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# SCHWARZ-KRYLOV SUBSPACE METHOD FOR MLFMM ANALYSIS OF ELECTROMAGNETIC WAVE SCATTERING PROBLEMS

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**Abstract**—In this paper, the high-order hierarchical basis functions are used for solving electromagnetic wave scattering problems. The multilevel fast multipole method (MLFMM) is applied to accelerate the matrix-vector product operation and the Schwarz method is employed to speed up the convergence rate of the Krylov subspace iterative methods. The efficiency of the proposed approach is studied on several numerical model problems and the comparison with conventional Krylov iterative methods is made. Numerical results demonstrate that the combination of the Schwarz method and the Krylov subspace iterative method is very effective with MLFMM and can reduce the overall simulation time significantly.

## 1. INTRODUCTION

Electromagnetic integral equations are often discretized with the method of moments (MoM) [1–5, 24–29]. In this process, the integral equation is first transformed into the corresponding matrix equation using the Galerkin-based MoM with subdomain basis functions such as Rao-Wilton-Glisson (RWG) functions [5]. Since it is convenient to model objects with arbitrary shape using triangular patches, RWG functions are widely used for representing unknown current distributions. However, the RWG functions have a poor convergence and need a large number of unknowns for a desired accuracy. To relieve this disadvantage, a remedy is to employ high-order basis functions.

The development of high-order basis functions for modeling electromagnetic fields has received intense attention recently because of permitting more accurate results with less effort than the low-order basis functions [6–9]. Divergence-conforming basis functions that impose normal continuity of a vector quantity between neighboring elements, such as the electric surface current density, are usually applied in MoM [6, 7] whereas curl-conforming functions that impose tangential continuity are applied in the finite element method (FEM) [8, 9]. Hierarchical basis functions which allow for much flexibility are employed as the high-order functions in this paper. The basis of order $m$ is a subset of the basis of order $m+1$, which allows mixing of different order bases in the same mesh. Thus, hierarchical bases combine the advantages of both low-order and high-order bases into a single flexible basis. This desirable property allows for selective field expansion using different order bases in different regions of the computational domain.

The formulation considered in this paper is the electric field integral equation (EFIE) [5] since it is the most general and does not require any assumption about the geometry of the object. The matrix associated with the resulting linear systems is hard to solve. It is basically impractical to solve EFIE matrix equations using direct methods because they have a memory requirement of $0(N^2)$ and computational complexity of $0(N^3)$, where $N$ refers to the number of unknowns. This difficulty can be circumvented by using Krylov iterative methods, and the required matrix-vector product operation can be efficiently evaluated by the multilevel fast multipole method (MLFMM) [10–14, 30, 31]. The use of MLFMM can reduce the complexity of the matrix-vector product operation to $O(N \log N)$.

It is well-known that EFIE provides a first-kind integral equation, which is ill-conditioned and gives rise to linear systems that are challenging to solve by iterative methods. Therefore, it is natural to use preconditioning techniques to improve the conditioning of the system and accelerate the convergence rate of iterative solvers. Simple preconditioners like the diagonal or diagonal blocks of the coefficient matrix can be effective only when the matrix has some degree of diagonal dominance [15]. Preconditioners based on incomplete factorizations have been successfully used on nonsymmetric dense systems [16] and hybrid integral formulations [17], but they are sensitive to indefiniteness in the EFIE matrix that leads to unstable triangular solvers and very poor preconditioners [18]. In this paper, the combination of Schwarz method with Krylov iterations is applied to speed up the convergence rate of iterative solvers. The key idea is divide the near-field matrix into two components, low-order part and high-order part, using Cholesky decomposition. The time for preconditioning is saved by solving the two components separately instead of solving near-field matrix directly as for preconditioning purpose.

This paper is organized as follows. Section 2 gives a brief introduction to the EFIE formulation and hierarchical basis functions. Section 3 describes the theory and implementation of the Schwarz-Krylov subspace iterative method in more details. Numerical experiments with a few electromagnetic scattering problems are presented to show the efficiency of the method in Section 4. Section 5 gives some conclusions.

