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TEMPORAL CAVITY OSCILLATIONS CAUSED BY A WIDE-BAND WAVEFORM

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Abstract—Excitation of the electromagnetic fields by a wide-band current surge, which has a beginning in time, is studied in a cavity bounded by a closed perfectly conducting surface. The cavity is filled with Debye or Lorentz dispersive medium. The fields are presented as the modal expansion in terms of the solenoidal and irrotational cavity modes with the time-dependent modal amplitudes, which should be found. Completeness of this form of solution has been proved earlier. The systems of ordinary differential equations with time derivative for the modal amplitudes are derived and solved *explicitly* under the initial conditions and in compliance with the causality principle. The solutions are obtained in the form of simple convolution (with respect to time variable) integrals. Numerical examples are exhibited as well.

1. INTRODUCTION

This study relates to the old problem of effective protection of electronic systems from the voltage and current surges. Such surges can be caused by various natural phenomena like strokes of lightning, electrostatic discharges, breakdown effects in industrial and household appliances, switching, *etc.* The surges are inevitably accompanied by onset of electromagnetic waveforms which are essentially distinct from the habitual time-harmonic oscillations and waves. They are able to entail dangerous failure of microwave appliances, navigation and defense systems, communication, modern medical equipment, computers, and so on. Besides, it becomes possible to develop nowadays effective ultra-wide band pulse generators which can be built in a very small volume. It arouses anxiety in connection with perilous threat in the aspect of possible destructive using them as a tool for electromagnetic terrorism [1–3].

As a data channel in modelling of the surges, the double-exponential pulse waveform is often used in the literature [4]. Consider a case when surge $s(t)$ has a beginning in time at t=0 :

$$s(t) = H(t)\ f(t) \tag{1}$$

where $H(t)$ is Heaviside step function, waveform $f(t)$ is just the normalized double-exponential function, which we specify as

$$f(t) = \left(e^{-\gamma_1 t} - e^{-\gamma_2 t}\right) / \left(e^{-\gamma_1 T} - e^{-\gamma_2 T}\right) \tag{2}$$

where $\gamma_2 > \gamma_1 > 0$ are two independent parameters, T is an intrinsic time (a free parameter as well), time variable $t>0$. Function $f(t)$ has only one extremum (maximum). It is convenient to superpose value $t=T$ with *maximum* of the function what yields $f(t)\,|_{t=T}=1$. Hence, equation $\frac{d}{dt} f(t)\,|_{t=T} = 0$ holds what generates a set of definitions as

$$\begin{aligned} T &= \rho\eta/\gamma_2 = \eta/\gamma_1\,, \\ \rho &= \gamma_2/\gamma_1 > 1, \\ \eta &= (\ln\rho)\,/\,(\rho - 1) < 1. \end{aligned} \tag{3}$$

Introduce dimensionless (scaled) time ξ as follows: $\xi = t/T$. Then the surge waveform (2) as a function of ξ seems as

$$f(\xi) = \left(e^{-\eta\xi} - e^{-\rho\eta\xi}\right) e^{\eta}\rho\,/\,(\rho - 1)\,. \tag{4}$$

Compare the formulas (2) and (4). The first one involves *three* independent parameters: γ_1, γ_2 and T. Function $f(\xi)$ involves only *one* parameter: ρ. In Fig. 1, time dependence of function $f(\xi)$ is exhibited for different numerical values of the parameter ρ.

It is evident that width of the frequency spectrum of the double exponential function can be effectively regulated by appropriate choice of the numerical value of parameter ρ. Just that function $f(\xi)$ will be used in our studies of the current surges further on.

Engineers have enough rich data storage and practical experience on the problem of protection from the surges. However, even the experts in this area estimate sometimes the actual state of affairs as "art rather than science." Progress essentially depends on further development of reliable theoretical fundamentals of electromagnetics. It should be oriented on study of just the *transient* processes in the time domain *(TD) directly.* New knowledge, which will be obtained on this way, is capable of being a base for perfection of fundamental concepts in the area of protection from the surges.

In this article, we demonstrate ability of an *evolutionary approach to electromagnetics (EAE)* for study of the transient processes in

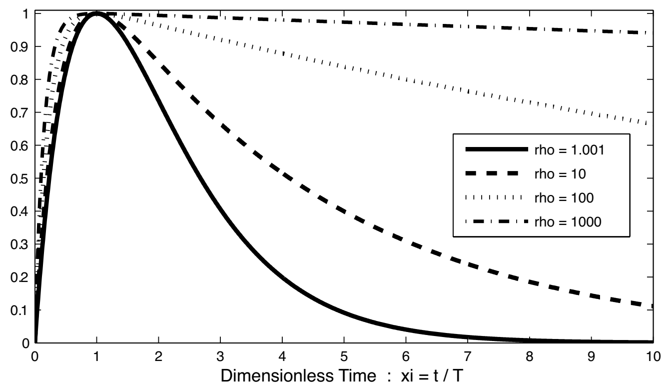

Figure 1. Surge form for different values of rho.

the microwave cavities [5–10]. The cavities serve as typical integral constituents which are present in any microwave system.

Study of the field waveforms excited by a current surge in a cavity volume is performed. Every surge has a beginning in time. Consequently, the solutions to the system of Maxwell's equations with ∂_t under appropriate *initial conditions* have been obtained in compliance with the *causality principle.* The solutions were found explicitly in elementary functions. The surge excites the solenoidal and the irrotational modal oscillations *both.* Noteworthy to note that the *irrotational* modal waveforms in the cavity *prevail* over the solenoidal ones almost per order. In realistic situations, some medium may be present in the cavity. Therefore, the modal waveforms were considered in the cavity filled completely by a *homogeneous* medium. The surges have a wide-band frequency spectrum, generally speaking. It enforces to supplement the system of Maxwell's equations with either Debye or Newton equation of motion for the polarization vector. Just these equations establish *dynamic* constitutive relation in the time domain between the polarization vector and the electric field, which gives rise to the polarization. The solutions have been obtained via simple convolution integrals, the properties of which were studied.

2. FORMULATION OF THE TIME DOMAIN PROBLEM

The system of vector differential Maxwell's equations

$$\begin{aligned} \nabla \times \mathcal{H}(\mathbf{r},t) &= \varepsilon_0 \partial_t \mathcal{E}(\mathbf{r},t) + \partial_t \mathcal{P}(\mathbf{r},t) + \sigma \mathcal{E}(\mathbf{r},t) + \mathcal{J}(\mathbf{r},t) \\ -\nabla \times \mathcal{E}(\mathbf{r},t) &= \mu_0 \partial_t \mathcal{H}(\mathbf{r},t) \end{aligned} \tag{5}$$

is under study for the macroscopic fields $\mathcal{E}(\mathbf{r},t)$, $\mathcal{H}(\mathbf{r},t)$, $\mathcal{P}(\mathbf{r},t)$, where $\mathbf{r}$ is position vector of a point of observation within a cavity of volume V and t is observation time. The term $\sigma\mathcal{E}$ is involved in the first equation (5) for modelling losses which are present in any real cavity. Conductivity σ is a given real-valued parameter (a constant). If the cavity is *hollow* (i.e., without a medium), polarization vector $\mathcal{P}$ vanishes under definition: $\mathcal{P}(\mathbf{r},t) \equiv 0$.

