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WIENER-HOPF ANALYSIS OF FINITE-LENGTH IMPEDANCE LOADING IN THE OUTER CONDUCTOR OF A COAXIAL WAVEGUIDE

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Abstract—A new canonical scattering problem consisting of the propagation of the dominant TEM mode at the finite-length impedance discontinuity in the outer conductor of a coaxial waveguide is solved. The contributions from the successive impedance discontinuities are accounted for through the solution of a modified Wiener-Hopf equation. Some graphical results displaying the reflection and transmission characteristic are presented.

1. INTRODUCTION

The discontinuities in coaxial waveguides are a very important topic in microwave theory and have been subjected to numerous past investigations. Most simple types of discontinuities such as steps in inner or outer conductors (see e.g., [1–4]) and wall impedance discontinuities [5–7] were analyzed and characterized. For example, in [5] and [6] the scattering of a shielded surface wave in a coaxial waveguide by a wall impedance discontinuity in the inner cylinder has been analyzed. These classical results are related mostly with isolated discontinuities, and fail when there are several of them close enough to interfere with each other.

In the present work we consider a new canonical scattering problem consisting of the propagation of the dominant TEM mode at the finite-length impedance discontinuity in the outer conductor of a coaxial waveguide (see Figure 1). The contributions from the successive impedance discontinuities are accounted for through the solution of a modified Wiener-Hopf equation. Notice that the present problem may also be thought a first order approximation for coaxial waveguides loaded with a shallow groove in the outer conductor.

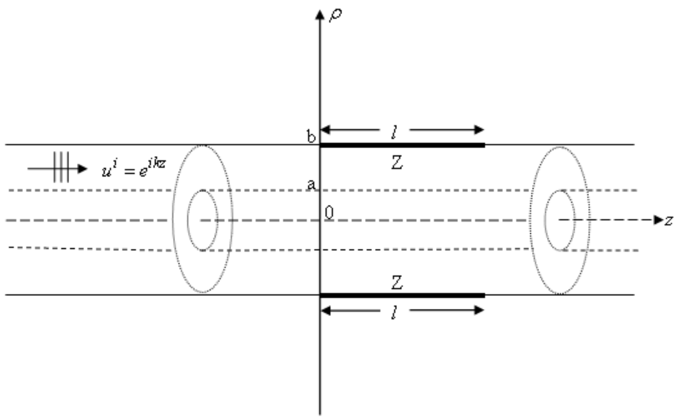


Figure 1. Coaxial cable with a finite length impedance loading in the outer conductor.

2. ANALYSIS

Consider a coaxial waveguide whose inner cylinder is of radius $\rho = a$, while the radius of the outer cylinder is $\rho = b$ with (ρ, ϕ, z) being the usual cylindrical coordinates. The part $0 < z < l$ of the outer conductor is characterized by constant surface impedance denoted by $Z = \eta Z_0$ with Z_0 being the characteristic impedance of the free space.

Let the incident TEM mode with angular frequency ω and propagating in the positive z direction be given by

$$u^i = \exp(ikz) \tag{1a}$$

where an $\exp(-i\omega t)$ time factor is assumed and suppressed. k is the propagation constant which is assumed to have a small imaginary part corresponding of medium with damping. The lossless case can be obtained by letting $\text{Im}k \to 0$ at the end of the analysis.

The total field $u^T(\rho, z)$ can be written as

$$u^T = u^i + u_1(\rho, z), \quad \rho \in (a, b), \quad z \in (-\infty, \infty) \tag{1b}$$

$u_1(\rho, z)$ appearing in (1b) is unknown function which satisfies the Helmholtz equation and the following boundary conditions and continuity relations

$$\left.\frac{\partial}{\partial\rho}[u^i + u_1(\rho, z)]\right|_{\rho=b} = 0 \qquad z \in (-\infty, 0) \cup (l, \infty) \tag{2a}$$

$$\frac{\partial}{\partial \rho}[u^i + u_1(\rho, z)]\bigg|_{\rho=a} = 0 \qquad z \in (-\infty, \infty) \tag{2b}$$

$$[\eta - \frac{1}{ik}\frac{\partial}{\partial \rho}]\left(u^i + u_1(\rho, z)\right)\bigg|_{\rho=b} = 0 \qquad z \in (0, l) \tag{2c}$$

