$

The fourth conserved quantity for Kerr law nonlinearity is given by [5, 12]

$$\begin{aligned} &\frac{i}{2}\int_{-\infty}^{\infty} (qq_t^* - q^* q_t)\, dx - \frac{i}{2}\frac{d}{dt}\int_{-\infty}^{\infty} x\,(qq_x^* - q^* q_x)\, dx \\ = &-\frac{3}{2}\int_{-\infty}^{\infty} \left(|q_x|^2 - |q|^4\right) dx \end{aligned} \tag{45}$$

3.1. Perturbation Terms

In presence of perturbation terms, the NLSE with Kerr law nonlinearity is given by

$$iq_t + \frac{1}{2} q_{xx} + |q|^2 q = i\epsilon R \tag{46}$$

Thus, the adiabatic parameter dynamics of the solitons with Kerr law nonlinearity are given by [4]

$$\frac{dA}{dt} = \frac{dB}{dt} = \frac{\epsilon}{2}\int_{-\infty}^{\infty}(q^*R + qR^*)dx \tag{47}$$

$$\frac{d\kappa}{dt} = \frac{\epsilon}{2A}\left[i\int_{-\infty}^{\infty}(q_x^*R - q_xR^*)dx - \kappa\int_{-\infty}^{\infty}(q^*R + qR^*)dx\right] \tag{48}$$

For the perturbation terms given by (31), the adiabatic parameter dynamics of the soliton is

$$\frac{dE}{dt} = \frac{2\epsilon A}{15}\left[15\delta A^{2m}B\left(\frac{1}{2}, m+1\right) - 10\beta\left(A^2+3\kappa^2\right) - 4\rho A^2\left(A^2+5\kappa^2\right) + 10A\left(3\sigma_1 - \sigma_2 A\right)\right] \tag{49}$$

$$\frac{dM}{dt} = -\frac{2\epsilon\kappa A}{15}\left[15\delta A^{2m}B\left(\frac{1}{2}, m+1\right) - 10\beta\left(A^2+3\kappa^2\right) - 4\rho A^2\left(A^2+5\kappa^2\right) + 10A\left(3\sigma_1 - \sigma_2 A\right)\right] \tag{50}$$

$$\frac{dA}{dt} = \frac{dB}{dt} = \frac{\epsilon A}{15}\left[15\delta A^{2m}B\left(\frac{1}{2}, m+1\right) - 10\left(A^2 + 3\kappa^2\right) - 4\rho A^2\left(A^2 + 5\kappa^2\right) + 10A\left(3\sigma_1 - \sigma_2 A\right)\right] \tag{51}$$

$$\frac{d\kappa}{dt} = -\frac{\epsilon A^2}{15}\left[10\kappa\left(2\beta - \sigma_2\right) + 8\mu A^2\right] \tag{52}$$

$$\begin{aligned}\frac{d\theta}{dt} = -\epsilon\Bigg[&\frac{2}{3}\gamma\kappa A\left(2A^2 + 3\kappa^2\right) + \frac{2}{3}\kappa A^3\left(\sigma + 3\lambda\right) \\ &-\frac{\xi A^3}{45}\left(A^2 + 75\kappa^2\right) + \frac{\eta A^3}{9}\left(A^2 - 15\kappa^2\right) + \frac{4\zeta A^3}{5}\left(A^2 - 5\kappa^2\right) \\ &+\frac{\chi A^3}{30}\left(42A^2 + 205\kappa^2\right) - \frac{\sigma_1\kappa A^4}{B^3} \\ &\frac{\psi A}{35}\left(155A^6 + 315\kappa^4 A^2 + 441\kappa^2 A^4 - 210\kappa^3 A^2 + 105\kappa^4\right)\Bigg]\end{aligned} \tag{53}$$

where in (49)–(51), $B(l, m)$ is the beta function. The change in the velocity of the Kerr law soliton is given by

$$\begin{aligned}v = -\kappa - \frac{\epsilon}{3}\Big[&3\alpha + 9\gamma\kappa^2 + A^2\left(3\lambda + 2\theta + 3\gamma\right) - 2\kappa A^2\left(\xi + \eta A + 4\zeta\right) \\ &-3A\left(2\sigma_1 + \sigma_2\right) + 12\chi\kappa\left(A^2 + \kappa^2\right) + 6\psi\kappa\left(7A^4 + 3\kappa^4\right)\Big]\end{aligned} \tag{54}$$

4. POWER LAW

The power law nonlinearity arises in various materials, including semiconductors. Moreover, this law of nonlinearity arises in nonlinear plasmas that solves the problem of small K-condensation in weak turbulence theory [1–4]. In this case, $F(s) = s^p$ so that the NLSE is

$$iq_t + \frac{1}{2}q_{xx} + |q|^{2p}q = 0 \tag{55}$$

In (55), it is necessary to have $0 < p < 2$ to prevent wave collapse [1, 5] and, in particular, $p \neq 2$ to avoid self-focussing singularity [1]. The soliton solution of (55) is given by [1, 3, 4]

$$q(x,t) = \frac{A}{\cosh^{\frac{1}{p}}\left[B(x - \bar{x}(t))\right]} e^{i(-\kappa x + \omega t + \theta)} \tag{56}$$

where

$$\kappa = -v \tag{57}$$

$$\omega = \frac{B^2}{2p^2} - \frac{\kappa^2}{2} \tag{58}$$

$$B = A^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} \tag{59}$$

The three integrals of motion, in this case, are respectively given by [1, 5]

$$E = \int_{-\infty}^{\infty} |q|^2 dx = \frac{A^2}{B} B\left(\frac{1}{2}, \frac{1}{p}\right) \tag{60}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (q^* q_x - q q_x^*) dx = -\frac{\kappa A^2}{B} B\left(\frac{1}{2}, \frac{1}{p}\right) \tag{61}$$

and

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - \frac{1}{p+1}|q|^{2p+2}\right] dx \\ &= \frac{A^2}{2Bp(p+1)(p+2)} \left\{\kappa^2 p(p+1)(p+2) + B^2(p+1) - 4pA^{2p}\right\} B\left(\frac{1}{2}, \frac{1}{p}\right) \end{aligned} \tag{62}$$

For power law nonlinearity, the fourth conserved quantity is given as

$$\begin{aligned} &\frac{i}{2}\int_{-\infty}^{\infty} (qq_t^* - q^* q_t)\, dx - \frac{i}{2}\frac{d}{dt}\int_{-\infty}^{\infty} x\,(qq_x^* - q^* q_x)\, dx \\ &= -\int_{-\infty}^{\infty} \left(\frac{3}{2}|q_x|^2 - \frac{2p+1}{p+1}|q|^4\right) dx \end{aligned} \tag{63}$$

4.1. Perturbation Terms

In presence of perturbation terms, the NLSE, with power law nonlinearity, is given by

$$iq_t + \frac{1}{2}q_{xx} + |q|^{2p}q = i\epsilon R[q, q^*] \tag{64}$$

The adiabatic parameter dynamics, in presence of perturbation terms are [5]

$$\frac{dA}{dt} = \frac{\epsilon A^{p-1}}{(2-p)B\left(\frac{1}{2},\frac{1}{p}\right)}\left(\frac{2p^2}{1+p}\right)^{\frac{p-1}{2p}}\int_{-\infty}^{\infty}(q^*R + qR^*)dx \tag{65}$$

$$\frac{dB}{dt} = \frac{\epsilon p B^{\frac{2p-2}{p}}}{(2-p)B\left(\frac{1}{2},\frac{1}{p}\right)}\left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}}\int_{-\infty}^{\infty}(q^*R + qR^*)dx \tag{66}$$

$$\frac{d\kappa}{dt} = \frac{\epsilon B^{\frac{p-2}{p}}}{B\left(\frac{1}{2},\frac{1}{p}\right)}\left(\frac{2p^2}{1+p}\right)^{\frac{1}{p}}\left[i\int_{-\infty}^{\infty}(q_x^*R - q_xR^*)dx - \kappa\int_{-\infty}^{\infty}(q^*R + qR^*)dx\right] \tag{67}$$