Electric current density $\mathcal{J}(\mathbf{r},t)$ is a *given* function of coordinates and time. Physically, function $\mathcal{J}$ generates the *forced* cavity oscillations. Let's present this "source function" as product:

$$\mathcal{J} = \varepsilon_0 \, \mathbf{F}(\mathbf{r}) \, s(t) \, T^{-1} \tag{6}$$

where $\mathbf{F}(\mathbf{r})$ is a *given* vector function of coordinates; it is assigned for description of a support (carrier) of the surge waveform (1). Vector $\mathbf{F}$ has the same physical dimension as the field $\mathcal{E}$ has it. Factor T^{-1} is installed heuristically in Eq. (6) to provide the same physical dimension of the term $\mathcal{J}$ as the other terms have it in Eq. (5).

It is supposed that the volume V is bounded by a closed surface S, which has the properties of perfect electric conductor. Hence, the boundary conditions hold over the surface S at any instant t as

$$\mathbf{r} \in S: \quad \mathbf{n} \times \mathcal{E}(\mathbf{r},t) = 0 \quad \mathbf{n} \cdot \mathcal{H}(\mathbf{r},t) = 0 \tag{7}$$

where $\mathbf{n}$ is the unit vector outward normal to S.

2.1. Constitutive Relations in the Time Domain

If the cavity is filled with a dielectric, polarization vector $\mathcal{P}(\mathbf{r},t)$ should be a function of the electric strength vector $\mathcal{E}(\mathbf{r},t\ )$: that is, $\mathcal{P}(\mathbf{r},t) = \mathcal{P}(\mathcal{E}(\mathbf{r},t))$. In the general case, vector function $\mathcal{P}(\mathcal{E})$ should be found via solving inhomogeneous Newton's equation of motion for polarization $\mathcal{P}$, in which field $\mathcal{E}$ specifies a force term. In electromagnetic field theory, dependence $\mathcal{P}(\mathcal{E})$ is named as the *constitutive relation.*

As a matter of fact, dependence of the vector function $\mathcal{P}(\mathcal{E})$ on the argument $\mathcal{E}$ is *nonlinear* in the general case because Newton's equation of motion for $\mathcal{P}$ is nonlinear in this case. However, if the field $\mathcal{E}$, which is applied to the atoms in a polarizable dielectric, is *much smaller* in comparison with the intrinsic *atomic* forces, then Newton's equation can be *linearized* by $\mathcal{E}$. In particular, *linear* Newton's equation, which is available for the dielectrics with *electronic* mechanism of polarization, has the following form:

$$\frac{d^2}{dt^2}\mathcal{P}(\mathbf{r},t) + 2\gamma\frac{d}{dt}\mathcal{P}(\mathbf{r},t) + \omega_0^2\,\mathcal{P}(\mathbf{r},t) = \varepsilon_0\omega_N^2\,\mathcal{E}(\mathbf{r},t)\,, \tag{8}$$

where the coefficients γ, ω_r^2, $\omega_0^2{=}\omega_r^2{-}\omega_N^2/3$, $\omega_N^2{=}Nq_e^2/\left(m_e\varepsilon_0\right)$ are some microscopic real-valued constants. Parameters γ and ω_r^2 depend on the atomic (or molecular) structure of the medium, N is the *volume density* of the polarized atoms (or molecules), m_e and q_e are the mass and unsigned charge of electron, respectively. Quantity ω_N^2 specifies *static* susceptibility χ_s of the medium as follows $\chi_s{=}\omega_N^2/\omega_0^2$ (see below). The media, polarization vector for which is governed by Eq. (8), are named as *Lorentz* materials.

In the case of so called *polar* dielectrics†, the "frictional force" $2\gamma d\mathcal{P}/dt$ prevails over the "inertial force" $d^2\mathcal{P}/dt^2$ and the latter may be *neglected* in equation (8). So, instead of the differential equation of the second order (8) one can operate with its simple *approximation,* which is known as *Debye's* equation:

$$\frac{d}{dt}\mathcal{P}\left(\mathbf{r},t\right)+\frac{1}{\tau_0}\,\mathcal{P}\left(\mathbf{r},t\right)=\varepsilon_0\frac{\chi_s}{\tau_0}\,\mathcal{E}\left(\mathbf{r},t\right), \tag{9}$$

where parameter $\tau_0=2\gamma/\omega_0^2$ is named as the *relaxation time.*

Independent space variables $(\mathbf{r})$ in the arguments of the vector functions $\mathcal{P}\left(\mathbf{r},t\right)$ and $\mathcal{E}\left(\mathbf{r},t\right)$ play role of a parameter with respect to the time derivative in the differential equations (8) and (9) both.

There are only *two* particular cases, in which Eqs. (8) and (9) have explicit solutions. The first one corresponds to the *static* fields. In this case, time derivative acts as multiplication by *zero* of the terms to which it is applied. It yields $\mathcal{P}\left(\mathbf{r}\right){=}\varepsilon_0\left(\omega_N^2/\omega_0^2\right)\mathcal{E}\left(\mathbf{r}\right){=}\varepsilon_0\chi_s\,\mathcal{E}\left(\mathbf{r}\right)$ where ω_N^2/ω_0^2 specifies static susceptibility χ_s. Indeed, if the field $\mathcal{E}$ does nod depend on time in Eqs. (8) and (9), then solution $\mathcal{P}$ to these equations both should not depend on time as well. Hence, time derivative cancels the terms, on which it acts at the left-hand sides. It yields simple *algebraic* constitutive relation for the polarization vector as $\mathcal{P}\left(\mathbf{r}\right){=}\varepsilon_0\chi_s\,\mathcal{E}\left(\mathbf{r}\right)$ and, in turn, for the electric flux density $\mathcal{D}{=}\varepsilon_0\mathcal{E}{+}\mathcal{P}{=}\varepsilon_0\varepsilon_s\mathcal{E}$, where $\varepsilon_s{=}1{+}\chi_s$ is static permittivity.

The second explicit solution can be obtained for the *time-harmonic* fields. If the field $\mathcal{E}\left(\mathbf{r},t\right)=\mathbf{E}\left(\mathbf{r}\right)e^{-i\omega t}$, where ω is a frequency parameter $(-\infty{<}\omega{<}\infty)$ and $\mathbf{E}\left(\mathbf{r}\right)$ is a given phasor, then the solutions to Eqs. (8) and (9) are also the time-harmonic fields, respectively, as

$$\begin{aligned}&\mathcal{P}_L=\varepsilon_0\chi^{(L)}\mathbf{E}\left(\mathbf{r}\right)e^{-i\omega t} \quad \text{and} \quad \mathcal{P}_D=\varepsilon_0\chi^{(D)}\mathbf{E}\left(\mathbf{r}\right)e^{-i\omega t},\\ &\chi^{(L)}=\tfrac{\chi_s}{1-(\omega/\omega_0)^2-i2\gamma\omega/\omega_0^2} \quad \text{and} \quad \chi^{(D)}=\tfrac{\chi_s}{1-i\omega\tau_0}\end{aligned} \tag{10}$$

where $\chi^{(L)}$ and $\chi^{(D)}$ are the *frequency-dependent* values of susceptibility for the Lorentzian and Debye media, respectively. Particular case $\omega{=}0$ results in the constitutive relation for the static field.

