

## TRANSMITTANCE AND FRACTALITY IN A CANTOR-LIKE MULTIBARRIER SYSTEM

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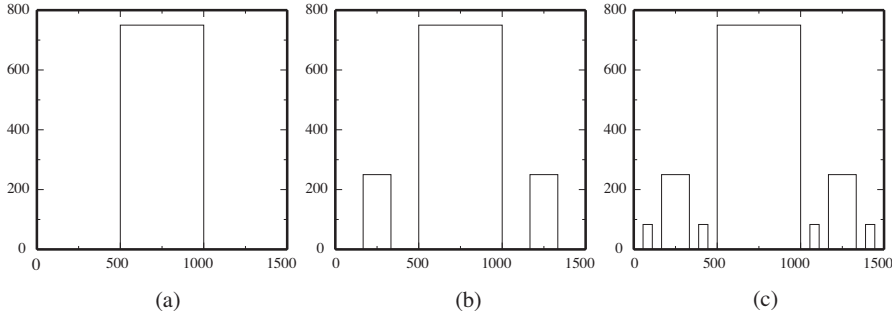
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**Abstract**—The transmittance is studied for a Cantor-like multibarrier system. The calculation are made in the framework of effective mass theory. Some typical values of effective masses and potentials are used in order to have an experimental reference. The techniques of Transfer Matrix are used to calculate the transmittance of the entire structure having some dozens of layers. The results display a complex structure of peaks and valleys. The set of maxima is studied with the tool of the  $q$ -dependent dimension  $D(q)$ . The set of transmittance maxima exhibits a fractal structure, or more exactly, a multifractal structure, i.e., a  $q$ -dependent dimension, characterized as usually with limit one when  $q$  parameter tends to  $-\infty$  but with a limit between 0 and 1 when tends to  $+\infty$ . This numerical experiment demonstrate that spatially bounded potential may exhibit spectrum with fractal character.

### 1. INTRODUCTION

The transmittance, energy levels and wave functions has been studied, among others, for periodic rectangular barriers, superlattices, quasiregular heterostructures and other multilayer one-dimensional systems. Nevertheless it is interesting to study non-periodic potentials, such as self-similar ones among others. In this case the term self-similar potential means that the height and/or the separation between barriers is not constant but it is constructed following some replicative rule. In our case we have chosen the Cantor's set rule [1]. The motivation for studying this kind of potentials comes out because it seems likely to find out that the transmittance reflects the self-imilar property of the potential through its fractal dimension. In the other hand Lavrinenko et al. [2, 3] studied the propagation of classical waves of the optical Cantor filter. This system is not a self-similar

system, because the refractive indices are not scaled. The authors observed that the optical spectra has been shown spectral scalability. In the last few years, a lot of experimental works, concerning the worth noting properties of porous silicon in chemical and biological sensing, have been reported [4]. Moretti et al. have compared the sensitivities of resonant optical biochemical sensor, based on both periodic and aperiodic porous silicon structures, such as Bragg and the Thue-Morse multilayer. They observed that the aperiodic multilayer is more sensitive than the periodic one. Then the task of finding similar systems with bigger sensitivity is important for applications.



**Figure 1.** Cantor-like potential. The main barrier has 3 times the energy of the other, and 3 times the length. Panel a/b/c shows the first/second/third generation.

## 2. METHODOLOGY

Figure 1 shows the first three generations, each generation having  $2^n - 1$  barriers. Each generation has a main barrier with an energy height  $H_0$  and spatial width  $L_0$ , for the second generation the other two barriers that appear have an energy height  $H_0/3$  and spatial width  $L_0/3$  and they are located in the middle third part where there is a zero potential. The third generation keeps the barriers of the second generation and adds barriers of height  $H_0/9$  and width  $L_0/9$  and they are located in the middle third part where there is a zero potential. With this algorithm one can construct the following generations. The transmittance for the potentials described above is computed, using the transfer matrix technique [5,6] applied to the one-dimensional Schrödinger equation [7].

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V(x) \Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t) \quad (1)$$

Using separation of variables reduces to the time-dependant Shrödinger equation,

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x) \quad (2)$$

Consider first a localized potential  $V$ , restricted to the interval  $(a, b)$ ; the general solution is

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } x < a \\ \psi_{ab}(x) & \text{if } a < x < b \\ Ce^{ikx} + De^{-ikx} & \text{if } b < x \end{cases} \quad (3)$$

where  $k \equiv \sqrt{2mE/\hbar}$ . When the time factor is included,  $A \exp(ikx)$  and  $C \exp(ikx)$  represent waves propagating to the right, while  $B \exp(-ikx)$  and  $D \exp(-ikx)$  represent waves propagating to the left.

To complete the problem, one solves Eq. (2) for  $\psi(x)$  in  $(a, b)$ . Then, invoking the appropriate boundary conditions at  $a$  and  $b$  [typically, continuity of  $\psi(x)$  and its derivative], one obtains two linear relations among the coefficients  $A$ ,  $B$ ,  $C$  and  $D$ . These can be solved for any two amplitudes in terms of the other two, and the result can be expressed as a matrix equation. Usually one chooses to write the outgoing amplitudes ( $B$  and  $C$ ) in terms of the incoming amplitudes ( $A$  and  $D$ ) using the so-called “ $S$  matrix”.

$$\begin{pmatrix} B \\ C \end{pmatrix} = S \begin{pmatrix} A \\ D \end{pmatrix}$$

We find it more convenient to express the amplitudes to the left of the barrier ( $A$  and  $B$ ) in terms of those to the right ( $C$  and  $D$ ):

$$\begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} C \\ D \end{pmatrix}$$