As stated before, $p \neq 2$, it is clearly seen in (65) and (66), that this restriction is a valid one. The adiabatic parameter dynamics of the power law solitons, due to perturbation terms, is given by

$$\begin{aligned}\frac{dE}{dt} = \frac{2\epsilon A^2}{B^2}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2p}}\Bigg[&2\delta BA^{2m}B\left(\frac{1}{2},\frac{m+1}{p}\right) \\ &-\frac{2\beta B}{p(p+2)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{1}{p}\right) \\ &-\frac{2\rho BA^2}{p(p+4)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{2}{p}\right) \\ &+2\sigma_1A^2\int_{-\infty}^{\infty}\frac{1}{\cosh^{\frac{2}{p}}s}\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds \\ &+\frac{\sigma_2A^2B^2}{p^2}\int_{-\infty}^{\infty}\left(\frac{\kappa^2p^2+1}{\cosh^{\frac{2}{p}}s}-\frac{p^2}{\cosh^{\frac{2}{p}+2}s}\right)\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds\Bigg]\end{aligned} \tag{68}$$

$$\begin{aligned}\frac{dM}{dt} = -\frac{2\epsilon\kappa A^2}{B^2}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2p}}\Bigg[&2\delta BA^{2m}B\left(\frac{1}{2},\frac{m+1}{p}\right) \\ &-\frac{2\beta B}{p(p+2)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{1}{p}\right)\end{aligned}$$

$$-\frac{2\rho BA^2}{p(p+4)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$+2\sigma_1A^2\int_{-\infty}^{\infty}\frac{1}{\cosh^{\frac{2}{p}}s}\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds$$
$$\left.+\frac{\sigma_2A^2B^2}{p^2}\int_{-\infty}^{\infty}\left(\frac{\kappa^2p^2+1}{\cosh^{\frac{2}{p}}s}-\frac{p^2}{\cosh^{\frac{2}{p}+2}s}\right)\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds\right] \quad (69)$$

$$\frac{dA}{dt}=\frac{\epsilon}{(2-p)B\left(\frac{1}{2},\frac{1}{p}\right)}\frac{A^{p+1}}{B^2}\left(\frac{2p^2}{p+1}\right)^{\frac{p-1}{2p}}\left[2\delta BA^{2m}B\left(\frac{1}{2},\frac{m+1}{p}\right)\right.$$
$$-\frac{2\beta B}{p(p+2)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{1}{p}\right)$$
$$-\frac{2\rho A^2B}{p(p+4)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$+2\sigma_1A^2\int_{-\infty}^{\infty}\frac{1}{\cosh^{\frac{2}{p}}s}\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds$$
$$\left.+\frac{\sigma_2A^2B^2}{p^2}\int_{-\infty}^{\infty}\left(\frac{\kappa^2p^2+1}{\cosh^{\frac{2}{p}}s}-\frac{p^2}{\cosh^{\frac{2}{p}+2}s}\right)\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds\right] \quad (70)$$

$$\frac{dB}{dt}=\frac{\epsilon p}{(2-p)B\left(\frac{1}{2},\frac{1}{p}\right)}\frac{A^{2p}}{B^2}\left(\frac{2p^2}{p+1}\right)^{\frac{2p-1}{2p}}\left[2\delta BA^{2m}B\left(\frac{1}{2},\frac{m+1}{p}\right)\right.$$
$$-\frac{2\beta B}{p(p+2)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{1}{p}\right)$$
$$-\frac{2\rho A^2B}{p(p+4)}\left(B^2+\kappa^2p^2+4\kappa^2p\right)B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$+2\sigma_1A^2\int_{-\infty}^{\infty}\frac{1}{\cosh^{\frac{2}{p}}s}\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds$$
$$\left.+\frac{\sigma_2A^2B^2}{p^2}\int_{-\infty}^{\infty}\left(\frac{\kappa^2p^2+1}{\cosh^{\frac{2}{p}}s}-\frac{p^2}{\cosh^{\frac{2}{p}+2}s}\right)\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds\right] \quad (71)$$

$$\frac{d\kappa}{dt} = -\frac{\epsilon B^{\frac{p-2}{p}}}{B\left(\frac{1}{2},\frac{1}{p}\right)}\left(\frac{2p^2}{p+1}\right)^{\frac{1}{p}}\left[\frac{4\beta}{3p^2}\kappa BAB\left(\frac{1}{2},\frac{1}{p}\right)+\frac{4\mu A^4B}{p(p+4)}B\left(\frac{1}{2},\frac{2}{p}\right)\right.$$
$$\left.+\frac{\sigma_2\kappa A^4}{p^2}\int_{-\infty}^{\infty}\left(\frac{\kappa^2p^2+1}{\cosh^{\frac{2}{p}}s}-\frac{p^2}{\cosh^{\frac{2}{p}+2}s}\right)\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds\right] \quad (72)$$

$$\frac{d\theta}{dt}=-\epsilon\left[\frac{\gamma\kappa A^2}{Bp(p+2)}\left\{2B^2+p(p+2)\kappa^2\right\}B\left(\frac{1}{2},\frac{1}{p}\right)+\frac{\kappa A^4}{B}(3\lambda+\sigma)B\left(\frac{1}{2},\frac{2}{p}\right)\right.$$
$$+\frac{\xi A^4}{4Bp(p+2)(p+4)}\left\{B^2(p-2)-5\kappa^2p(p+2)(p+4)\right\}B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$+\frac{\eta A^4}{4Bp(p+2)(p+4)}\left\{B^2(3p+2)-5\kappa^2p(p+2)(p+4)\right\}B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$+\frac{3\zeta A^4}{4Bp(p+2)(p+4)}\left\{B^2(p+2)-\kappa^2p(p+2)(p+4)\right\}B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$+\frac{\psi A^2B}{4p^2(p+2)(3p+2)}\left\{6B^2(4p+3)+35\kappa^2p(3p+2)\right\}B\left(\frac{1}{2},\frac{1}{p}\right)$$
$$+\frac{\chi A^2}{2Bp^6(p+2)(3p+2)(5p+2)}\left\{3\kappa^2p^6(p+2)(3p+2)(5p+2)\right.$$
$$-18\kappa^3B^2p^5(3p+2)(5p+2)+27\kappa^2B^4p^4(4p+3)(5p+2)$$
$$+27\kappa^4B^2p^5(3p+2)(5p+2)$$
$$\left.3B^6\left(16p^6+64p^5+60p^4+15p^3-30p^2-32p-8\right)\right\}B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$\left.+\frac{\sigma_1\kappa A^4}{B^3}\int_{-\infty}^{\infty}\frac{s}{\cosh^{\frac{2}{p}}s}\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}}s_1}\right)ds\right]\tag{73}$$