In the region, $\rho \in (a, b)$ where the field $u_1(\rho, z)$ satisfies the Helmholtz equation in the range $z \in (-\infty, \infty)$.

$$\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{\partial^2}{\partial z^2} + k^2\right) u_1(\rho, z) = 0 \qquad z \in (-\infty, \infty) \tag{3a}$$

Its Fourier transform is:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + K^2(\alpha)\right) F(\rho, \alpha) = 0 \tag{3b}$$

with

$$F(\rho, \alpha) = \int_{-\infty}^{\infty} u_1(\rho, \alpha) e^{i\alpha z} dz. \tag{3c}$$

Here, $K(\alpha)$ denotes the square-root function

$$K(\alpha) = \sqrt{k^2 - \alpha^2}, \tag{4}$$

which is defined in the complex α-plane, cut along $\alpha = k$ to $\alpha = k + i\infty$ and $\alpha = -k$ to $\alpha = -k - i\infty$, such that $K(0) = k$.

$F(\rho, \alpha)$ which is the Fourier transform of $u_1(\rho, z)$ can also be written as

$$F(\rho, \alpha) = F_-(\rho, \alpha) + F_1(\rho, \alpha) + e^{i\alpha l} F_+(\rho, \alpha) \tag{5a}$$

with,

$$F_-(\rho, \alpha) = \int_{-\infty}^{0} u_1(\rho, \alpha) e^{i\alpha z} dz, \tag{5b}$$

$$F_1(\rho, \alpha) = \int_{0}^{l} u_1(\rho, \alpha) e^{i\alpha z} dz, \tag{5c}$$

$$F_+(\rho, \alpha) = \int_{l}^{\infty} u_1(\rho, \alpha) e^{i\alpha(z-l)} dz \tag{5d}$$

Owing to the analytical properties of Fourier integrals, $F_+(\rho,\alpha)$ and $F_-(\rho,\alpha)$ are yet unknown and regular functions of α in the half-planes $\mathrm{Im}(\alpha) > \mathrm{Im}(-k)$ and $\mathrm{Im}(\alpha) < \mathrm{Im}(k)$, respectively, while $F_1(\rho,\alpha)$ defined by (5c) is an entire function.

The solution of the homogeneous differential equation in (3b) is:

$$F(\rho,\alpha) = A(\alpha)J_0(K\rho) + B(\alpha)Y_0(K\rho). \tag{6}$$

with J_n and Y_n being the usual Bessel and Neumann functions of order n. The spectral coefficients $A(\alpha)$ and $B(\alpha)$ are to be determined through the boundary conditions in (2a) and (2b). Hence, (6) can be cast into the following form:

$$F_-(\rho,\alpha) + F_1(\rho,\alpha) + e^{i\alpha l}F_+(\rho,\alpha) = -\frac{T_2(a,\rho,\alpha)}{T_1(a,b,\alpha)}\dot{F}_1(b,\alpha) \tag{7a}$$

where the $(\cdot)$ denotes the derivative with respect to ρ i.e., $\dot{F}_1(b,\alpha) = \frac{\partial}{\partial\rho}F_1(\rho,\alpha)\Big|_{\rho=b}$. $T_1(a,b,\alpha)$ and $T_2(a,b,\alpha)$ appearing in (7a) are entire functions of α, defined by

$$T_1(a,b,\alpha) = K^2(\alpha)\left[J_1(Ka)Y_1(Kb) - J_1(Kb)Y_1(Ka)\right] \tag{7b}$$
$$T_2(a,b,\alpha) = K(\alpha)\left[J_1(Ka)Y_0(Kb) - J_0(Kb)Y_1(Ka)\right]. \tag{7c}$$