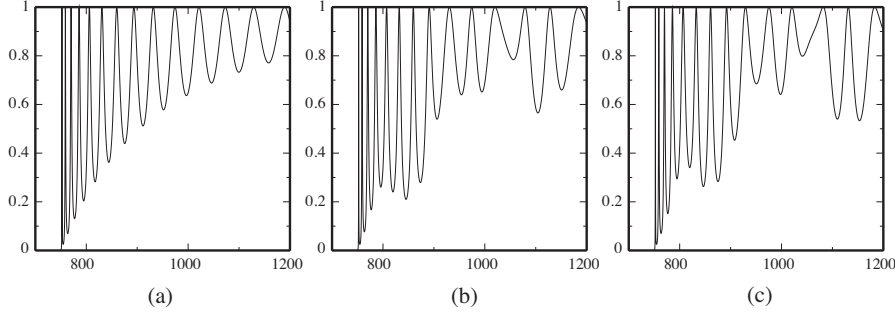
This  $2 \times 2$  matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is called the “transfer matrix”.

For this particular work, the energy of the main barrier  $H_0$  is taken as 750 meV and the segment  $L_0$  is of length 500 Å.

Figure 2 shows the transmittance for the first three generations. The behavior of the transmittance for high order generations, above the 6th one, remains approximately the same. This is not surprising



**Figure 2.** Transmittance for Cantor-like potential. Panel a/b/c shows the first/second/third generation.

considering that the added barriers are of the order of  $H_0/3^n$  and  $L_0/3^n$  or smaller and their contribution is negligible, even when there is  $2^n$  of these barriers.

The transmittance, as expected, exhibits a complex structure characterized by many maxima and minima in a relatively small interval of energy. The maxima are related with the discretized spectrum of the set of wells inside the structure. We have collected all these maxima and treated them as a set of points in order to calculate its fractal dimensions  $D(q)$ .

### 3. $Q$ DEPENDENT DIMENSIONS

Following Rasband [8], Pérez-Álvarez [9–11] and coworkers, and the references they cite, we use the so called  $q$ -dependent fractal dimensions in order to describe the strange character of the spectrum as a set of points. The main point is to cover the set with a collection of boxes of size  $\epsilon$  and to study the Information magnitude

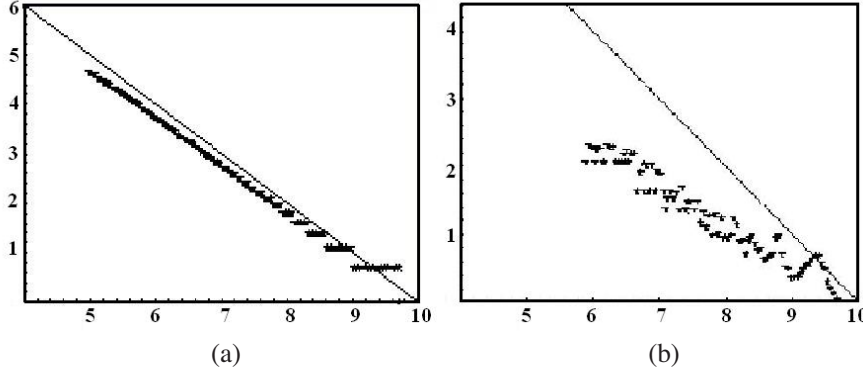
$$I(\epsilon, q) = \frac{1}{1-q} \log \sum_{j=1}^{N(\epsilon)} p_j^q, \quad (4)$$

and from this the *generalised box-counting dimension*

$$D(q) = - \lim_{\epsilon \rightarrow 0} \frac{I(\epsilon, q)}{\log [\epsilon]}. \quad (5)$$

In these formulas  $p_j$  is the fraction of points in the  $j$ -th box.  $N(\epsilon)$  is the number of boxes of size  $\epsilon$ .

In practice the limit in (5) is calculated as the slope of the so called *scaled part* of the curve  $I(\epsilon, q)$  vs  $\log[\epsilon]$ . The reader is addressed to references [10, 12] in order to face some practical aspects and tricks on this matter.



**Figure 3.**  $I(\epsilon, q)$  vs  $\log[\epsilon]$  curves for  $q = 0$  (left panel) and  $q = 40$  (right panel). A straight line with slope  $-1$  is added as a guide for the eyes. It is clearly seen that dimension at  $q = 40$  is less than unity.

Figure 3 depicts the  $I(\epsilon, q)$  vs  $\log[\epsilon]$  curves for  $q = 0$  and  $q = 40$ . The figures are restricted to the scaled parts of these curves. The  $D(q)$  calculations take as input 725 eigenvalues obtained from the 7th generation. The conclusion is obvious: the spectrum has  $D(0) \approx 1$  but a dimension clearly less than unity for  $q = 40$ , i.e., it has a *distribution* of fractal dimensions. The detailed analysis for  $q \in (-\infty, +\infty)$  shows that  $D(q)$  goes continuously from 1 at  $-\infty$  to  $\approx 0.6$  at  $+\infty$ , as it should be because of very basic principles [12].

#### 4. CONCLUSION

The numerical experiment we present in this paper demonstrates that the spectrum of elementary excitations in spatially bounded potential can have a fractal character. It is not at all trivial. Mathematicians have proved some theorems about the fractality of the spectrum of Schrödinger-like hamiltonians in some quasiregular sequences [13, 14] but, as far as we know, there is no solid result on potentials defined on bounded intervals. We hope our results open a new field of interest for physicists and mathematicians.

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