The change in the velocity of the perturbed soliton is given by

$$v=-\kappa-\frac{\epsilon}{B\left(\frac{1}{2},\frac{1}{p}\right)}\left[\alpha B\left(\frac{1}{2},\frac{1}{p}\right)-\frac{6\gamma B^2}{p^2(p+2)}B\left(\frac{1}{2},\frac{1}{p}\right)\right.$$
$$-\frac{3\gamma}{p^2}\left(B^2+\kappa^2p^2\right)B\left(\frac{1}{2},\frac{1}{p}\right)$$
$$+\frac{A^2}{2}(3\lambda+2\theta-2\xi\kappa-2\eta\kappa-8\rho\kappa)B\left(\frac{1}{2},\frac{2}{p}\right)$$
$$-\frac{4\chi\kappa}{p^2(p+2)}\left\{p^2(p+2)\left(B^2-\kappa^2\right)-4B^2(p+1)\right\}B\left(\frac{1}{2},\frac{1}{p}\right)$$
$$-\frac{4\chi\kappa B^2}{p^2}B\left(\frac{1}{2},\frac{2}{p}\right)-\frac{2\psi\kappa}{p^3}\left(3\kappa^4p^3+3B^4-10\kappa^2B^2p\right)B\left(\frac{1}{2},\frac{1}{p}\right)$$
$$-\frac{4\psi\kappa B}{p^2(p+2)}\left\{2Bp+5\kappa^2p(p+2)-\left(4p^3+4p^2+3p+4\right)\right\}B\left(\frac{1}{2},\frac{1}{p}\right)$$
$$-\frac{24\psi\kappa B^4}{p^3(p+4)}(p+1)(13p+7)B\left(\frac{1}{2},\frac{2}{p}\right)$$

$$
-\frac{2\sigma_1 A^2}{B^2}\int_{-\infty}^{\infty}\frac{1}{\cosh^{\frac{2}{p}} s}\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}} s_1}\right)ds
$$
$$
\left.+\frac{2\sigma_2 A^2}{pB}\int_{-\infty}^{\infty}\frac{s\tanh s}{\cosh^{\frac{2}{p}} s}\left(\int_{-\infty}^{s}\frac{ds_1}{\cosh^{\frac{2}{p}} s_1}\right)ds\right] \tag{74}
$$

5. PARABOLIC LAW

For the parabolic law, $F(s) = s + \nu s^2$ where ν is a constant. This law is for constant ν is also known as the cubic-quintic nonlinearity. The term with ν is large for the case of p-toluene sulfonate crystals. It arises in the nonlinear interaction between Langmuir waves and electrons and describes the nonlinear interaction between the high frequency Langmuir waves and the ion-acoustic waves by pondermotive forces [4, 16].

There was little attention paid to the propagation of optical beams in the fifth order nonlinear media, since no analytic solutions were known and it seemed that chances of finding any material with significant fifth order term was slim. However, recent developments have rekindled interest in this area. The optical susceptibility of CdS_xSe_{1-x}-doped glasses was experimentally shown to have a considerable $\chi^{(5)}$, the fifth order susceptibility. It was also demonstrated that there exists a significant $\chi^{(5)}$ nonlinearity effect in a transparent glass in intense femtosecond pulses at 620 nm [4, 16].

It is necessary to consider nonlinearities higher than the third order to obtain some knowledge of the diameter of the self-trapping beam. It was recognized in 1960s and 70s that saturation of the nonlinear refractive index plays a fundamental role in the self-trapping phenomenon. Higher order nonlinearities arise by retaining the higher order terms in the nonlinear polarization tensor [16].

The form of the NLSE here is

$$
iq_t + \frac{1}{2}q_{xx} + \left(|q|^2 + \nu\,|q|^4\right)q = 0 \tag{75}
$$

The solution of (67) is now written as [2, 13]

$$
q(x,t) = \frac{A}{\left[1 + a\cosh\left\{B\left(x - \bar{x}(t)\right)\right\}\right]^{\frac{1}{2}}}e^{i(-\kappa x + \omega t + \theta)} \tag{76}
$$

where

$$
\kappa = -v \tag{77}
$$
$$
\omega = \frac{A^2}{4} - \frac{\kappa^2}{2} \tag{78}
$$

$$B = \sqrt{2}A \tag{79}$$

$$a = \sqrt{1 + \frac{4}{3}\nu A^2} \tag{80}$$

For the soliton given by (76), the corresponding integrals of motion are

$$E = \int_{-\infty}^{\infty} |q|^2 dx = \frac{A^2}{aB} F\left(1, 1, \frac{3}{2}; \frac{a-1}{2a}\right) B\left(1, \frac{1}{2}\right) \tag{81}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} \left(qq_x^* - q^*q_x\right) dx = -\frac{\kappa A^2}{aB} F\left(1, 1, \frac{3}{2}; \frac{a-1}{2a}\right) B\left(1, \frac{1}{2}\right) \tag{82}$$

and

$$\begin{aligned} H = & \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - \frac{1}{2}|q|^4 - \frac{\nu}{3}|q|^6\right] dx \\ = & \frac{\sqrt{2}A}{12a^2}\left[3a^2A^2F\left(3, 1, \frac{5}{2}; \frac{a-1}{2a}\right) B\left(1, \frac{3}{2}\right)\right. \\ & +6\kappa a^2 F\left(1, 1, \frac{3}{2}; \frac{a-1}{2a}\right) B\left(1, \frac{1}{2}\right) \\ & \left. -3aF\left(2, 2, \frac{5}{2}; \frac{a-1}{2a}\right) B\left(2, \frac{1}{2}\right) - \nu A^4 F\left(3, 3, \frac{7}{2}; \frac{a-1}{2a}\right) B\left(3, \frac{1}{2}\right)\right] \end{aligned} \tag{83}$$

where in (81)–(83), $F(\alpha, \beta; \gamma; z)$ is the Gauss' hypergeometric function. The fourth conserved quantity for the parabolic law nonlinearity is given as

$$\begin{aligned} & \frac{i}{2}\int_{-\infty}^{\infty} \left(qq_t^* - q^*q_t\right) dx - \frac{i}{2}\frac{d}{dt}\int_{-\infty}^{\infty} x\left(qq_x^* - q^*q_x\right) dx \\ & = -\int_{-\infty}^{\infty} \left(\frac{3}{2}|q_x|^2 - c_1|q|^4\right) dx \end{aligned} \tag{84}$$

where

$$c_1 = \frac{3aF\left(3, 1, \frac{5}{2}; \frac{a-1}{2a}\right) B\left(1, \frac{3}{2}\right) + aF\left(1, 1, \frac{3}{2}; \frac{a-1}{2a}\right) B\left(1, \frac{1}{2}\right)}{2F\left(2, 2, \frac{5}{2}; \frac{a-1}{2a}\right) B\left(1, \frac{1}{2}\right)} \tag{85}$$

5.1. Perturbation Terms

In presence of perturbation terms, the NLSE, with parabolic law nonlinearity, is given by

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^2 + \nu|q|^4\right) q = i\epsilon R[q, q^*] \tag{86}$$