† In particular, H_2O, N_2, O_2, O_3, C, CO, SO_2, HCl, CH_3CN, *etc.* are *polar* molecules.

In mathematically rigorous studies of the *transient* electromagnetic processes, the equation of motion (either (8) or (9)) should be solved *simultaneously* with Maxwell's equations (5), Eventually, these motion equations play role of *dynamic* (because d/dt is present therein) constitutive relation $\mathcal{P}(\mathcal{E})$, which establishes formation of the polarization vector $\mathcal{P}(\mathbf{r}, t)$ under action the applied field $\mathcal{E}(\mathbf{r}, t)$ on the medium.

2.2. Initial Conditions and Causality Principle

Partial differential equations (5) belong the *hyperbolic* kind. So, they must be supplemented with some *initial conditions* like

$$\mathbf{r} \in V,\ t{=}0: \quad \mathcal{E}(\mathbf{r}, 0) = 0 \quad \mathcal{H}(\mathbf{r}, 0) = 0. \tag{11}$$

Since the signal (6) is *zero* while $t<0$, the forced oscillations should be found in compliance with the *causality principle:*

$$\mathcal{E}(\mathbf{r}, t) = 0 \quad \mathcal{H}(\mathbf{r}, t) = 0, \quad \text{if} \quad t < 0. \tag{12}$$

To define a space of solutions, impose a generally accepted physical requirement that electromagnetic field energy is finite. Here is one of the versions of this condition:

$$\int_{t_1}^{t_2} dt \int_{V'} (\varepsilon_0 \mathcal{E} \cdot \mathcal{E}^* + \mu_0 \mathcal{H} \cdot \mathcal{H}^*)\, dv < \infty \tag{13}$$

where $0 \le t_1 < t_2 < \infty$, $V' \subseteq V$, the dot stands for the scalar multiplication of the vectors, the star stands for complex conjugation. The latter means, in turn, that we operate within a class of complex-valued vector functions of space and time. Eq. (13) suggests to choose a Hilbert space L_2 as the space of solutions to the problem under study.

3. FIELD EXPANSIONS OVER A MODAL BASIS

One can find all the mathematical details on the *EAE* in publications [5–10]. Schematically, lay-out of the method looks as follows.

The First Step. Let's take a Hilbert space L_2 as the space of solutions. A basis in L_2 can be derived *with keeping* ∂_t in the system Maxwell's equations. It yields *expansions* for the fields as projection on the basis elements:

$$\begin{aligned} \mathcal{H}(\mathbf{r}, t) &= \sum_{n=1}^{\infty} i h'_n(t)\, \mathbf{H}'_n + \sum_{n=1}^{\infty} i h''_n(t)\, \mathbf{H}''_n + \sum_{n=1}^{\infty} b_n(t)\, \nabla \psi_n \\ \mathcal{E}(\mathbf{r}, t) &= \sum_{n=1}^{\infty} e'_n(t)\, \mathbf{E}'_n + \sum_{n=1}^{\infty} e''_n(t)\, \mathbf{E}''_n + \sum_{n=1}^{\infty} a_n(t)\, \nabla \varphi_n \end{aligned} \tag{14}$$

where i is imaginary unit. Completeness of such presentation of the electromagnetic fields in any cavity with perfectly conducting its surface has been proved in our previous publications relying on Weyl Theorem about orthogonal detachments of Hilbert space [11]. Herein, the bold-face *vectors* are the basis elements which depend on the space variables $(\mathbf{r})$ *solely*. They are specified as the *eigensolutions* to well studied boundary eigenvalue problems normalized in a proper way:

$$\begin{aligned} \nabla \times \mathbf{H}'_n(\mathbf{r}) &= -i\,\omega'_n \varepsilon_0 \mathbf{E}'_n(\mathbf{r}) \\ \nabla \times \mathbf{E}'_n(\mathbf{r}) &= i\,\omega'_n \mu_0 \mathbf{H}'_n(\mathbf{r}) \\ \mathbf{n} \times \mathbf{E}'_n(\mathbf{r})\,|_S &= 0 \end{aligned} \tag{15}$$

where ω'_n's are positive real eigenvalues, $\mathbf{n}-$ the unit vector outward normal to the cavity surface S, and

$$\begin{aligned} \nabla \times \mathbf{H}''_n(\mathbf{r}) &= -i\,\omega''_n \varepsilon_0 \mathbf{E}''_n(\mathbf{r}) \\ \nabla \times \mathbf{E}''_n(\mathbf{r}) &= i\,\omega''_n \mu_0 \mathbf{H}''_n(\mathbf{r}) \\ \mathbf{n} \cdot \mathbf{H}''_n(\mathbf{r})\,|_S &= 0 \end{aligned} \tag{16}$$

where ω''_n's are the positive real eigenvalues as well.

Normalize solutions to the problems (15) and (16) both as

$$\frac{\varepsilon_0}{V} \int_V |\mathbf{E}_n|^2 \, dv = \frac{\mu_0}{V} \int_V |\mathbf{H}_n|^2 \, dv = 1\,. \tag{17}$$

The solutions to the problems (15) and (16) are the *solenoidal* vectors. Without loss of generality one can consider the vectors $\mathbf{E}'_n(\mathbf{r})$ and $\mathbf{E}''_n(\mathbf{r})$ as *real-valued* functions of coordinates. Then the vectors $\mathbf{H}'_n(\mathbf{r})$ and $\mathbf{H}''_n(\mathbf{r})$ should be pure imaginary. Solutions to the problems (15) and (16) can be associated with the sets of the TM- and TE-modes in a case of a cylindrical cavity of arbitrary cross section.

The subsets $\{\nabla\varphi_n(\mathbf{r})\}$ and $\{\nabla\psi_n(\mathbf{r})\}$ involve the *irrotational* vectors as their elements. Their generating functions $\varphi_n(\mathbf{r})$ and $\psi_n(\mathbf{r})$ are normalized *eigensolutions* to Dirichlet and Neumann boundary eigenvalue problems, respectively:

$$\begin{aligned} \nabla^2\varphi_n(\mathbf{r}) + \kappa_n^2\,\varphi_n(\mathbf{r}) &= 0 \\ \varphi_n(\mathbf{r})\,|_S &= 0 \\ \frac{\kappa_n^2\varepsilon_0}{V}\int_V |\varphi_n|^2\, dv &= 1; \end{aligned} \tag{18}$$

$$\begin{aligned} \nabla^2\psi_n(\mathbf{r}) + \nu_n^2\,\psi_n(\mathbf{r}) &= 0 \\ \partial_{\mathbf{n}}\psi_n(\mathbf{r})\,|_S &= 0 \\ \frac{\nu_n^2\mu_0}{V}\int_V |\psi_n|^2\, dv &= 1 \end{aligned} \tag{19}$$

where κ_n^2, ν_n^2 are the eigenvalues and $\partial_{\mathbf{n}}$ is normal derivative.

