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SCATTERING PROPERTIES OF THE STRIP WITH FRACTIONAL BOUNDARY CONDITIONS AND COMPARISON WITH THE IMPEDANCE STRIP

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Abstract—In this paper two-dimensional problem of plane-wave diffraction by a "fractional strip" is studied. "Fractional strip" is introduced as a strip with fractional boundary conditions (FBC) involving fractional derivatives of the field components. FBC describe intermediate boundary between perfect electric conductor (PEC) and perfect magnetic conductor (PMC). It is shown that "fractional strip" has scattering properties similar to the well-known impedance strip. For one important case of fractional order equal to 0.5 the solution of the wave diffraction problem by a "fractional strip" can be found analytically. Detailed comparison analysis of the physical characteristics of the scattered fields for both fractional and impedance strips is presented. The relation between the fractional order and the value of impedance is derived. It is shown that in a wide range of input parameters the physical characteristics of the "fractional strip" are similar to the strip with pure imaginary impedance.

1. FORMULATION OF THE PROBLEM

Consider a two-dimensional problem of electromagnetic wave diffraction by a strip located at the plane $y = 0$ and infinite along the axis z. The width of the strip is $2a$. In this paper E-polarization case is discussed. An incident plane wave is described by the function $\vec{E}^i(x, y) = \vec{z}E^i(x, y) = \vec{z}e^{-\imath k(x\alpha_0 + y\sqrt{1-\alpha_0^2})}$ where $\alpha_0 = cos\theta_0$, θ_0 is the incidence angle and $k = \frac{2\pi}{\lambda}$ is the wave number. Time dependence is assumed to be $e^{-\imath\omega t}$ and deprecated throughout the paper.

Boundary conditions are fractional boundary conditions (FBC) defined as application of fractional derivative to the electric field component [1]

$$D_{ky}^{\nu} E_z(x,y)|_{y\to\pm 0} = 0, \quad -a < x < a \tag{1}$$

For convenience the fractional differentiation is applied in respect to a dimensionless variable ky. Operator $D_y^{\nu} f(y)$ in (1) is defined by the integral of Riemann-Liouville [2]

$$D_y^{\nu} f(y) \equiv \ {}_{-\infty}D_y^{\nu} f(y) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dy} \int_{-\infty}^{y} \frac{f(t)}{(y-t)^{\nu}} dt, \tag{2}$$

The fractional order ν varies from 0 to 1, and $\Gamma(\nu)$ is the Gamma function. Function $E_z(x,y)$ in Equation (1) is the z-component of the total electric field $\vec{E} = \vec{E}^i + \vec{E}^{1,s}$ — a sum of the incident plane wave $\vec{E}^i(x,y)$ and the scattered field $\vec{E}^{1,s} = \vec{z}E^{1,s}$. For the value $\nu = 0$ the strip with FBC (1) corresponds to a perfectly electric conducting (PEC) strip, and for $\nu = 1$ we get a perfectly magnetic conducting (PMC) strip [3]. For intermediate values $0 < \nu < 1$ FBC describe fractional boundary with specific properties studied in this paper. FBC yield to utilization of fractional Green's function (FGF) G^{ν} [4] and the fractional Green's theorem [1, 5, 6]. In this case the scattered field can be presented as [1]

$$E_z^{1,s}(x,y) := \int_{-\infty}^{\infty} f^{1-\nu}(x') G^{\nu}(x-x',y) dx' \tag{3}$$

where $f^{1-\nu}(x')$ is an unknown function which we name "fractional potential density". The presentation (3) is a result of application of the fractional Green's theorem which generalizes the classic Green's theorem. FGF G^{ν} is defined in two-dimensional case as [4–6]

$$\begin{aligned} G^{\nu}(x-x',y) &= -\frac{\imath}{4} D_{ky}^{\nu} H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right) \\ &= -\imath \frac{e^{sign(y)\imath\pi\nu/2}}{4\pi} \int_{-\infty}^{\infty} e^{\imath k((x-x')\alpha+|y|\sqrt{1-\alpha^2})} (1-\alpha^2)^{(\nu-1)/2} d\alpha, \end{aligned} \tag{4}$$

Here $H_0^{(1)}(x)$ is the Hankel function of the first kind of zeroth order.