The adiabatic parameter dynamics of the solitons with parabolic law nonlinearity is

$$\frac{dA}{dt} = \epsilon a^2\frac{\sqrt{2}}{2}\int_{-\infty}^{\infty}(q^*R+qR^*)dx \tag{87}$$

$$\frac{dB}{dt} = \epsilon a^2\int_{-\infty}^{\infty}(q^*R+qR^*)dx \tag{88}$$

$$\frac{d\kappa}{dt} = \epsilon\frac{B}{A^2E}\left[i\int_{-\infty}^{\infty}(q_x^*R-q_xR^*)dx-\kappa\int_{-\infty}^{\infty}(q^*R+qR^*)dx\right] \tag{89}$$

Here, the adiabatic variation of the soliton energy and linear momentum are given by [4]

$$\begin{aligned}\frac{dE}{dt} &= \frac{\epsilon\sqrt{2}}{4a}F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\\ &\left[\frac{2\delta A^{2m+1}}{2^m a^{m-1}}F\left(m+1,m+1,m+\frac{3}{2};\frac{a-1}{2a}\right)B\left(m+1,\frac{1}{2}\right)\right.\\ &-4\beta aA^3\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)-2\beta\kappa^2aAF\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\\ &-2\rho A^5F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)-\rho\kappa^2A^3F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\\ &+\sigma_1\sqrt{2}a^2A^2\int_{-\infty}^{\infty}\frac{1}{1+a\cosh s}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds\\ &\left.-\sigma_2a^3A^3\int_{-\infty}^{\infty}\frac{\sinh s}{(1+a\cosh s)^2}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds\right]\end{aligned} \tag{90}$$

$$\begin{aligned}\frac{dM}{dt} &= -\frac{\epsilon\kappa\sqrt{2}}{4a}F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\\ &\left[\frac{2\delta A^{2m+1}}{2^m a^{m-1}}F\left(m+1,m+1,m+\frac{3}{2};\frac{a-1}{2a}\right)B\left(m+1,\frac{1}{2}\right)\right.\\ &-4\beta aA^3F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)-2\beta\kappa^2aAF\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\\ &-2\rho A^5F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)-\rho\kappa^2A^3F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\\ &+\sigma_1\sqrt{2}a^2A^2\int_{-\infty}^{\infty}\frac{1}{1+a\cosh s}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds\\ &\left.-\sigma_2a^3A^3\int_{-\infty}^{\infty}\frac{\sinh s}{(1+a\cosh s)^2}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds\right]\end{aligned} \tag{91}$$

Now, the adiabatic variation of the soliton amplitude, width and frequency, on using (87)–(89) are given by

$$\frac{dA}{dt} = \frac{\epsilon}{2}\left[\frac{2\delta A^{2m+1}}{2^m a^{m-1}} F\left(m+1, m+1, m+\frac{3}{2}; \frac{a-1}{2a}\right) B\left(m+1, \frac{1}{2}\right)\right.$$
$$-4\beta a A^3 F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)-2\beta\kappa^2 aAF\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)$$
$$-2\rho A^5 F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)-\rho\kappa^2 A^3 F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)$$
$$+\sigma_1\sqrt{2}a^2A^2\int_{-\infty}^{\infty}\frac{1}{1+a\cosh s}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds$$
$$\left.-\sigma_2 a^3 A^3\int_{-\infty}^{\infty}\frac{\sinh s}{(1+a\cosh s)^2}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds\right] \tag{92}$$

$$\frac{dB}{dt} = \frac{\epsilon\sqrt{2}}{2}\left[\frac{2\delta A^{2m+1}}{2^m a^{m-1}} F\left(m+1, m+1, m+\frac{3}{2}; \frac{a-1}{2a}\right) B\left(m+1, \frac{1}{2}\right)\right.$$
$$-4\beta a A^3 F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)-2\beta\kappa^2 aAF\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)$$
$$-2\rho A^5 F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)-\rho\kappa^2 A^3 F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)$$
$$+\sigma_1\sqrt{2}a^2A^2\int_{-\infty}^{\infty}\frac{1}{1+a\cosh s}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds$$
$$\left.-\sigma_2 a^3 A^3\int_{-\infty}^{\infty}\frac{\sinh s}{(1+a\cosh s)^2}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds\right] \tag{93}$$

and

$$\frac{d\kappa}{dt} = -\frac{\epsilon}{aF\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)}\left[2\sqrt{2}\beta\kappa aAF\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\right.$$
$$-2\sqrt{2}\beta\kappa aAF\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)$$
$$-2\sqrt{2}\beta\kappa AF\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)$$
$$+\sqrt{2}\mu A^3 F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)$$
$$\left.-\sigma_2\kappa a^3\int_{-\infty}^{\infty}\frac{\sinh s}{(1+a\cosh s)^2}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds\right] \tag{94}$$

$$
\begin{aligned}
\frac{d\theta}{dt} = & -\frac{\epsilon\gamma\kappa A^2}{2aB}\left\{F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)+2\kappa^2 F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\right\} \\
& -\frac{\epsilon\kappa A^4}{4a^2B}(3\lambda+\sigma)F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right) \\
& -\frac{\epsilon\xi A^4}{768a^4B}\left[96a^2B^2F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)\right. \\
& -96a^2\kappa^2F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right) \\
& -B^2\left\{24a^2F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)-8aF\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right)B\left(3,\frac{1}{2}\right)\right. \\
& \left.-9\left(a^2-1\right)F\left(4,4,\frac{9}{2};\frac{a-1}{2a}\right)B\left(4,\frac{1}{2}\right)\right\} \\
& +2B^2\left\{3\left(a^2-1\right)F\left(4,4,\frac{9}{2};\frac{a-1}{2a}\right)B\left(4,\frac{1}{2}\right)\right. \\
& \left.\left.+16aF\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right)B\left(3,\frac{1}{2}\right)-24a^2F\left(2,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\right\}\right] \\
& -\frac{\epsilon\eta A^4}{768a^4B}\left[96a^2B^2F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)\right. \\
& -480a^2\kappa^2F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right) \\
& +B^2\left\{3\left(a^2-1\right)F\left(4,4,\frac{9}{2};\frac{a-1}{2a}\right)B\left(4,\frac{1}{2}\right)\right. \\
& \left.\left.+16aF\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right)B\left(3,\frac{1}{2}\right)-24a^2F\left(2,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\right\}\right] \\
& -\frac{\epsilon\zeta}{768a^4B}\left[1152a^2\kappa^2B^2F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\right. \\
& -384a^2B^2F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right) \\
& +2B^2\left\{24a^2F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\right. \\
& \left.-8aF\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right)B\left(3,\frac{1}{2}\right)-9\left(a^2-1\right)F\left(4,4,\frac{9}{2};\frac{a-1}{2a}\right)B\left(4,\frac{1}{2}\right)\right\} \\
& -2B^2\left\{3\left(a^2-1\right)F\left(4,4,\frac{9}{2};\frac{a-1}{2a}\right)B\left(4,\frac{1}{2}\right)\right. \\
& \left.\left.+16aF\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right)B\left(3,\frac{1}{2}\right)-24a^2F\left(2,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\epsilon\chi A^2}{32a^2B}\left[3B^4\left\{4a^2F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\right.\right.\\
&-4F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)+F\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right)B\left(3,\frac{1}{2}\right)\\
&-12a^2F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)+6aF\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)\\
&\left.+9a^2F\left(5,1,\frac{7}{2};\frac{a-1}{2a}\right)B\left(1,\frac{5}{2}\right)\right\}+24a\kappa^2F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)\\
&\left.+58a^2\kappa^2B^2F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)\right]\\
&\frac{\epsilon\psi A^2}{128a^3B}\left[3B^6\left\{196a^2F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)\right.\right.\\
&+81F\left(5,3,\frac{9}{2};\frac{a-1}{2a}\right)B\left(3,\frac{3}{2}\right)-225a^2F\left(7,1,\frac{9}{2};\frac{a-1}{2a}\right)B\left(1,\frac{7}{2}\right)\\
&-252aF\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(2,\frac{3}{2}\right)+420a^2F\left(5,1,\frac{7}{2};\frac{a-1}{2a}\right)B\left(1,\frac{5}{2}\right)\\
&\left.-270aF\left(6,2,\frac{9}{2};\frac{a-1}{2a}\right)B\left(2,\frac{5}{2}\right)\right\}\\
&+108a\kappa^2B^4\left\{4a^2F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\right.\\
&-4F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right)B\left(2,\frac{1}{2}\right)+F\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right)B\left(3,\frac{1}{2}\right)\\
&-12a^2F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)+6aF\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(1,\frac{5}{2}\right)\\
&\left.+9a^2F\left(5,1,\frac{7}{2};\frac{a-1}{2a}\right)B\left(1,\frac{5}{2}\right)\right\}+432a^2B^2\kappa^4F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right)B\left(1,\frac{3}{2}\right)\\
&\left.+6aF\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right)B\left(1,\frac{5}{2}\right)+9a^2F\left(5,1,\frac{7}{2};\frac{a-1}{2a}\right)B\left(1,\frac{5}{2}\right)\right]\\
&+\frac{\epsilon\sigma_1\kappa A^4}{B^3}\int_{-\infty}^{\infty}\frac{1}{1+a\cosh s}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right)ds
\end{aligned}
\tag{95}
$$