By using the boundary conditions in (2c) and substituting $\rho = b$ in the Equation (7a) one obtains the following modified Wiener-Hopf equation valid in the strip $\mathrm{Im}(-k) < \mathrm{Im}(\alpha) < \mathrm{Im}(k)$:

$$ik\eta F_-(b,\alpha)+V(\alpha)\dot{F}_1(b,\alpha)+ik\eta e^{i\alpha l}F_+(b,\alpha) = k\eta\left[\frac{e^{i(k+\alpha)l}-1}{k+\alpha}\right] \tag{8a}$$

with

$$V(\alpha) = \frac{\chi(\alpha)}{T_1(a,b,\alpha)} \tag{8b}$$

and

$$\chi(\alpha) = T_1(a,b,\alpha) + ik\eta T_2(a,b,\alpha) \tag{8c}$$

The modified Wiener-Hopf equation in (8a) can be rewritten as

$$ik\eta R_-^*(\alpha) + V(\alpha)\dot{F}_1(b,\alpha) + ik\eta e^{i\alpha l}S_+(\alpha) = 0 \tag{9a}$$

with

$$R_-^*(\alpha) = F_-(b,\alpha) - \frac{i}{k+\alpha} \tag{9b}$$

and

$$S_+(\alpha) = F_+(b,\alpha) + \frac{ie^{ikl}}{k+\alpha} \tag{9c}$$

Now multiplying (9a) first by $1/V_-(\alpha)$ and then by $e^{-i\alpha l}/V_+(\alpha)$ we obtain the following equations, after applying the well known Wiener-Hopf decomposition procedure and the Liouville theorem:

$$\frac{R^*_-(\alpha)}{V_-(\alpha)} = \frac{1}{2\pi i}\int\limits_{\mathcal{L}-} \frac{S_+(\tau)}{V_-(\tau)}\frac{e^{i\tau l}}{(\tau-\alpha)}d\tau - \frac{i}{k+\alpha}\frac{1}{V_+(k)} \tag{10a}$$

and

$$\frac{S_+(\alpha)}{V_+(\alpha)} = -\frac{1}{2\pi i}\int\limits_{\mathcal{L}+} \frac{R^*_-(\tau)}{V_+(\tau)}\frac{e^{-i\tau l}}{(\tau-\alpha)}d\tau \tag{10b}$$

The evaluation of the above integrals by using Jordan's lemma and the residue theorem yields

$$\frac{R^*_-(\alpha)}{V_-(\alpha)} = \sum_{m=0}^{\infty}\frac{V_+(\beta_m)e^{i\beta_m l}S_+(\beta_m)}{V'(\beta_m)(\beta_m-\alpha)} - \frac{i}{k+\alpha}\frac{1}{V_+(k)} \tag{11a}$$

$$\frac{S_+(\alpha)}{V_+(\alpha)} = \sum_{m=0}^{\infty}\frac{V_+(\beta_m)e^{i\beta_m l}R^*_-(-\beta_m)}{V'(\beta_m)(\beta_m+\alpha)} \tag{11b}$$

where β_n are the simple zeros of $\chi(\alpha)$, lying in the upper half-plane:

$$\chi(\pm\beta_n) = 0, \qquad \mathrm{Im}\beta_n > \mathrm{Im}k, \tag{11c}$$

and the dash denotes the derivative with respect to α, i.e., $V'(\beta_n) = \frac{\partial}{\partial\alpha}V(\alpha)\big|_{\alpha=\beta_n}$.

Substituting (11a) and (11b) into (9a), the formal solution of the modified Wiener-Hopf equation is obtained in terms of the yet unknown coefficients $S_+(\beta_n)$ and $R^*_-(-\beta_n)$:

$$\begin{aligned}\dot{F}_1(b,\alpha) = {} & -ik\eta\frac{1}{V_+(\alpha)}\sum_{m=0}^{\infty}\frac{V_+(\beta_m)e^{i\beta_m l}S_+(\beta_m)}{V'(\beta_m)(\beta_m-\alpha)} + ik\eta\frac{1}{V_+(\alpha)}\frac{i}{k+\alpha}\frac{1}{V_+(k)} \\ & -ik\eta\frac{e^{i\alpha l}}{V_-(\alpha)}\sum_{m=0}^{\infty}\frac{V_+(\beta_m)e^{i\beta_m l}R^*_-(-\beta_m)}{V'(\beta_m)(\beta_m+\alpha)}.\end{aligned} \tag{12}$$