At the series (14), *every* vector satisfies already the boundary conditions (7). Unknown scalar coefficients e'_n, e''_n, a_n, h'_n, h''_n, b_n are the *modal amplitudes,* physically. It is evident that they should be *time-dependent* with necessity. It is worthwhile to note that the time-harmonic field concept interprets all the irrotational modes as some *static* fields. It means, in turn, that the modal amplitudes a_n and b_n of the irrotational modes should be some constants. Within the frames of the *EAE*, however, we'll show later on that they can vary in time.

The coordinate part $\mathbf{F}(\mathbf{r})$ of the "force function" (6) can be projected onto the basis elements what yields

$$\mathbf{F}(\mathbf{r}) = \sum_{n=1}^{\infty} A'_n \mathbf{E}'_n + \sum_{n=1}^{\infty} A''_n \mathbf{E}''_n + \sum_{n=1}^{\infty} A^0_n \nabla\varphi_n \tag{20}$$

where the constant "force amplitudes" are specified as

$$\begin{aligned} A'_n &= (\varepsilon_0/V) \int_V \left(\mathbf{F}\cdot\mathbf{E}'^*_n\right) dv \\ A''_n &= (\varepsilon_0/V) \int_V \left(\mathbf{F}\cdot\mathbf{E}''^*_n\right) dv \\ A^0_n &= (\varepsilon_0/V) \int_V \left(\mathbf{F}\cdot\nabla\varphi^*_n\right) dv\,. \end{aligned} \tag{21}$$

The Second Step at the *EAE*. A problem should be posed for definition of the modal amplitudes. It can be performed by projecting Maxwell's equations (5) onto the same basis elements. Since the time derivative *has been kept* while the expansions (14) were derived, it yields a system of ordinary *evolutionary* (i.e., with d/dt) differential equations for the modal amplitudes. Projecting the initial conditions (11) onto the basis elements yields appropriate initial conditions for the modal amplitudes. Combination them results in well studied in mathematics Cauchy problem. It can be solved by various analytical or numerical methods, or by their combinations.

One can find below several examples on solving of the Cauchy problems for the time-dependent modal amplitudes.

4. WAVEFORMS IN THE HOLLOW LOSSY CAVITY

Let us first put, for simplicity sake, that the dielectric material is absent within the cavity volume V. It means that one can substitute $\mathcal{P} \equiv 0$ in Eq. (5). Projecting the Maxwell's equations onto the basis elements and combination the results with the initial conditions yields four sets of *uncoupled*[‡] Cauchy problems for the modal amplitudes as

$$\begin{cases} \frac{d}{dt}e'_n + 2\gamma_0 e'_n + \omega'_n h'_n = -\frac{s(t)}{T}A'_n & e'_n(0) = 0 \\ \frac{d}{dt}h'_n - \omega'_n e'_n = 0 & h'_n(0) = 0 \end{cases} \tag{22}$$

$$\begin{cases} \frac{d}{dt}e''_n + 2\gamma_0 e''_n + \omega''_n h''_n = -\frac{s(t)}{T}A''_n & e''_n(0) = 0 \\ \frac{d}{dt}h''_n - \omega''_n e''_n = 0 & h''_n(0) = 0 \end{cases} \tag{23}$$

$$\left\{ \frac{d}{dt}a_n + 2\gamma_0 a_n = -\frac{s(t)}{T}A^0_n \quad a_n(0) = 0 \right. \tag{24}$$

$$\left\{ \frac{d}{dt}b_n = 0 \quad b_n(0) = 0, \right. \tag{25}$$

where $\gamma_0 = \sigma/(2\varepsilon_0)$ is a lossy parameter, the subscripts $n = 1, 2, \ldots$ identify the cavity modes. Homogeneous (i.e., *zero*-valued) initial conditions for the modal amplitudes follow from Eq. (11).

The problem (25) has evident solution: $b_n(t) = 0$, $n = 1, 2, \ldots$. So, the series $\sum_{n=1}^{\infty} b_n \nabla\psi_n(\mathbf{r})$ can be *omitted* at the expansion (14) for magnetic field[§]. However, the amplitudes $a_n(t)$ vary in time under driving by the impressed force: see the problem (24). Thereby, the series $\sum_{n=1}^{\infty} a_n(t)\nabla\varphi_n(\mathbf{r})$ should be *held* in the field expansion (14).

The Irrotational Modes. Solving the problem (24) is elementary: it yields the following convolution integral:

$$a_n(\xi) = -A^0_n H(\xi)\int_0^{\xi} e^{2\rho_0\eta(x-\xi)} f(x)\,dx \tag{26}$$

where $\xi = t/T$ is dimensionless time, $H(\xi)$ is Heaviside step function, the surge waveform $f(x)$ is specified by formula (4).

The Solenoidal Modes. In the equations (22) and (23), the coefficients ω'_n and ω''_n have physical dimension and sense of the eigenfrequencies of the *solenoidal* modes in a hollow *loss-free* cavity (i.e., when $\gamma_0 \propto \sigma = 0$). The subscript n orders the eigenvalues in order of

[‡] This fact means physically that *every* mode evolves *individually* under driving by the force $\mathcal{J}(t)$. *Any* frequency band of the force function has *not* effect on this fact. Mode coupling may happen if some *irregularity* of a medium is present in the cavity.

[§] The amplitudes b_n's are zeros because there is *not* a source of *magnetic* kind in Maxwell's equations (5). But if one installs a time-dependent source of *magnetic* kind $\mathcal{I}$ in the second equation (5), the situation changes. The amplitudes b_n's will be varying in time under driving by the force $\mathcal{I}$.

increasing their numerical values. If the cavity is performed as shorted piece of a waveguide of arbitrary cross section, then the pairs of the modal amplitudes $\{e_n'(t), h_n'(t)\}$ can be associated with TM- modes and the pairs $\{e_n''(t), h_n''(t)\}$ be corresponding to the TE- modes. When the waveguide has rectangular cross section (i.e., a boxed cavity), the eigenfrequencies ω_n' and ω_n'' coincide and are equal to

$$\omega_n' = \omega_n'' \equiv \omega_{p,q,s} = \frac{\pi}{\sqrt{\varepsilon_0 \mu_0}} \sqrt{\left(\frac{p}{a}\right)^2 + \left(\frac{q}{b}\right)^2 + \left(\frac{s}{d}\right)^2} > 0 \tag{27}$$

where p, q, s are integers and a, b, d are the edges of the box.

It is convenient to rewrite the Cauchy problem (22) in a vector form via introducing a vector $X(\xi)$ composed of the modal amplitudes $e_n'(\xi)$ and $h_n'(\xi)$ as follows

$$X_n'(\xi) = \text{col}\left(e_n'(\xi), h_n'(\xi)\right) \tag{28}$$

where notation col means "column" here and henceforward. So,

$$\tfrac{d}{d\xi} X_n' + Q X_n' = -A_n' F_2 f(\xi) \quad X_n'(0) = \text{col}(0, 0) \tag{29}$$

where $F_2 = \text{col}(1, 0)$, coefficient Q is 2×2 matrix as

$$Q = \begin{pmatrix} 2\rho_0 \eta & \varpi_n' \\ -\varpi_n' & 0 \end{pmatrix} \tag{30}$$

and $\varpi_n' = \omega_n' T$, $\rho_0 = \gamma_0/\gamma_1$. Solving the problem (29) yields

$$X_n'(\xi) = -A_n' H(\xi) \int_0^{\xi} \left[e^{(x-\xi)Q} * F_2\right] f(x)\, dx \tag{31}$$

where $f(x) = f(\xi)_{\xi = x}$. Calculation of the vector convolution integral[‖] in formula (31) will be discussed in the next section in detail.