Also, the change in velocity of the perturbed soliton, due to parabolic law nonlinearity, in presence of such perturbation terms, is given by

$$
\begin{aligned}
v \;=\; &-\kappa-\frac{\epsilon}{4a^2F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)}\\
&\left[\left(\alpha+3\gamma\kappa^2\right)4a^2F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right)B\left(1,\frac{1}{2}\right)\right.
\end{aligned}
$$

$$+6\gamma a^2 A^2 F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right) B\left(1,\frac{3}{2}\right)$$
$$+aA^2\left(3\lambda+2\theta+2\xi\kappa-2\eta\kappa-8\rho\kappa\right) F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right) B\left(2,\frac{1}{2}\right)$$
$$-16\chi\kappa a A^2 F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right) B\left(2,\frac{1}{2}\right)$$
$$-40\chi\kappa a^2 A^2 F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right) B\left(1,\frac{3}{2}\right)$$
$$+16\chi\kappa a^2\left(\kappa^2+2A^2\right) F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right) B\left(1,\frac{1}{2}\right)$$
$$-8\psi\kappa a^2 A\left(45A^3+20\kappa^2 A+\kappa^4\right) F\left(1,1,\frac{3}{2};\frac{a-1}{2a}\right) B\left(1,\frac{1}{2}\right)$$
$$+8\psi\kappa a A^2\left(51A^2+10\kappa^2\right) F\left(2,2,\frac{5}{2};\frac{a-1}{2a}\right) B\left(2,\frac{1}{2}\right)$$
$$+8\psi\kappa a^2 A^2\left(186A^2+25\kappa^2\right) F\left(3,1,\frac{5}{2};\frac{a-1}{2a}\right) B\left(1,\frac{3}{2}\right)$$
$$-114\psi\kappa A^4 F\left(3,3,\frac{7}{2};\frac{a-1}{2a}\right) B\left(3,\frac{1}{2}\right)$$
$$-1008\psi\kappa a A^4 F\left(4,2,\frac{7}{2};\frac{a-1}{2a}\right) B\left(2,\frac{3}{2}\right)$$
$$-1134\psi\kappa a^2 A^4 F\left(5,1,\frac{7}{2};\frac{a-1}{2a}\right) B\left(2,\frac{3}{2}\right)$$
$$-4\sigma_1 a^3\int_{-\infty}^{\infty}\frac{s}{1+a\cosh s}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right) ds$$
$$\left.+4\sqrt{2}\sigma_2 a^3 A\int_{-\infty}^{\infty}\frac{s\sinh s}{(1+a\cosh s)^2}\left(\int_{-\infty}^{s}\frac{ds_1}{1+a\cosh s_1}\right) ds\right] \quad (96)$$

6. DUAL-POWER LAW

This model is used to describe the saturation of the nonlinear refractive index. Also, this serves as a basic model to describe the solitons in photovoltaic-photorefractive materials such as $LiNbO_3$. In this case, $F(s) = s^p + \nu s^{2p}$ so that the NLSE with dual-power law of nonlinearity is given by [4]

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^{2p} + \nu\,|q|^{4p}\right) q = 0 \quad (97)$$

Equation (97) supports solitary waves of the form [4]

$$q(x,t) = \frac{A}{\left[1 + b\cosh\left\{B\left(x - \bar{x}(t)\right)\right\}\right]^{\frac{1}{2p}}} e^{i(-\kappa x + \omega t + \theta)} \tag{98}$$

where

$$\kappa = -v \tag{99}$$

$$\omega = \frac{A^{2p}}{2p+2} - \frac{\kappa^2}{2} \tag{100}$$

$$B = A^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2p}} \tag{101}$$

$$b = \sqrt{1 + \frac{\nu B^2}{2p^2}\frac{(1+p)^2}{1+2p}} \tag{102}$$

For the dual-power law case, solitons exist for

$$-\frac{2p^2}{B^2}\frac{1+2p}{(1+p)^2} < \nu < 0 \tag{103}$$

In this case, the integrals of motion are

$$E = \int_{-\infty}^{\infty} |q|^2 dx = \frac{2A^2}{B 2^{\frac{1}{p}} b^{\frac{1}{p}}} F\left(\frac{1}{p}, \frac{1}{p}; \frac{1}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right) \tag{104}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} \left(q q_x^* - q^* q_x\right) dx = -\frac{2\kappa A^2}{B 2^{\frac{1}{p}} b^{\frac{1}{p}}} F\left(\frac{1}{p}, \frac{1}{p}; \frac{1}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right) \tag{105}$$

and

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_x|^2 - \frac{|q|^{2p+2}}{p+1} - \nu\frac{|q|^{4p+2}}{2p+1}\right] dx \\ &= \frac{A^2}{2^{\frac{1}{p}} b^{\frac{1}{p}}} \left[\frac{B}{4p^2} F\left(2 + \frac{1}{p}, \frac{1}{p}; \frac{3}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{3}{2}\right)\right. \\ &\quad + \frac{\kappa}{B} F\left(\frac{1}{p}, \frac{1}{p}; \frac{1}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right) \\ &\quad - \frac{A^{2p}}{Bb(p+1)} F\left(1 + \frac{1}{p}, 1 + \frac{1}{p}; \frac{3}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{p+1}{p}, \frac{1}{2}\right) \\ &\quad \left. - \frac{\nu A^{4p}}{2Bb^2(2p+1)} F\left(2 + \frac{1}{p}, 2 + \frac{1}{p}; \frac{5}{2} + \frac{1}{p}; \frac{b-1}{2b}\right) B\left(\frac{2p+1}{p}, \frac{1}{2}\right)\right] \end{aligned} \tag{106}$$