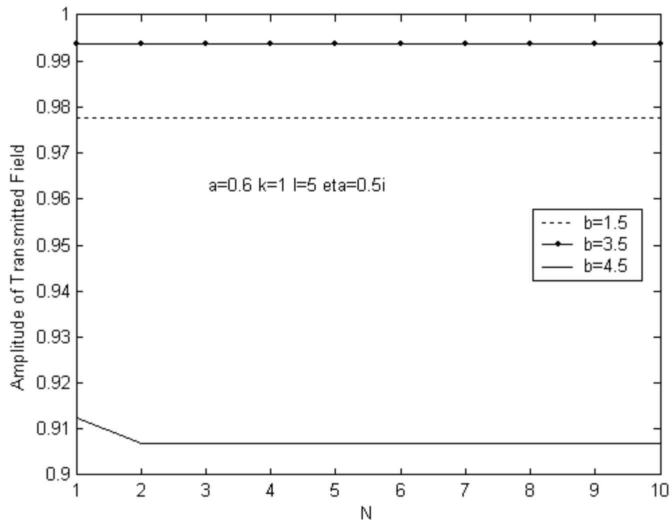


Figure 2. Amplitude of the transmitted field versus the truncation number N.

In order to determine these unknown coefficients one has to replace $\alpha = -\alpha_r$ in (11a) and $\alpha = \beta_r$ in (11b). This gives

$$\frac{R_-^*(-\alpha_r)}{V_-(-\alpha_r)} = \sum_{m=0}^{\infty} \frac{V_+(\beta_m) e^{i\beta_m l} S_+(\beta_m)}{V'(\beta_m)(\beta_m + \alpha_r)} - \frac{i}{(k - \alpha_r)\, V_+(k)} \quad (13a)$$

$$\frac{S_+(\beta_r)}{V_+(\beta_r)} = \sum_{m=0}^{\infty} \frac{V_+(\beta_m) e^{i\beta_m l} R_-^*(-\beta_m)}{V'(\beta_m)(\beta_m + \beta_r)} \quad (13b)$$

This infinite system of algebraic equations is solved numerically. All the numerical results were derived by truncating the infinite series and the infinite systems of linear algebraic equations after the first N terms. Figure 2 shows the variation of the modulus of the transmitted field against the truncation number N. It is seen that the amplitude of the diffracted field becomes insensitive to the increase of the truncation number after $N = 3$.

3. THE SCATTERED FIELD

The transmitted field in the region $\rho \in (a, b)$ and $z > l$ is obtained by taking the inverse Fourier Transform of $F_+(\rho, \alpha)$. By using (7a) we write:

$$u_1(\rho,\alpha) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{T_2(a,\rho,\alpha)}{T_1(a,b,\alpha)}\dot{F}_1(b,\alpha)+F_-(\rho,\alpha)+F_1(\rho,\alpha)\right]e^{-i\alpha z}d\alpha. \tag{14a}$$

The application of the residue theorem yields for $z > l$,

$$u_1(\rho,\alpha) = i\sum_{m=0}^{\infty} res\left(\frac{T_2(a,\rho,\alpha)e^{-i\alpha z}}{T_1(a,b,\alpha)}\dot{F}_1(b,\alpha)\right)\Bigg|_{\alpha=-\xi_m}. \tag{14b}$$

The transmission coefficient $\mathcal{T}$ of the fundamental mode is defined as the complex coefficient multiplying the travelling wave term $\exp(ikz)$ and is computed from the contribution of the first pole at $\alpha = -k$. The result is :

$$\mathcal{T} = \frac{ib}{k}\frac{\dot{F}_1(b,-k)}{(b^2-a^2)} \tag{15}$$

Similarly, the expression of the reflected field in the region $\rho \in (a,b)$ and $z < 0$ can be obtained by taking the inverse Fourier transform of $F_-(\rho,\alpha)$. By using again (7a) and evaluating the resulting integral through the residue theorem we obtain

$$u_1(\rho,\alpha) = -i\sum_{m=0}^{\infty} res\left(\frac{T_2(a,\rho,\alpha)e^{-i\alpha z}}{T_1(a,b,\alpha)}\dot{F}_1(b,\alpha)\right)\Bigg|_{\alpha=\xi_m}. \tag{16}$$