## 2. HIGH-ORDER HIERARCHICAL BASIS FUNCTIONS

Consider an arbitrarily shaped 3D conducting object illuminated by an incident field $\mathbf{E}^i$, the EFIE is given by

$$-\frac{k\eta i}{4\pi}\hat{t}\cdot\int_s \bar{\mathbf{G}}\left(\mathbf{r},\mathbf{r}'\right)\cdot\mathbf{J}(\mathbf{r}')dS' = \hat{t}\cdot\mathbf{E}^i(\mathbf{r}) \tag{1}$$

where $\mathbf{J}(\mathbf{r})$ denotes the unknown surface current density and $\bar{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ is defined by

$$\bar{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \left[\bar{\mathbf{I}} - \frac{1}{k^2}\nabla\nabla'\right]\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{2}$$

The EFIE can be solved by MoM. The conducting surface is subdivided into small triangular patches and the unknown current $\mathbf{J}(\mathbf{r})$ is first expanded as

$$\mathbf{J}(\mathbf{r}) =_{i=1}^{N} I_n\mathbf{f}(\mathbf{r}) \tag{3}$$

where $N$ is the number of unknowns and $\mathbf{f}(\mathbf{r})$ denotes the vector basis functions, and $I_n$ denotes the unknown expansion coefficients. Applying Galerkin's method resulting in the EFIE impedance matrix equation

$$_{n=1}^{N} Z_{mn}I_n = b_m, \qquad m = 1, 2, \ldots, N \tag{4}$$

where

$$Z_{mn} = jk\int_s \mathbf{f}_m(\mathbf{r})\cdot\int_{s'}\left(\mathbf{I} + \frac{1}{k^2}\nabla\nabla'\right)G\left(\mathbf{r},\mathbf{r}'\right)\cdot\mathbf{f}_n(\mathbf{r}')dsds' \tag{5}$$

and

$$b_m = \frac{1}{\eta} \int_s \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{E}^i(\mathbf{r}) ds \tag{6}$$

For a given accuracy, the use of high-order basis functions allows us to use larger triangular patches to discretize the object. A set of high-order hierarchical basis functions based on the hierarchical tangential vector finite elements (TVFE) in [8] is introduced into MoM. The hierarchical mixed-order TVFE of order 1.5 is characterized by the following eight vector basis functions:

$$\bar{\mathbf{W}}_1^e = \xi_2 \nabla \xi_3 - \xi_3 \nabla \xi_2, \ \bar{\mathbf{W}}_2^e = \xi_3 \nabla \xi_1 - \xi_1 \nabla \xi_3, \ \bar{\mathbf{W}}_3^e = \xi_1 \nabla \xi_2 - \xi_2 \nabla \xi_1 \tag{7}$$

$$\bar{\mathbf{W}}_4^e = (\xi_2 - \xi_3)(\xi_2 \nabla \xi_3 - \xi_3 \nabla \xi_2), \ \bar{\mathbf{W}}_5^e = (\xi_3 - \xi_1)(\xi_3 \nabla \xi_1 - \xi_1 \nabla \xi_3) \tag{8}$$

$$\begin{aligned} &\bar{\mathbf{W}}_6^e = (\xi_1 - \xi_2)(\xi_1 \nabla \xi_2 - \xi_2 \nabla \xi_1), \ \bar{\mathbf{W}}_7^2 = \xi_3 (\xi_1 \nabla \xi_2 - \xi_2 \nabla \xi_1), \\ &\bar{\mathbf{W}}_8^2 = \xi_2 (\xi_3 \nabla \xi_1 - \xi_1 \nabla \xi_3) \end{aligned} \tag{9}$$

$\xi_1$, $\xi_2$, $\xi_3$ are the simplex coordinates. A set of three vector basis functions in the Equation (7) is the mixed-order TVFE of order 0.5 and (7) $\sim$ (9) is the mixed-order TVFE of order 1.5. This shows that the above presented vector basis functions are hierarchical.