For dual-power law nonlinearity, the fourth conserved quantity is given as

$$\begin{aligned} &\frac{i}{2}\int_{-\infty}^{\infty}\left(qq_t^* - q^*q_t\right)dx - \frac{i}{2}\frac{d}{dt}\int_{-\infty}^{\infty} x\left(qq_x^* - q^*q_x\right)dx \\ &= -\int_{-\infty}^{\infty}\left(\frac{3}{2}|q_x|^2 - c_2|q|^4\right)dx \end{aligned} \quad (107)$$

where c_2 is given by

$$c_2 = \frac{3aF\left(\frac{1}{p}+2, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right)B\left(\frac{1}{p}, \frac{3}{2}\right)+aF\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; \frac{b-1}{2b}\right)B\left(\frac{1}{p}, \frac{1}{2}\right)}{(p+1)F\left(\frac{1}{p}+1, \frac{1}{p}+1, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right)B\left(\frac{1}{p}+1, \frac{1}{2}\right)} \quad (108)$$

6.1. Perturbation Terms

In presence of perturbation terms, the NLSE with dual-power law nonlinearity is given by

$$iq_t + \frac{1}{2}q_{xx} + \left(|q|^{2p} + \nu|q|^{4p}\right)q = i\epsilon R[q, q^*] \quad (109)$$

The parameter dynamics is now given by

$$\frac{dA}{dt} = \frac{\epsilon}{pLA^{p-1}}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2p}}\int_{-\infty}^{\infty}(q^*R + qR^*)dx \quad (110)$$

$$\frac{dB}{dt} = \frac{\epsilon}{L}\int_{-\infty}^{\infty}(q^*R + qR^*)dx \quad (111)$$

$$\frac{d\kappa}{dt} = \frac{\epsilon}{E}\left[i\int_{-\infty}^{\infty}(q_x^*R - q_xR^*)dx - \kappa\int_{-\infty}^{\infty}(q^*R + qR^*)dx\right] \quad (112)$$

where E is the energy given by (104) while

$$\begin{aligned} L = &\left[\frac{(b-1)(2p+1)}{2\nu(1+p)}\right]^{\frac{1}{p}} B\left(\frac{1}{2}, \frac{1}{p}\right) \\ &\left\{\frac{2\nu^2}{bp^3}\frac{(p+1)^3}{(b-1)(2p+1)^2}F\left(\frac{1}{2}, \frac{1}{p}; \frac{1}{2}+\frac{1}{p}; \frac{1-b}{1+b}\right)\right. \\ &-\frac{2}{B^2}F\left(\frac{1}{2}, \frac{1}{p}; \frac{1}{2}+\frac{1}{p}; \frac{1-b}{1+b}\right) \\ &\left.-\frac{2\nu}{bp^2}\frac{(p+1)^2}{(b-1)^2(p+2)(2p+1)}F\left(\frac{1}{2}, \frac{1}{p}; \frac{1}{2}+\frac{1}{p}; \frac{1-b}{1+b}\right)\right\} \end{aligned} \quad (113)$$

The adiabatic dynamics of energy and linear momentum for dual-power law solitons are given by

$$\begin{aligned}
\frac{dE}{dt} =& \frac{\epsilon(2-p)A^{2(1-p)}}{2^{\frac{1-p}{p}}b^{\frac{1}{p}}L}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}}F\left(\frac{1}{p},\frac{1}{p};\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\\
&\left[\frac{4\delta A^{2m+2}}{B2^{\frac{m+1}{p}}b^{\frac{m+1}{p}}}F\left(\frac{m+1}{p},\frac{m+1}{p},\frac{m+1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{m+1}{p},\frac{1}{2}\right)\right.\\
&-\frac{4\beta A^2}{2^{\frac{1}{p}}b^{\frac{1}{p}}}\left\{\frac{1}{4p^2}BF\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{5}{2}\right)\right.\\
&\left.+\frac{\kappa^2}{B}F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right\}\\
&-\frac{4\rho A^4}{2^{\frac{2}{p}}b^{\frac{2}{p}}}\left\{\frac{1}{4p^2}BF\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)\right.\\
&\left.+\frac{\kappa^2}{B}F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right\}\\
&+\frac{2\sigma_1A^4}{B^2}\int_{-\infty}^{\infty}\frac{1}{(1+b\cosh s)^{\frac{1}{p}}}\left(\int_{-\infty}^{s}\frac{ds_1}{1+b\cosh s_1}\right)ds\\
&\left.-\frac{\sigma_2A^4b}{pB}\int_{-\infty}^{\infty}\frac{\sinh s}{(1+b\cosh s)^{\frac{1}{p}+1}}\left\{\int_{-\infty}^{s}\frac{ds_1}{(1+b\cosh s_1)^{\frac{1}{p}}}\right\}ds\right] \qquad (114)
\end{aligned}$$

$$\begin{aligned}
\frac{dM}{dt} =& -\frac{\epsilon\kappa(2-p)A^{2(1-p)}}{2^{\frac{1-p}{p}}b^{\frac{1}{p}}L}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}}F\left(\frac{1}{p},\frac{1}{p};\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\\
&\left[\frac{4\delta A^{2m+2}}{B2^{\frac{m+1}{p}}b^{\frac{m+1}{p}}}F\left(\frac{m+1}{p},\frac{m+1}{p},\frac{m+1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{m+1}{p},\frac{1}{2}\right)\right.\\
&-\frac{4\beta A^2}{2^{\frac{1}{p}}b^{\frac{1}{p}}}\left\{\frac{1}{4p^2}BF\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{5}{2}\right)\right.\\
&\left.+\frac{\kappa^2}{B}F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right\}\\
&-\frac{4\rho A^4}{2^{\frac{2}{p}}b^{\frac{2}{p}}}\left\{\frac{1}{4p^2}BF\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)\right.\\
&\left.+\frac{\kappa^2}{B}F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right\}
\end{aligned}$$

$$+\frac{2\sigma_1 A^4}{B^2}\int_{-\infty}^{\infty}\frac{1}{(1+b\cosh s)^{\frac{1}{p}}}\left(\int_{-\infty}^{s}\frac{ds_1}{1+b\cosh s_1}\right)ds$$
$$\left.-\frac{\sigma_2 A^4 b}{pB}\int_{-\infty}^{\infty}\frac{\sinh s}{(1+b\cosh s)^{\frac{1}{p}+1}}\left\{\int_{-\infty}^{s}\frac{ds_1}{(1+b\cosh s_1)^{\frac{1}{p}}}\right\}ds\right] \quad (115)$$