From the contribution of the first root occurring at $\alpha = k$ we can define the reflection coefficient of the dominant mode as:

$$\mathcal{R} = \frac{ib}{k}\frac{\dot{F}_1(b,k)}{(b^2-a^2)}. \tag{17}$$

4. COMPUTATIONAL RESULTS

In this section some computational results displaying the effect of various parameters such as the radii of the inner and outer cylinders, the frequency, the surface impedance and its length on the propagation of the fundamental mode are presented. In what follows the surface impedance is assumed to be purely capacitive.

Figure 3 displays the amplitude of the reflected and transmitted fields against the radius of the inner cylinder. It is seen that the transmitted field amplitude decreases when the spacing between the inner and outer cylinders diminishes. When the surface impedance is capacitive, the curves related to the reflected and transmitted fields are shifted to the left for increasing values of $|\eta|$. Hence, by assigning higher capacitive values to the surface impedance one can reduce the transmitted energy even for bigger spacing between the cylinders.

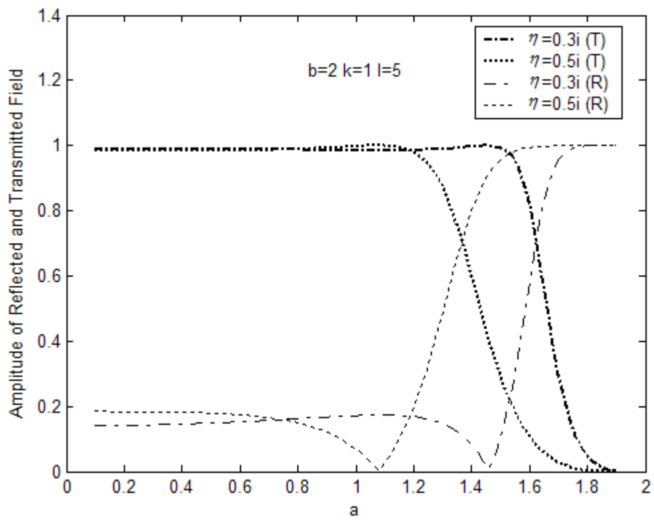


Figure 3. Amplitude of the reflection and transmission coefficients versus the radius of the inner cylinder a.

Figure 4 shows the variation of the reflection coefficient amplitude with respect to the wave number. By increasing the purely capacitive impedance values we can increase the maxima of reflected field amplitude and the nulls are shifted slightly to the right.

From Figure 5 it is seen that for increasing values of the purely capacitive impedance the transmitted field is reduced. For bigger outer cylinder radius the transmitted field amplitude curves become "sharper" and a more rapid decrease in the transmitted energy is observed.

Figure 6(a) and Figure 6(b) depict the variation of the reflection coefficient with the width of the impedance zone for different values of the outer cylinder radius b and for different impedance values, respectively. It is seen that below a certain value of the outer cylinder radius (Figure 6(a)), and above a certain purely capacitive impedance value (Figure 6(b)) the incident energy is almost totally reflected and consequently the transmitted field is notably attenuated.

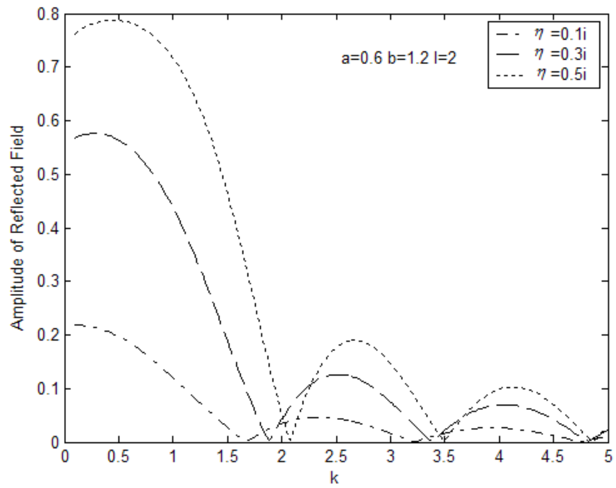


Figure 4. Amplitude of the reflected field versus the wavenumber (frequency).