Divergence-conforming bases $\bar{\mathbf{f}}_\beta^e$ on 2-D elements can be obtained by forming the cross product of the associated curl-conforming bases $\bar{\mathbf{W}}_\beta^e$ with the unit vector $\hat{n}$ normal to the element and defined as $\bar{\mathbf{f}}_\beta^e = \bar{\mathbf{W}}_\beta^e \times \hat{\mathbf{n}}$. Thus, the divergence-conforming bases of order 0.5 on the above Lars $S$. Andersen's curl-conforming basis functions are

$$\begin{aligned} &\bar{\mathbf{f}}_1^e = (\xi_2 \nabla \xi_3 - \xi_3 \nabla \xi_2) \times \hat{\mathbf{n}}, \ \bar{\mathbf{f}}_2^e = (\xi_3 \nabla \xi_1 - \xi_1 \nabla \xi_3) \times \hat{\mathbf{n}}, \\ &\bar{\mathbf{f}}_3^e = (\xi_1 \nabla \xi_2 - \xi_2 \nabla \xi_1) \times \hat{\mathbf{n}} \end{aligned} \tag{10}$$

with

$$\nabla \xi_2 = \frac{\hat{n} \times \bar{\mathbf{I}}_1}{J}, \quad \nabla \xi_3 = \frac{\hat{n} \times \left(\bar{\mathbf{I}}_2 - \bar{\mathbf{I}}_1\right)}{J}$$

where $J$ is the Jacobian, $\bar{\mathbf{I}}_i$ ($i = 1, 2, 3$) represents the edge vector opposite to the nodes 1, 2 and 3, respectively.

As described in the above, the hierarchical basis functions of order 1.5 can then be obtained by forming the cross product of (7)–(9) with the unit vector $\hat{\mathbf{n}}$ and defined as

$$\begin{aligned} &\bar{\mathbf{f}}_1^e = \frac{1}{J}\left(\xi_2 \bar{\mathbf{I}}_3 - \xi_3 \bar{\mathbf{I}}_2\right), \ \bar{\mathbf{f}}_2^e = \frac{1}{J}\left(\xi_3 \bar{\mathbf{I}}_1 - \xi_1 \bar{\mathbf{I}}_3\right), \ \bar{\mathbf{f}}_3^e = \frac{1}{J}\left(\xi_1 \bar{\mathbf{I}}_2 - \xi_2 \bar{\mathbf{I}}_1\right) \\ &\bar{\mathbf{f}}_4^e = \frac{1}{J}(\xi_2 - \xi_3)\left(\xi_2 \bar{\mathbf{I}}_3 - \xi_3 \bar{\mathbf{I}}_2\right), \ \bar{\mathbf{f}}_5^e = \frac{1}{J}(\xi_3 - \xi_1)\left(\xi_3 \bar{\mathbf{I}}_1 - \xi_1 \bar{\mathbf{I}}_3\right) \\ &\bar{\mathbf{f}}_6^e = \frac{1}{J}(\xi_1 - \xi_2)\left(\xi_1 \bar{\mathbf{I}}_2 - \xi_2 \bar{\mathbf{I}}_1\right), \ \bar{\mathbf{f}}_7^e = \frac{1}{J}\xi_1\left(\xi_2 \bar{\mathbf{I}}_3 - \xi_3 \bar{\mathbf{I}}_2\right), \ \bar{\mathbf{f}}_8^e = \frac{1}{J}\xi_2\left(\xi_3 \bar{\mathbf{I}}_1 - \xi_1 \bar{\mathbf{I}}_3\right) \end{aligned} \tag{11}$$

The hierarchical basis functions are ideally suited for employing an efficient selective field expansion where different order basis functions are employed in different regions of the computational domain. Hence, for a uniform mesh, the lowest order basis functions can be employed in regions where the field is expected to experience smooth variation whereas a higher order basis functions can be employed in regions where the field is expected to vary rapidly. Similarly, for a nonuniform mesh, the lowest order basis functions can be employed where the mesh is dense while a higher order basis functions can be employed where the mesh is coarse. Regions where higher order basis functions are employed can be fixed a priori or an adaptive scheme can be developed where lowest order basis functions are initially employed throughout the computational domain and higher order basis functions are subsequently employed in regions where the error is estimated to be large.

A generalization of the hierarchical basis functions of order 2.5 and even higher order ones can be obtained in a similar way. Finally, the element of the impedance matrix $Z_{mn}$ is evaluated with the presented hierarchical basis functions.