Now, the adiabatic parameter dynamics of the soliton amplitude, width and frequency are respectively given by [4, 5]

$$\begin{aligned}\frac{dA}{dt} =& \frac{\epsilon}{pLA^{p-1}}\left(\frac{p+1}{2p^2}\right)^{\frac{1}{2p}}\\ &\left[\frac{4\delta A^{2m+2}}{B2^{\frac{m+1}{p}}b^{\frac{m+1}{p}}}F\left(\frac{m+1}{p},\frac{m+1}{p},\frac{m+1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{m+1}{p},\frac{1}{2}\right)\right.\\ &-\frac{4\beta A^2}{2^{\frac{1}{p}}b^{\frac{1}{p}}}\left\{\frac{1}{4p^2}BF\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{5}{2}\right)\right.\\ &\left.+\frac{\kappa^2}{B}F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right\}\\ &-\frac{4\rho A^4}{2^{\frac{2}{p}}b^{\frac{2}{p}}}\left\{\frac{1}{4p^2}BF\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)\right.\\ &\left.+\frac{\kappa^2}{B}F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right\}\\ &+\frac{2\sigma_1 A^4}{B^2}\int_{-\infty}^{\infty}\frac{1}{(1+b\cosh s)^{\frac{1}{p}}}\left(\int_{-\infty}^{s}\frac{ds_1}{1+b\cosh s_1}\right)ds\\ &\left.-\frac{\sigma_2 A^4 b}{pB}\int_{-\infty}^{\infty}\frac{\sinh s}{(1+b\cosh s)^{\frac{1}{p}+1}}\left\{\int_{-\infty}^{s}\frac{ds_1}{(1+b\cosh s_1)^{\frac{1}{p}}}\right\}ds\right] \quad (116)\end{aligned}$$

$$\begin{aligned}\frac{dB}{dt} =& \frac{\epsilon}{L}\left[\frac{4\delta A^{2m+2}}{B2^{\frac{m+1}{p}}b^{\frac{m+1}{p}}}F\left(\frac{m+1}{p},\frac{m+1}{p},\frac{m+1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{m+1}{p},\frac{1}{2}\right)\right.\\ &-\frac{4\beta A^2}{2^{\frac{1}{p}}b^{\frac{1}{p}}}\left\{\frac{B}{4p^2}F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\right.\\ &\left.+\frac{\kappa^2}{B}F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right\}\\ &-\frac{4\rho A^4}{2^{\frac{2}{p}}b^{\frac{2}{p}}}\left\{\frac{1}{4p^2}BF\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)\right.\end{aligned}$$

$$+\frac{\kappa^2}{B}F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\Bigg\}$$
$$+\frac{2\sigma_1 A^4}{B^2}\int_{-\infty}^{\infty}\frac{1}{(1+b\cosh s)^{\frac{1}{p}}}\left(\int_{-\infty}^{s}\frac{ds_1}{1+b\cosh s_1}\right)ds$$
$$-\frac{\sigma_2 A^4 b}{pB}\int_{-\infty}^{\infty}\frac{\sinh s}{(1+b\cosh s)^{\frac{1}{p}+1}}\left\{\int_{-\infty}^{s}\frac{ds_1}{(1+b\cosh s_1)^{\frac{1}{p}}}\right\}ds\Bigg] \quad (117)$$

$$\frac{d\kappa}{dt} = \frac{\epsilon}{F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)}$$
$$\Bigg[4\beta\kappa B\left\{\frac{B}{p}F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\right.$$
$$-\frac{B}{p}F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)$$
$$\left.+\frac{B}{2bp}F\left(\frac{1}{p}+1,\frac{1}{p}+1,\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{1}{2}\right)\right\}$$
$$-\frac{4A^2B^2\mu}{2^{\frac{1}{p}}b^{\frac{1}{p}}p^2}F\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)$$
$$-2^{\frac{1}{p}}b^{\frac{1}{p}+1}\sigma_2\kappa A^2\int_{-\infty}^{\infty}\frac{\sinh s}{(1+b\cosh s)^{\frac{1}{p}+1}}\left\{\int_{-\infty}^{s}\frac{ds_1}{(1+b\cosh s_1)^{\frac{1}{p}}}\right\}ds\Bigg] \quad (118)$$

$$\frac{d\theta}{dt} = -\frac{\epsilon\gamma\kappa A^2}{Bp^2a^{\frac{1}{p}}2^{\frac{1}{p}}}\left[B^2F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\right.$$
$$\left.+2\kappa^2p^2F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right]$$
$$-\frac{\epsilon\kappa A^4}{Ba^{\frac{2}{p}}2^{\frac{2}{p}}}(\sigma+3\lambda)F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)$$
$$-\frac{\epsilon\xi A^4}{8p^3Ba^{\frac{2}{p}}2^{\frac{2}{p}}}\left[4pB^2F\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)\right.$$
$$-20p^3\kappa^2F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)$$
$$-B^2\left\{F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right.$$

$$-\frac{2p}{p+2}F\left(\frac{2}{p}+1,\frac{2}{p}+1,\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+1,\frac{1}{2}\right)$$
$$-\frac{p(2p+1)\,(a^2-1)}{4a^2(p+1)}F\left(\frac{2}{p}+2,\frac{2}{p}+2,\frac{2}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+2,\frac{1}{2}\right)\Bigg\}$$
$$-2B^2p\left\{F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right.$$
$$-\frac{2}{a(p+2)}F\left(\frac{2}{p}+1,\frac{2}{p}+1,\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+1,\frac{1}{2}\right)$$
$$\left.-\frac{(a^2-1)}{4a^2(p+1)}F\left(\frac{2}{p}+2,\frac{2}{p}+2,\frac{2}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+2,\frac{1}{2}\right)\right\}\Bigg]$$
$$-\frac{\epsilon\eta A^4}{8p^2Ba^{\frac{2}{p}}2^{\frac{2}{p}}}\Bigg[4B^2F\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)$$
$$-20p^2\kappa^2F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)$$
$$-B^2\left\{F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right.$$
$$-\frac{2}{a(p+2)}F\left(\frac{2}{p}+1,\frac{2}{p}+1,\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+1,\frac{1}{2}\right)$$
$$\left.-\frac{(a^2-1)}{4a^2(p+1)}F\left(\frac{2}{p}+2,\frac{2}{p}+2,\frac{2}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+2,\frac{1}{2}\right)\right\}\Bigg]$$
$$+\frac{\epsilon\zeta A^4}{8p^3Ba^{\frac{2}{p}}2^{\frac{2}{p}}}\Bigg[48\kappa^2F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)$$
$$-16pB^2F\left(\frac{2}{p}+2,\frac{2}{p},\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{3}{2}\right)$$
$$+2B^2\left\{F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right.$$
$$-\frac{2p}{a(p+2)}F\left(\frac{2}{p}+1,\frac{2}{p}+1,\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+1,\frac{1}{2}\right)$$
$$\left.-\frac{p(2p+1)\,(a^2-1)}{4a^2(p+1)}F\left(\frac{2}{p}+2,\frac{2}{p}+2,\frac{2}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+2,\frac{1}{2}\right)\right\}$$
$$-2B^2p\left\{F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\right.$$
$$-\frac{2}{a(p+2)}F\left(\frac{2}{p}+1,\frac{2}{p}+1,\frac{2}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+1,\frac{1}{2}\right)$$