The problem (23) is the same Cauchy problem as (22). It is enough to replace the primes ($'$) in formula (31) with double primes ($''$) in order to obtain the solution like (31) to that problem (23).

5. CALCULATION OF A VECTOR CONVOLUTION INTEGRAL

There is a wide set of various methods for calculation of the exponential functions with a matrix argument [12]. We use here a method based

‖ Product $\left[e^{(x-\xi)Q} * F_2\right]$ of the 2×2 matrix exponential function $\exp\left[(x - \xi) Q\right]$ and the constant $2-$ component vector F_2 results in a $2-$ component vector function which depends on the variable $(x - \xi)$ at the integrand in formula (31).

on *Langrage interpolation.* Let A be a $m \times m$ constant matrix and F_m is m-component vector as $F_m = \mathrm{col}\,(1, 0, ..., 0)$. Then product $(*)$ of the matrix exponential and the vector F_m is a *finite* series as

$$\exp\left[(x-\xi)\,A\right] * F_m = \sum_{r=1}^{m} \mathcal{F}_r e^{(x-\xi)\lambda_r} \tag{32}$$

where λ_r's are the eigenvalues of the matrix A. They are solutions to the characteristic equation for the matrix A:

$$\det\left(\lambda U - A\right) = 0 \tag{33}$$

where U is the identity matrix of the same order m as the matrix A. The coefficients $\mathcal{F}_r$'s in (32) are $m-$ component *constant* vectors, which should be calculated as follows

$$\mathcal{F}_r = \left(\prod_{s=1,\dots,m}^{s\neq r} \frac{\lambda_s U - A}{\lambda_s - \lambda_r} \right) * F_m\,. \tag{34}$$

Formula (34) is available when all the eigenvalues λ_r's are *distinct* what is true in all our cases¶.

Thus, a vector convolution integral (like present in formula (31) above) can be calculated easily as

$$\int_0^{\xi} e^{(x-\xi)A} * F_m\, f(x)\, dx = \sum_{r=1}^{m} \mathcal{F}_r \Lambda_r(\xi) \tag{35}$$

where the scalar multipliers $\Lambda_r(\xi)$'s are elementary integrals:

$$\Lambda_r(\xi) = \int_0^{\xi} e^{(x-\xi)\lambda_r} f(x)\; dx\,. \tag{36}$$

6. EVOLUTION EQUATIONS IN PRESENCE OF A MEDIUM

Maxwell's equations (5) involve the polarization vector $\mathcal{P}$ which specifies a reaction of a medium on presence of the field inside the cavity. Vector $\mathcal{P}(\mathbf{r}, t)$ of "electric" kind can be presented as a modal expansion similarly to used for the field $\mathcal{E}(\mathbf{r}, t)$ in Eq. (20), i.e.,

$$\varepsilon_0^{-1}\mathcal{P}(\mathbf{r}, t) = \sum_{n=1}^{\infty} p'_n(t)\,\mathbf{E}'_n + \sum_{n=1}^{\infty} p''_n(t)\,\mathbf{E}''_n + \sum_{n=1}^{\infty} \mathfrak{a}_n(t)\,\nabla\varphi_n \tag{37}$$

¶ Otherwise, this method should be somewhat modified.

where $p'_n(t), p''_n(t), \mathfrak{a}_n(t)$ are unknown modal amplitudes.

The same procedure of projecting of Eq. (5) onto the elements of the modal basis yields a set of the Cauchy problems as

$$\begin{cases} \frac{d}{dt}e'_n+\frac{d}{dt}p'_n+2\gamma_0 e'_n+\omega'_n h'_n=-\frac{s(t)}{T}A'_n & e'_n(0)=0 \\ \frac{d}{dt}h'_n-\omega'_n e'_n=0 & h'_n(0)=0 \end{cases} \tag{38}$$

$$\begin{cases} \frac{d}{dt}e''_n+\frac{d}{dt}p''_n+2\gamma_0 e''_n+\omega''_n h''_n=-\frac{s(t)}{T}A''_n & e''_n(0)=0 \\ \frac{d}{dt}h''_n-\omega''_n e''_n=0 & h''_n(0)=0 \end{cases} \tag{39}$$

$$\left\{ \frac{d}{dt}a_n+\frac{d}{dt}\mathfrak{a}_n+2\gamma_0 a_n=-\frac{s(t)}{T}A^0_n, \quad a_n(0)=0 \right. \tag{40}$$

However, every system of the evolutionary equations in the set (38), (39) and (40) is *unclosed* now in the sense that number of unknown functions is *more* per one then number of the differential equations. In order to enclose every system of the equations, it is necessary to supplement everyone with a result of projecting of appropriate *equation of motion* for the polarization vector (37) and the initial condition onto the elements of the modal basis.

6.1. Waveforms in the Cavity with Debye Medium

Projection of the Debye equation onto the solenoidal basis elements yields appropriate evolutionary equations as

$$\frac{d}{dt}p'_n+\frac{1}{\tau_0}p'_n=\frac{\chi_s}{\tau_0}e'_n\, p'_n(0)=0, \tag{41}$$

$$\frac{d}{dt}p''_n+\frac{1}{\tau_0}p''_n=\frac{\chi_s}{\tau_0}e''_n\, p''_n(0)=0\,. \tag{42}$$

The same procedure with the elements $\nabla\varphi_n$ of the modal basis yields

$$\frac{d}{dt}\mathfrak{a}_n+\frac{1}{\tau_0}\mathfrak{a}_n=\frac{\chi_s}{\tau_0}a_n \quad \mathfrak{a}_n(0)=0\,. \tag{43}$$

The Irrotational Modes. Let's solve at first the problems (40) and (43) simultaneously. Introduce a 2-component vector $Y_n(\xi)$ (where $\xi = t/T$ as before) composed of the modal amplitudes of the *irrotational* modes $a_n(\xi)$ and $\mathfrak{a}_n(\xi)$ as

$$Y_n(\xi)=\mathrm{col}\left(a_n(\xi),\mathfrak{a}_n(\xi)\right)\,. \tag{44}$$

Joint the problems (40) and (43). It yields vector Cauchy problem as

$$\frac{d}{d\xi}Y_n+D_0Y_n=-\,A_n^0\,s\left(\xi\right)\quad Y_n\left(0\right)=\mathrm{col}\left(0,0\right)\,. \tag{45}$$