## 3. SCHWARZ-KRYLOV SUBSPACE ITERATIVE METHOD

In this section, hierarchical basis functions are used to analyze the 3D scattering problems in electromagnetics. After discretizing the EFIE, we obtain a dense algebraic linear system, which can be written into the following form:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{12}$$

where $\mathbf{A} \in \mathbf{C}^{n\times n}$, $\mathbf{b}$, $\boldsymbol{x} \in \boldsymbol{C}^n$. In order to efficiently solve Equation (12) by iterative methods, preconditioning techniques are applied based on near-field part of impedance matrix $\mathbf{A}_N$. An efficient and popular preconditioner is to solve a near-field sparse matrix equation which can be written by

$$\mathbf{A}_N\mathbf{x} = \mathbf{r} \tag{13}$$

where $\mathbf{r}$ is the residual at each iteration. By numbering the unknowns from the low-order group to the high-order group, the system (13) can be written into block form as:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \tag{14}$$

Here the unknown vector $\mathbf{x}_1$ corresponds to the low-order basis functions, while $\mathbf{x}_2$ corresponds to the high-order basis functions.

It is well known that the low frequency components of the iteration error can be captured by the hierarchical low-order basis functions, while the high frequency components can be well expressed by the hierarchical high-order basis functions. Moreover, it is the low frequency part $\mathbf{x}_1$ that hinders the convergence of iterative methods most, though it occupies only a small fraction of the whole solution. It is assumed that if the solution of $\mathbf{x}_1$ is obtained, $\mathbf{x}_2$ can be solved efficiently with an iterative solver. The above idea can be realized easily with the Schwarz method. The Schwarz method was originally introduced by Schwarz in 1870 [19], which is a method often used in the domain decomposition area. However, our research is focused on the solution of hierarchical MoM systems and we treat each basis functions group as a non-overlapping domain. More details of the method are given in the following:

Through block Cholesky decomposition, the near-field matrix $\mathbf{A}_N$ can be written as:

$$\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & 0\\ \mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{11} & 0\\ 0 & \mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\\ 0 & \mathbf{I}\end{bmatrix} \tag{15}$$

If we define

$$\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\\ 0 & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix},\quad \begin{bmatrix}\bar{\mathbf{b}}_1\\ \bar{\mathbf{b}}_2\end{bmatrix}=\begin{bmatrix}\mathbf{I} & 0\\ -\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix} \tag{16}$$

Then equation system (13) are partitioned into two equations

$$\mathbf{A}_{11}\mathbf{y}_1=\bar{\mathbf{b}}_1 \tag{17}$$

$$\left[\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right]\mathbf{y}_2=\bar{\mathbf{b}}_2 \tag{18}$$

By the above decomposition, the linear system (13) is separated into two sub-linear systems. We can solve them individually according to their characteristics. In Equation (17), $\mathbf{A}_{11}$ corresponds to the low-order hierarchical basis functions and is seriously ill-conditioned. As a result, it is hard to solve Equation (17) with iterative methods. However, since it occupies only a small fraction of the near-field matrix $\mathbf{A}_N$, we can solve it quickly with a suitable direct solver without much memory requirement and CPU time. Therefore, the main task is to solve Equation (18) in order to realize preconditioning operation defined in Equation (13). In this framework, because most of low frequency components of the solution are eliminated from $\mathbf{y}_2$ (which is equal to $\mathbf{x}_2$), Equation (18) can be solved efficiently with a suitable iterative solver. In this paper, the Krylov subspace iterative methods are adopted.

The Krylov subspace methods are a kind of iterative methods that search for solution $\mathbf{x}_m$ at the $m$th iteration within the subspace $\mathbf{x}_0 + K_m(\mathbf{A}, \mathbf{r}_0)$, where $\mathbf{r}_0 = \mathbf{b} - \mathbf{A}\mathbf{x}_0$ is the associated initial residual vector, $K_m(\mathbf{A}, \mathbf{r}_0)$ is the Krylov subspace of dimension $m$ and is defined by

$$K_m(\mathbf{A}, \mathbf{r}_0) = \mathit{span}\left\{\mathbf{r}_0, \mathbf{A}\mathbf{r}_0, \mathbf{A}^2\mathbf{r}_0, \ldots, \mathbf{A}^{m-1}\mathbf{r}_0\right\} \tag{19}$$

In exact arithmetic, this kind of methods can reach exact solution at most $N$ iterations. The systematic description of this kind of methods can be seen in [20]. Among these methods, the conjugate gradient method (CG), and the generalized minimum residual method (GMRES) are most commonly used, since both two methods are stable in convergence. The CG method is only suitable for positive definite matrices, while the GMRES method is suitable for any matrices of full rank. In this paper, the combination of the Schwarz method and GMRES is investigated to solve the EFIE based on hierarchical basis functions. For convenience, we call it the Schwarz-GMRES method.