$$
\begin{aligned}
&-\frac{(a^2-1)}{4a^2(p+1)}F\left(\frac{2}{p}+2,\frac{2}{p}+2,\frac{2}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p}+2,\frac{1}{2}\right)\Bigg\}\Bigg]\\
&-\frac{\epsilon\chi A^2}{32p^4Ba^{\frac{1}{p}}2^{\frac{1}{p}}}\left[6B^4\left\{4a^2p^2F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right.\right.\\
&-4ap^2F\left(\frac{1}{p}+1,\frac{1}{p}+1,\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{1}{2}\right)\\
&+p^2F\left(\frac{1}{p}+2,\frac{1}{p}+2,\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+2,\frac{1}{2}\right)\\
&-a^2p(2p+1)F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\\
&-2ap(2p+1)F\left(\frac{1}{p}+3,\frac{1}{p}+1,\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{3}{2}\right)\\
&\left.-a^2(2p+1)^2F\left(\frac{1}{p}+4,\frac{1}{p},\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{5}{2}\right)\right\}\\
&+96\kappa^2p^4A^2F\left(\frac{2}{p},\frac{2}{p},\frac{2}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{2}{p},\frac{1}{2}\right)\\
&\left.+116\kappa^2p^2B^2F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\right]\\
&+\frac{\epsilon\psi A^2}{64p^6a^2Ba^{\frac{1}{p}}2^{\frac{1}{p}}}\left[3B^6\left\{4a^2p^2(4p+3)^2F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\right.\right.\\
&+9p^2(2p+1)^2F\left(\frac{1}{p}+4,\frac{1}{p}+2,\frac{1}{p}+\frac{7}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+2,\frac{3}{2}\right)\\
&-a^2(2p+1)^2(4p+1)^2F\left(\frac{1}{p}+6,\frac{1}{p},\frac{1}{p}+\frac{7}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{7}{2}\right)\\
&-12ap^2(2p+1)(4p+3)F\left(\frac{1}{p}+3,\frac{1}{p}+1,\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{3}{2}\right)\\
&+4a^2p(2p+1)(4p+1)(4p+3)F\left(\frac{1}{p}+4,\frac{1}{p},\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{5}{2}\right)\\
&\left.-6ap(2p+1)^2(4p+1)F\left(\frac{1}{p}+5,\frac{1}{p}+1,\frac{1}{p}+\frac{7}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{5}{2}\right)\right\}\\
&+432\kappa^4p^4B^2F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)\\
&+108\kappa^2p^2B^4\left\{4a^2p^2F\left(\frac{1}{p},\frac{1}{p},\frac{1}{p}+\frac{1}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{1}{2}\right)\right.\\
&-4ap^2F\left(\frac{1}{p}+1,\frac{1}{p}+1,\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{1}{2}\right)
\end{aligned}
$$

$$+p^2 F\left(\frac{1}{p}+2, \frac{1}{p}+2, \frac{1}{p}+\frac{5}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}+2, \frac{1}{2}\right)$$
$$-a^2 p(2p+1) F\left(\frac{1}{p}+2, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{3}{2}\right)$$
$$-2ap(2p+1) F\left(\frac{1}{p}+3, \frac{1}{p}+1, \frac{1}{p}+\frac{5}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}+1, \frac{3}{2}\right)$$
$$-a^2(2p+1)^2 F\left(\frac{1}{p}+4, \frac{1}{p}, \frac{1}{p}+\frac{5}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{5}{2}\right)\Bigg\}$$
$$-288\kappa^3 p^4 B^2 F\left(\frac{1}{p}+2, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{3}{2}\right)$$
$$+192\kappa^4 p^6 B^2 F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{1}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right)\Bigg]$$
$$+\frac{\epsilon\sigma_1\kappa A^4}{B^3}\int_{-\infty}^{\infty}\frac{1}{(1+b\cosh s)^{\frac{1}{p}}}\left\{\int_{-\infty}^{s}\frac{ds_1}{(1+b\cosh s_1)^{\frac{1}{p}}}\right\} ds \quad (119)$$

Finally, the velocity change for solitons with dual-power law, in presence of such perturbations, is given by

$$v = -\kappa - \frac{\epsilon}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right)}$$
$$\Bigg[\left(\alpha+3\gamma\kappa^2\right) F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right)$$
$$+\frac{3\gamma B^2}{4p^2} F\left(\frac{1}{p}+2, \frac{1}{p}, \frac{1}{p}+\frac{5}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{3}{2}\right)$$
$$+\frac{A^2}{2^{\frac{1}{p}} b^{\frac{1}{p}}}(3\lambda+\theta+\xi\kappa-\eta\kappa-4\rho\kappa) F\left(\frac{2}{p}, \frac{2}{p}, \frac{2}{p}+\frac{1}{2}; \frac{b-1}{2b}\right) B\left(\frac{2}{p}, \frac{1}{2}\right)$$
$$-\frac{4\chi\kappa B^2}{2pb} F\left(\frac{1}{p}+1, \frac{1}{p}+1, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}+1, \frac{1}{2}\right)$$
$$-\frac{\chi\kappa B^2(4p+1)}{p^2} F\left(\frac{1}{p}+2, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{3}{2}\right)$$
$$+\frac{4\chi\kappa}{p}\left(p\kappa^2+B^2\right) F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right)$$
$$-\frac{6\kappa\psi}{p^2}\Big\{6(4p+3)B^4+40p\kappa^2 B^2+12\kappa^4 p^2$$
$$+3B^4\Big\} F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{b-1}{2b}\right) B\left(\frac{1}{p}, \frac{1}{2}\right)$$

$$+\frac{\psi\kappa B^2}{2bp^3}\left(11B^2+10p^2B^2+20\kappa^2p^2\right)$$
$$F\left(\frac{1}{p}+1,\frac{1}{p}+1,\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{1}{2}\right)$$
$$+\frac{\psi\kappa B^2}{p^3}\left\{5p\left(4p+1\right)\kappa^2-3\left(8p^2+13p+2\right)B^2\right\}$$
$$F\left(\frac{1}{p}+2,\frac{1}{p},\frac{1}{p}+\frac{3}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{3}{2}\right)$$
$$-\frac{3\psi\kappa B^4}{8p^2b^2}(12p+7)F\left(\frac{1}{p}+2,\frac{1}{p}+2,\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+2,\frac{1}{2}\right)$$
$$-\frac{3\psi\kappa B^4}{bp^3}(2p+1)(6p+1)F\left(\frac{1}{p}+3,\frac{1}{p}+1,\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p}+1,\frac{3}{2}\right)$$
$$-\frac{3\psi\kappa B^4}{8p^4}(2p+1)\left(48p^2+14p+1\right)F\left(\frac{1}{p}+4,\frac{1}{p},\frac{1}{p}+\frac{5}{2};\frac{b-1}{2b}\right)B\left(\frac{1}{p},\frac{5}{2}\right)$$
$$-\frac{\sigma_1 bA^2 2^{\frac{1}{p}}}{B^2}\int_{-\infty}^{\infty}\frac{s}{(1+b\cosh s)^{\frac{1}{p}}}\left(\int_{-\infty}^{s}\frac{ds_1}{1+b\cosh s_1}\right)ds$$
$$\left.+\frac{\sigma_2 A^2 2^{\frac{1}{p}}b^{\frac{1}{p}}}{2B}\int_{-\infty}^{\infty}\frac{s\sinh s}{(1+b\cosh s)^{\frac{p+1}{p}}}\left\{\int_{-\infty}^{s}\frac{ds_1}{(1+b\cosh s_1)^{\frac{1}{p}}}\right\}ds\right] \quad (120)$$

7. CONCLUSIONS

This paper studied the non-Kerr law solitons in presence of perturbation terms by the aid of soliton perturbation theory (SPT). The adiabatic parameter dynamics was obtained by using this theory. The fourth conservation law, that is not so widely known, is also put to use in this paper to obtain the adiabatic variation of the phase of the soliton. Thus one can conclude that SPT is complete in the sense that the dynamics of all the soliton parameters can be laid down. Until the fourth conserved quantity was known, one had to resort to variational principle, particularly to get the dynamics of the soliton phase. Not any more! Thus SPT, from now on, is totally an independent technique, to study its parameter dynamics.

The parameter dynamics will be later on used to study the further issues of optical solitons. They include the collision induced frequency and timing jitter, amplitude jitter, four-wave mixing, intra-channel interaction of solitons and other such phenomena.

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