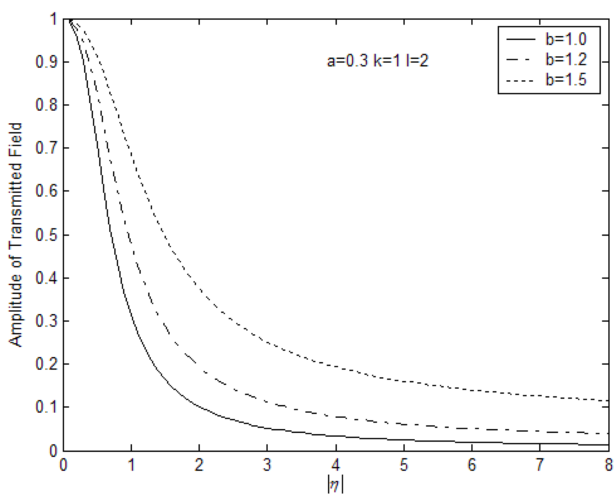
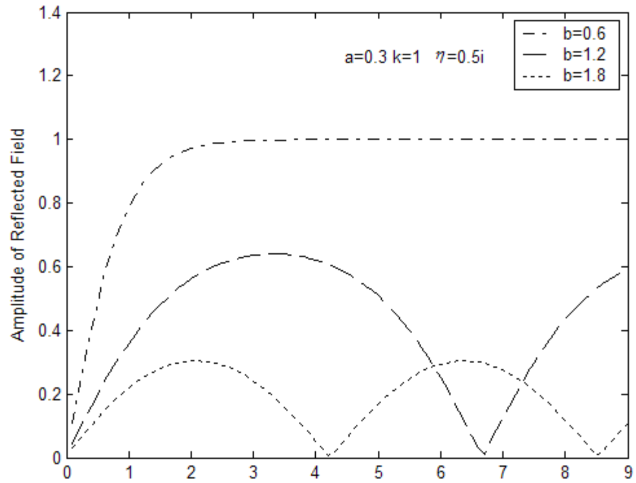
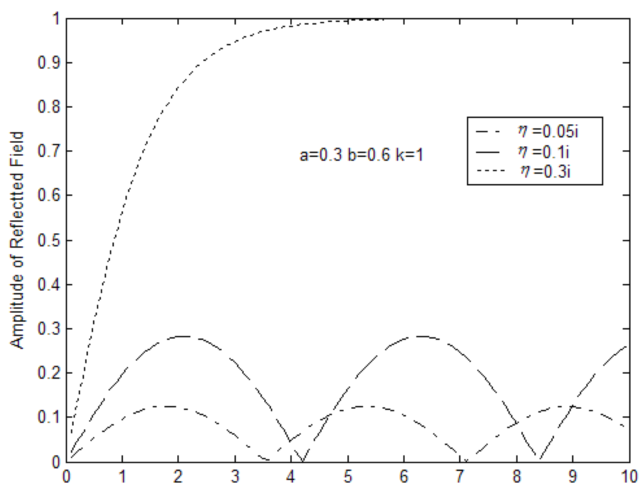


Figure 5. Reflected and transmitted field versus the impedance loading.



(a)



(b)

Figure 6. (a) Reflected field amplitudes versus the impedance length l, for different values of b. (b) Reflected field amplitudes versus the impedance length l, for different values of η.

5. CONCLUDING REMARKS

A new scattering problem consisting of the propagation of the dominant TEM mode at the finite-length impedance discontinuity in the outer conductor of a coaxial waveguide is considered. The contributions from the successive impedance discontinuities are accounted for through the solution of a modified Wiener-Hopf equation. The influence of various parameters such as the radius of the inner cylinder, the frequency, the impedance values and the length of the impedance zone on the outer cylinder on the reflection and transmission characteristics are displayed graphically.

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