During the iterative solution of Equation (18), the matrix-vector product $\left[\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right]\mathbf{r}_2$ is needed. This involves the solution of the following sub-linear systems:

$$\mathbf{A}_{11}\mathbf{p} = \mathbf{r}_2 \tag{20}$$

which is solved with a direct solver. In this paper, the multifrontal solver is applied, and the $\mathbf{A}_{11}$ is factorized only once and the factorizations can be used repeatedly throughout the iteration. Once $\mathbf{y}_1$, $\mathbf{y}_2$ are solved, $\mathbf{x}_1$, $\mathbf{x}_2$ can then be obtained as:

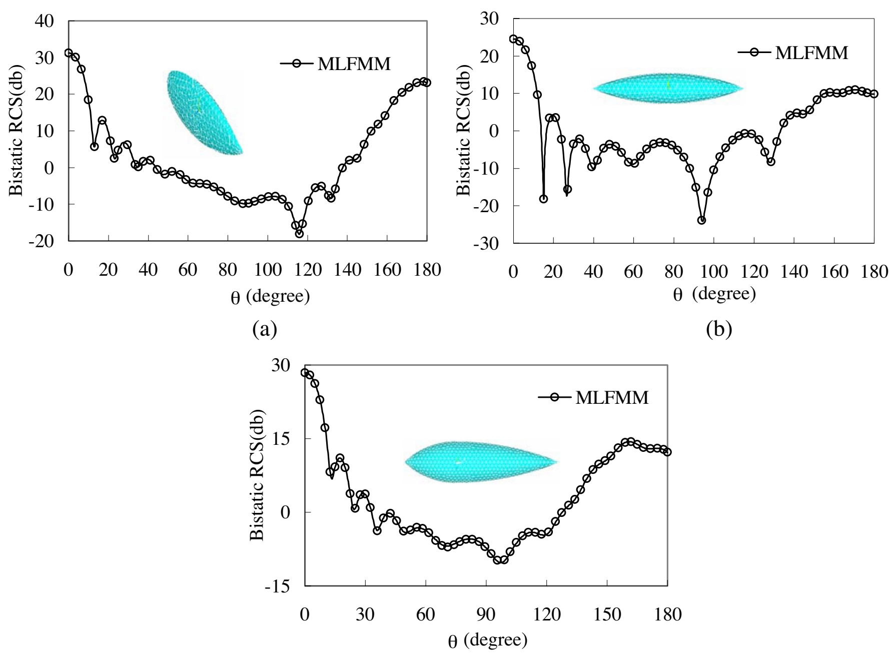
$$\mathbf{x}_2 = \mathbf{y}_2, \quad \mathbf{x}_1 = \mathbf{y}_1 - \mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{y}_2 \tag{21}$$

## 4. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we show some numerical results that illustrate the effectiveness of the proposed Schwarz-GMRES method for the solution of linear systems arising from the discretization of EFIE formulation in electromagnetic scattering problems. In the implementation of the Schwarz preconditioner, we use the sparse approximate inverse (SAI) preconditioned GMRES method [22] for iteratively solving Equation (18), and use the direct multifrontal method for solving Equation (17). We compare the Schwarz-GMRES method with

GMRES methods with the incomplete LU decomposition (ILU) [16] preconditioner, with the SSOR [23] preconditioner, and without a preconditioner.

To illustrate the performance of the Schwarz method, we calculate the RCS of three conducting geometries from [21], which is shown in Figs. 1(a)–(c). They consist of a single-ogive with 3690 unknowns at 6 GHz, an almond with 3510 unknowns at 1.19 GHz, and a double-ogive with 27130 unknowns at 15 GHz. The numerical results of bistatic RCS for vertical polarization are also displayed in Figs. 1(a)–(c) for these three geometries. All experiments are conducted on a Pentium 4 with 1 GB local memory and run at 2.9 GHz in single precision. The iteration process is terminated when the 2-norm residual error is reduced by $10^{-3}$, and the limit of the maximum number of iterations is set as 2000.




(a) (b) (c)

**Figure 1.** The Bistatic RCS (a) for an almond at 1.19 GHz, (b) for a single-ogive at 6 GHz, (c) for a double-ogive at 15 GHz.