Herein, coefficient D_0 is the following 2×2 constant matrix:

$$D_0=\begin{pmatrix} 2\rho_0\eta+\chi_s\eta/\upsilon & -\eta/\upsilon \\ -\chi_s\eta/\upsilon & \eta/\upsilon \end{pmatrix} \tag{46}$$

where a new notation is introduced as $\upsilon=\gamma_1\tau_0$. Solution to the problem (45) is obtained as the vector convolution integral:

$$Y_n\left(\xi\right)=-A_n^0H\left(\xi\right)\int_0^\xi e^{(x-\xi)D_0}F_2f\left(x\right)\,dx, \tag{47}$$

which has been already studied above in *Section V.*

The Solenoidal Modes. Solve now the problems (38) and (41) simultaneously. It is naturally to introduce a 3-component vector composed of the modal amplitudes as follows

$$Y_n'\left(\xi\right)=\mathrm{col}\left(e_n'\left(\xi\right),h_n'\left(\xi\right),p_n'\left(\xi\right)\right)\,. \tag{48}$$

The problems (38) and (41), taken simultaneously, are equivalent to the following vector Cauchy problem:

$$\frac{d}{d\xi}Y_n'+D\,Y_n'=-A_n's\left(\xi\right)F_3\quad Y_n'\left(0\right)=\mathrm{col}\left(0,0,0\right) \tag{49}$$

where $F_3=\mathrm{col}\left(1,0,0\right)$. The constant matrix D is found as

$$D=\begin{pmatrix} 2\rho_0\eta+\chi_s\eta/\upsilon & \varpi_n' & -\eta/\upsilon \\ -\varpi_n' & 0 & 0 \\ -\chi_s\eta/\upsilon & 0 & \eta/\upsilon \end{pmatrix} \tag{50}$$

where $\rho_0=\gamma_0/\gamma_1,\upsilon=\gamma_1\tau_0$, $\varpi_n'=\omega_n'T$. Solution to the problem (49) is expressible via the following vector convolution integral:

$$Y_n'\left(\xi\right)=-A_n'H\left(\xi\right)\int_0^\xi e^{(x-\xi)D}f\left(x\right)\,dx \tag{51}$$

The modal amplitudes $e_n''\left(\xi\right)$, $h_n''\left(\xi\right)$ and $p_n''\left(\xi\right)$ can be found by formula (51) via replacing there primes ($'$) by double-primes ($''$).

6.2. The Cavity Filled with Lorentz Medium

We should operate with the same unclosed systems of differential equations (38), (39) and (40). In order to enclose them, it is necessary to supplement these systems with appropriate projections of the motion equation (8) and the initial conditions. The initial conditions are homogeneous. The differential equations are obtained as

$$\frac{d^2}{dt^2}p'_n+2\gamma\frac{d}{dt}p'_n+\frac{3\omega_r^2}{3+\chi_s}\,p'_n = \frac{3\chi_s}{3+\chi_s}\omega_r^2 e'_n \tag{52}$$

$$\frac{d^2}{dt^2}p''_n+2\gamma\frac{d}{dt}p''_n+\frac{3\omega_r^2}{3+\chi_s}\,p''_n = \frac{3\chi_s}{3+\chi_s}\omega_r^2 e''_n \tag{53}$$

$$\frac{d^2}{dt^2}\mathfrak{a}_n+2\gamma\frac{d}{dt}\mathfrak{a}_n+\frac{3\omega_r^2}{3+\chi_s}\,\mathfrak{a}_n = \frac{3\chi_s}{3+\chi_s}\omega_r^2 a_n \tag{54}$$

Factually, Eqs. (52)–(54) establish *dynamic* relations between the modal amplitudes of the polarization vector (37) and the modal amplitudes of the electric field (14). Herein, we used standard definition of *static* susceptibility $\chi_s=\omega_N^2/\omega_0^2$ what yields $\omega_0^2 = \omega_r^2\, 3/\left(3+\chi_s\right)$ and $\omega_N^2 = \chi_s\omega_r^2\, 3/\left(3+\chi_s\right)$.

The Irrotational Modes. Rearrange first differential equation of the second order (54) to equivalent pair of the first order equations:

$$\begin{aligned} \frac{d}{d\xi}\mathfrak{a}_n - \mathfrak{q}_n &= 0 \\ \frac{d}{d\xi}\mathfrak{q}_n-\frac{3\chi_s w^2}{3+\chi_s}a_n+\frac{3w^2}{3+\chi_s}\mathfrak{a}_n+2\eta\rho_a\,\mathfrak{q}_n &= 0 \end{aligned} \tag{55}$$

where $\mathfrak{q}_n = \frac{d}{d\xi}\mathfrak{a}_n$ and $w = \omega_r T$ is "atomic" dimensionless frequency, $\rho_a = \gamma/\gamma_1$ is atomic dimensionless lossy parameter. Introduce 3− component vector function $Z_n\left(\xi\right)$ as

$$Z_n\left(\xi\right) = \mathrm{col}\left(a_n\left(\xi\right), \mathfrak{a}_n\left(\xi\right), \mathfrak{q}_n\left(\xi\right)\right) \tag{56}$$

Join now Eq. (40) with Eq. (55) and write them as Cauchy problem:

$$\begin{aligned} \frac{d}{d\xi}Z_n + L_0 Z_n &= -s\left(\xi\right) A_n^0 F_3 \\ Z_n\left(0\right) &= \mathrm{col}\left(0,0,0\right) \end{aligned} \tag{57}$$

where L_0 and F_3 are constant matrix and vector, respectively:

$$\begin{aligned} L_0 &= \begin{pmatrix} 2\rho_0\eta & 0 & 1 \\ 0 & 0 & -1 \\ -\dfrac{3\chi_s w^2}{3+\chi_s} & \dfrac{3w^2}{3+\chi_s} & 2\rho_a\eta \end{pmatrix} \\ F_3 &= \text{col}\,(1,0,0) \end{aligned} \tag{58}$$

Solution to the problem (57) is also vector convolution integral as

$$Z_n\,(\xi) = -A_n^0 H\,(\xi) \int_0^\xi e^{(x-\xi)L_0} f\,(x)\;dx\,. \tag{59}$$

The Solenoidal Modes. Differential equation (52) (with appropriate initial conditions) should be solved simultaneously with the problem (38). This set of equations can be rearranged to a vector Cauchy problem for a 4-component vector function of the dimensionless time introduced as

$$Z'_n\,(\xi) = \text{col}\left(e'_n\,(\xi)\,, h'_n\,(\xi)\,, p'_n\,(\xi)\,, q'_n\,(\xi)\right) \tag{60}$$

where $q'_n = \frac{d}{d\xi} p'_n$. The Cauchy problem has standard forma as

$$\begin{aligned} \frac{d}{d\xi} Z'_n + L\,Z'_n &= -A'_n s\,(\xi)\,F_4 \\ Z'_n\,(0) &= \text{col}\,(0,0,0,0)\;. \end{aligned} \tag{61}$$

The constant matrix L and vector F_4 are found as

$$\begin{aligned} L &= \begin{pmatrix} 2\eta\rho_0 & \varpi'_n & 0 & w \\ -\varpi'_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -w \\ -\dfrac{3\chi_s w}{3+\chi_s} & 0 & \dfrac{3w}{3+\chi_s} & 2\eta\rho_a \end{pmatrix} \\ F_4 &= \text{col}\,(1,0,0,0)\,. \end{aligned} \tag{62}$$

Notation of all the parameters, which are present in the matrix elements, are the same as specified above. Solution to the problem (61) is the vector convolution integral as

$$Z'_n\,(\xi) = -A'_n H\,(\xi) \int_0^\xi e^{(x-\xi)L} f\,(x)\;dx\,. \tag{63}$$

In a similar way, one can introduce a vector

$$Z''_n\,(\xi) = \text{col}\left(e''_n\,(\xi)\,, h''_n\,(\xi)\,, p''_n\,(\xi)\,, q''_n\,(\xi)\right) \tag{64}$$

composed of the modal amplitudes which are present in the equations (39) and (53). These equations, taken jointly, compose Cauchy problem which coincides with (61) with accuracy to notation. Its solution is the same as in (63) if one put double prime ($''$) everywhere where the prime ($'$) stands.