Figures 2(a)–(c) depicts the convergence history of the Schwarz-GMRES method, the ILU preconditioned GMRES method (ILU-GMRES), and the SSOR preconditioned GMRES method (SSOR-

GMRES). It demonstrates that both the ILU-GMRES method and the SSOR-GMRES method converge much faster than the GMRES method without a preconditioner. But the Schwarz-GMRES method is the most efficient one on all three examples. More detailed comparisons can be found in Table 1, in which the solution cost of the Schwarz-GMRES method and other preconditioned GMRES methods is evaluated in terms of number of matrix-vector products and CPU time, where $*$ refers to no convergence after maximum 2000 iterations and 's' denotes second. It can be observed that both the SSOR-GMRES and the ILU-GMRES methods can not convergence in the maximum iterations for most cases, while the Schwarz-GMRES method converges for all cases with the smallest computational cost. This shows the robustness and efficiency of the proposed Schwarz-GMRES method.

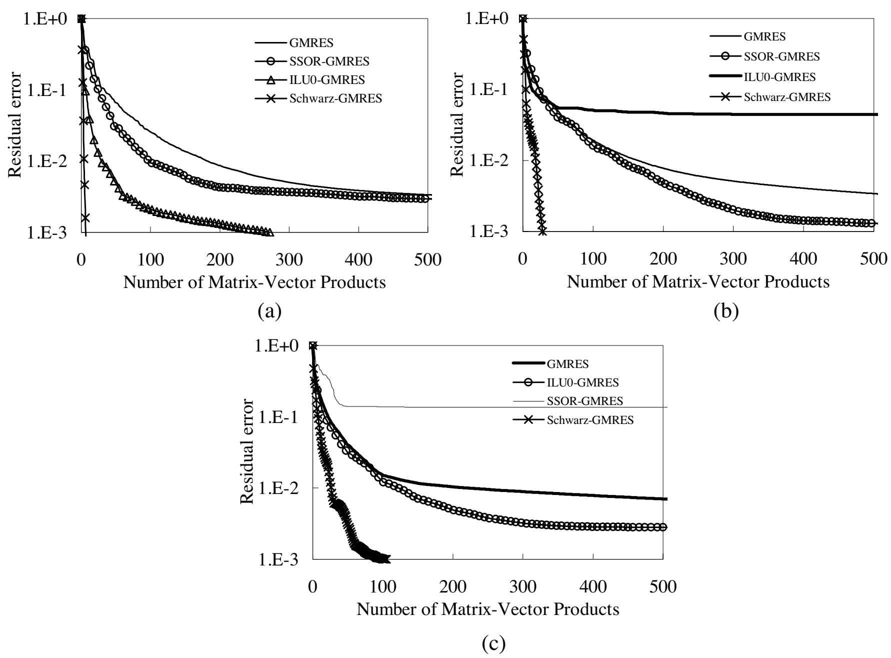



**Figure 2.** The convergence history (a) on the almond example, (b) on the single-ogive example, (c) on the double-ogive example.

**Table 1.** Cumulated number of matrix-vector products and the overall simulation time in the bistatic RCS calculation.

| Examples | GMRES | SSOR-GMRES | ILU0-GMRES | Schwarz-GMRES |
|---|---|---|---|---|
| Almond | * | * | 281(109.84s) | 7(57.94s) |
| Ogive | * | 1068(284.34s) | * | 29(19.97s) |
| Dbogive | * | * | * | 103(4125.58s) |

## 5. CONCLUSIONS

In this paper, a set of high-order hierarchical basis functions introduced in the FEM is proposed for EFIE solved using the MLFMA with a reduced computational complexity. It leads to a significant reduction in the number of unknowns without compromising the accuracy of geometry modeling. A kind of Schwarz method is proposed to act as the preconditioner in order to accelerate the Krylov subspace iterative methods for solving EFIE. Our numerical results show that for linear systems with highly indefinite symmetric matrices, the combination of Schwarz method and Krylov subspace iterative methods can circumvent the low-frequency instability problem and thus significantly reduce the iteration number and the total simulation time. A considerable improvement in convergence rate is achieved compared with classical preconditioned Krylov subspace iterative methods. It can be concluded that the Schwarz-Krylov method is highly efficient when applied to the large dense linear systems resulting from the hierarchical EFIE.

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