6.3. Numerical Examples

For uniformity of presentation and convenience of comparison of the numerical results, let us take the constants A'_n, A''_n, and A^0_n, which stand in front of the convolution integrals, as equal to 1. Take for calculations dimensionless lossy parameter $\rho_0 = \gamma_0/\gamma_1 = 0.1$, where $\gamma_0 = \sigma/(2\varepsilon_0)$ and see formula (2) for γ_1. Parameter $\varpi'_n \equiv \omega_{pqs} T = 4\pi$. In Fig. 2–Fig. 7, calculations are performed for the value of parameter $\rho = 1.001$; remind that $\rho = \gamma_2/\gamma_1$. In Fig. 8–Fig. 12, the result of calculations are exhibited for $\rho = 1000$ provided that all other parameters are saved.

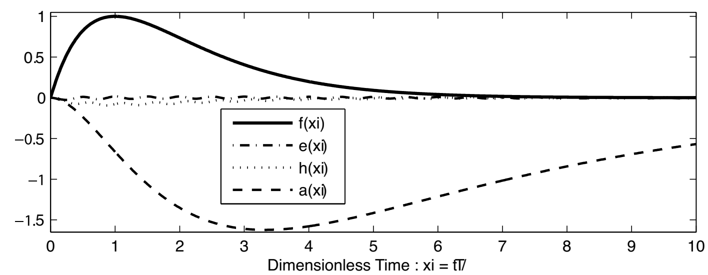

Figure 2. The surge function $f(xi)$ and amplitudes of the irrotational $a(xi)$ and solenoidal modes $e(xi), h(xi)$ for hollow cavity.

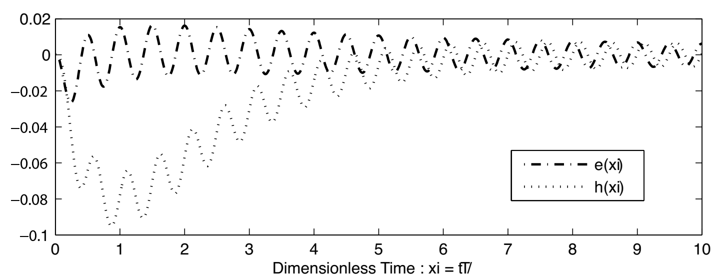

Figure 3. Scaled amplitudes $e(xi)$ and $h(xi)$ of the solenoidal mode taken from Fig. 2.

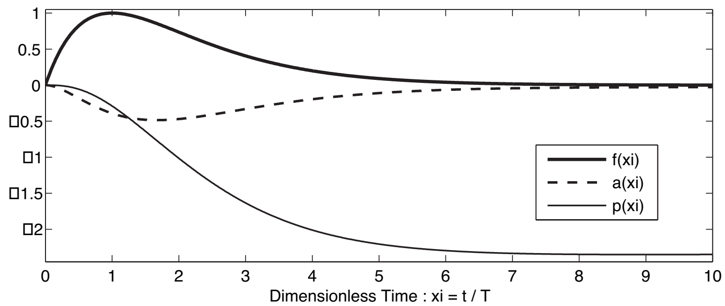

Figure 4. The surge function $f(xi)$, amplitude of the field irrotational mode $a(xi)$ and the amplitude $p(xi)$ of the irrotational mode of polarization vector for a cavity filled with Debye medium.

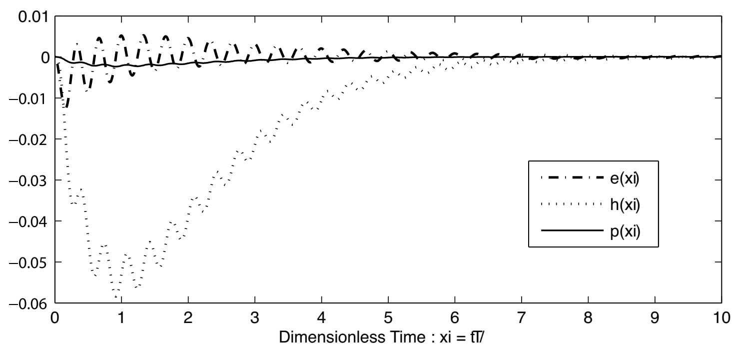

Figure 5. The amplitudes of the solenoidal modes $e(xi), h(xi)$ and $p(xi)$ for the cavity filled with Debye medium.

In Fig. 2 and Fig. 3, the surge function $f(\xi)$, amplitude of the irrotational mode $a_n(\xi)$, and amplitudes $e_n(\xi)$, $h_n(\xi)$ of the solenoidal mode are exhibited for the *hollow* cavity. In Fig. 2, amplitudes $e_n(\xi)$, $h_n(\xi)$ fluctuate in vicinity of *zero.*

In Fig. 3, amplitudes $e_n(\xi)$, $h_n(\xi)$ are shown with scaling. As will readily be observed, magnitude of the *irrotational* mode essentially prevails over the magnitudes of the *solenoidal* modes.

In Fig. 4, numerical results of calculation for the cavity, which is filled with *Debye* medium with dimensionless relaxation time $\upsilon=\gamma_1\tau_0=50$ are presented. The surge function $f(\xi)$ is exhibited for comparison with the amplitudes of the *irrotational* modes of electric field $a_n(\xi)$ and polarization vector $p_n(\xi)$.

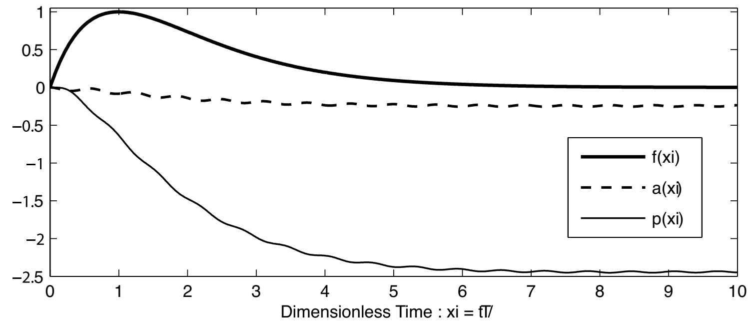

Figure 6. The surge function $f(xi)$, the amplitude $a(xi)$ of the irrotational mode for electric field and the irrotational modal amplitude $p(xi)$ of the polarization vector for a cavity filled with Lorentz medium.

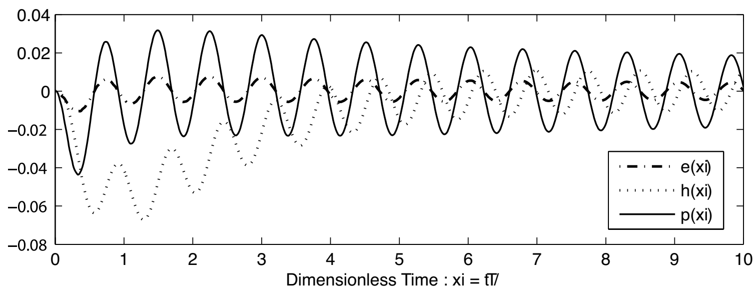

Figure 7. Scaled amplitudes $e(xi)$ and $h(xi)$ taken from Fig. 6 with zoom.

In Fig. 5, the amplitudes of the *solenoidal* modes of electromagnetic field $e_n(\xi)$, $h_n(\xi)$ and of polarization vector $p_n(\xi)$ are shown for the same cavity, which is filled with the same *Debye* medium.

In Fig. 6 and Fig. 7, the results of calculations are presented for the cavity filled with *Lorentzian* medium, which has the following parameters: $w=\omega_r T=2\pi$, $\chi_s=10$.

In Fig. 8–Fig. 12, the results are presented for the case when the surge is specified by parameter $\rho=1000$. All other parameters needed have the same numerical values, which were used in Fig. 2–Fig. 7.

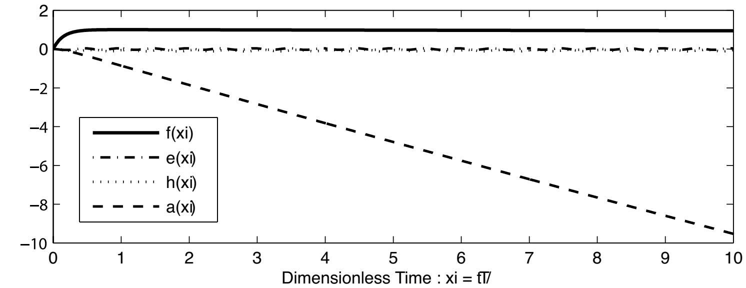

Figure 8. The surge function $f(xi)$ for $rho = 1000$ and amplitudes of the irrotational $a(xi)$ and solenoidal modes $e(xi), h(xi)$ for hollow cavity.

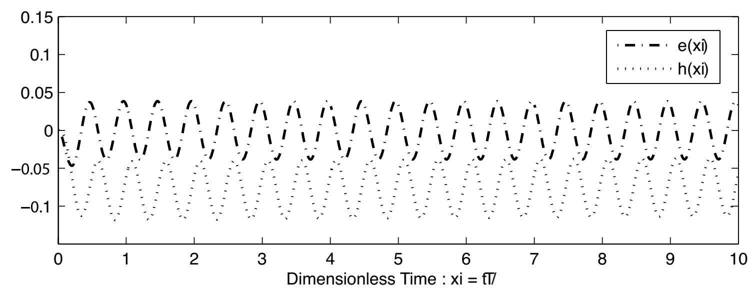

Figure 9. Scaled amplitudes $e(xi)$ and $h(xi)$ of the solenoidal mode taken from Fig. 8.

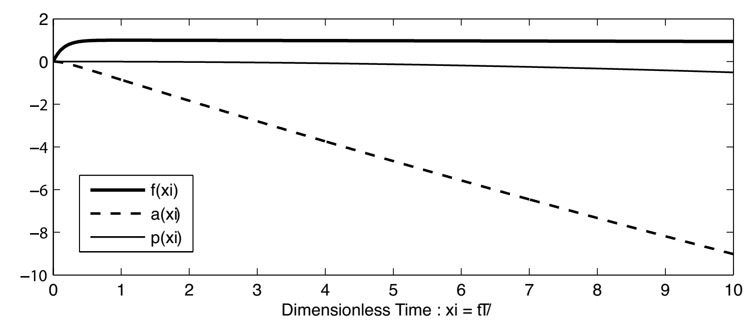

Figure 10. The surge function $f(xi)$ for $rho = 1000$, amplitude of the field irrotational mode $a(xi)$ and the amplitude $p(xi)$ of the irrotational mode of polarization vector for a cavity filled with Debye medium.

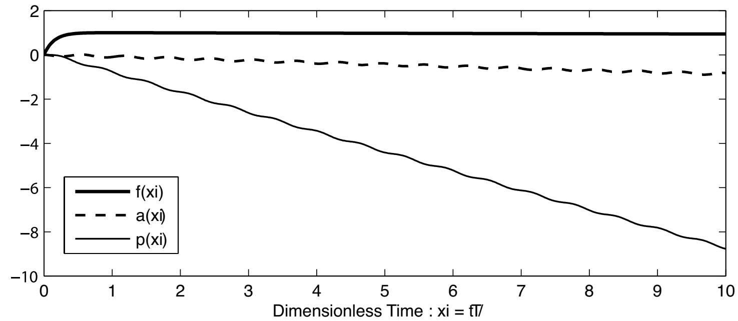

Figure 11. The surge function $f(xi)$, amplitude of the field irrotational mode $a(xi)$ and the irrotational modal amplitude $p(xi)$ of the polarization vector for a cavity filled with Lorentz medium.

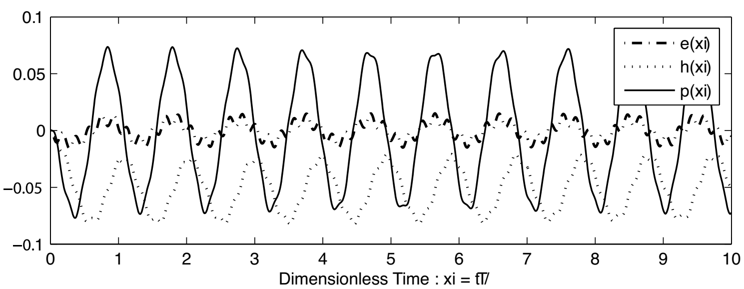

Figure 12. Scaled amplitudes of the solenoidal modes $e(xi)$ and $h(xi)$ for a cavity filled with Lorentz medium.

7. DISCUSSION

1. Classical time-harmonic field concept always interprets the irrotational mode sets as some *static* fields. The evolutionary approach (EAE) reveals situations when the amplitudes of the irrotational modes can be time-variant. And what is more, a surge can excite the irrotational modes which prevail over the the solenoidal mode amplitudes *per orders* in magnitude.

2. Solution to the problem of cavity excitation by the surge has been obtained explicitly, in quadratures. These quadrature formulas are some simple convolution integrals where the surge is present at the integrand. In this article, we used the double-exponential function as a surge model. However, one can substitute at the integrand any other surge function: e.g., a surge detected *experimentally* as a real function of time.

3. Two cases has been analyzed specially when the cavity is filled (i) by a polar dielectric and (ii) by a Lorentz medium. Every real surge has a wide-band frequency spectrum, generally speaking. Therefore, one should use in the time domain appropriate *dynamic* constitutive relation between the polarization vector and the electric field which entails the polarization phenomenon. In the case of a polar dielectric, the system of Maxwell's equations have been solved *simultaneously* with Debye equation for the polarization macroscopic vector. In the case of a Lorentz dielectric, the system of Maxwell's equations have been supplemented and solved *simultaneously* with Newton motion equation for the polarization vector.

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