The second diffraction problem we consider in this paper is a diffraction of the E-polarized plane wave on a strip defined by the impedance boundary conditions (IBC) [3]:

$$\frac{\partial}{\partial y}E_z \pm \frac{\imath k}{\eta}E_z = 0, \quad \text{for} \quad y \to \pm 0, \ -a < x < a, \tag{5}$$

where η is the impedance of the strip normalized by the impedance of the free space. For value $\eta = 0$ we get a PEC strip, and for $\eta = -\imath\infty$ it is a PMC strip. Commonly used technique is to search the scattered field as a sum of two potentials: single layer potential and double layer potential, i.e., [3, 8, 9]

$$E_z^{2,s}(x,y) := -\frac{\imath}{4}\int_{-\infty}^{\infty}\left[f_e(x') + f_m(x')\frac{\partial}{\partial y}\right]G(x-x',y)dx' \tag{6}$$

where $G(x-x',y)$ is the Green's function of the free space defined the two-dimensional case as

$$G(x-x',y) := H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right)$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{\imath k((x-x')\alpha + |y|\sqrt{1-\alpha^2})}\frac{1}{\sqrt{1-\alpha^2}}d\alpha \tag{7}$$

Two unknown densities $f_e(x')$, $f_m(x')$ in presentation (6) have certain physical meaning - they correspond to the densities of surface electric and magnetic currents, respectively. It should be noted, that presentation (6) is not the only possible way to search a scattered field. It is possible to utilize one single layer potential or double-layer potential only [9]. Each presentation results in the different integral equations (IE) to evaluate unknown potential densities and different methods to solve the IEs.

2. SOLUTION OF THE DIFFRACTION PROBLEMS

Following the method described in the works [1, 9, 11, 12] we present the scattered field $E_z^{1,s}(x,y)$ (3) via the Fourier transform (FT) $F^{1-\nu}(\alpha)$ of the fractional potential density $f^{1-\nu}(x)$:

$$E_z^{1,s}(x,y) = -\imath\frac{e^{\pm\imath\pi\nu/2}}{4\pi}\int_{-\infty}^{\infty}F^{1-\nu}(\alpha)e^{\imath k(x\alpha \pm y\sqrt{1-\alpha^2})}(1-\alpha^2)^{(\nu-1)/2}d\alpha, \tag{8}$$

$$F^{1-\nu}(\alpha) = \int_{-1}^{1}\tilde{f}^{1-\nu}(\xi)e^{-\imath\epsilon\alpha\xi}d\xi, \ \tilde{f}^{1-\nu}(\xi) \equiv af^{1-\nu}(a\xi) \tag{9}$$

where $\epsilon = ka$, $\xi = x/a$.

Satisfying the function $E_z(x,y)$ FBC (1) and taking into account (8) we get IE in respect to the function $F^{1-\nu}(\alpha)$ [1]:

$$\frac{1}{\epsilon}\int_{-\infty}^{\infty}F^{1-\nu}(\alpha)\frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{\nu-1/2}d\alpha$$
$$= -4\pi e^{\imath\pi/2(1-\nu)}(1-\alpha_0^2)^{\nu/2}\frac{\sin\epsilon(\beta+\alpha_0)}{\epsilon(\beta+\alpha_0)},\ \text{for } -\infty<\beta<\infty \quad (10)$$

IE for impedance strip case in respect to functions $F_e(\alpha)$ and $F_m(\alpha)$, which are the Fourier transforms of $f_e(x)$ and $f_m(x)$, respectively, can be obtained in the following form [9]:

$$\begin{aligned}
-\frac{\eta}{\epsilon}F_e(\beta) &= 4\imath\frac{\sin\epsilon(\beta+\alpha_0)}{\epsilon(\beta+\alpha_0)} \\
&\quad+\frac{1}{\epsilon\pi}\int_{-\infty}^{\infty}F_e(\alpha)\frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{-1/2}d\alpha \\
\frac{1}{\eta}F_m(\beta) &= 4\sqrt{1-\alpha_0^2}\frac{\sin\epsilon(\beta+\alpha_0)}{\epsilon(\beta+\alpha_0)} \\
&\quad-\frac{1}{\pi}\int_{-\infty}^{\infty}F_m(\alpha)\frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{1/2}d\alpha
\end{aligned} \quad (11)$$

where

$$\begin{aligned}
F_e(\alpha) &= \int_{-1}^{1}\tilde{f}_e(\xi)e^{-\imath\epsilon\alpha\xi}d\xi,\quad \tilde{f}_e(\xi)\equiv af_e(a\xi) \\
F_m(\alpha) &= \int_{-1}^{1}\tilde{f}_m(\xi)e^{-\imath\epsilon\alpha\xi}d\xi,\ \tilde{f}_m(\xi)\equiv f_m(a\xi)
\end{aligned} \quad (12)$$

For the impedance boundary we have to solve two independent IE (11) in spite of one IE (10) for the "fractional strip". As one can see the kernel of the "fractional" IE $\frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{\nu-1/2}$ generalizes kernels of the IE for the impedance boundary: for $\nu=0$ and $\nu=1$ the kernel in (10) equals to $\frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{-1/2}$ and $\frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{1/2}$, respectively. In order to solve considered IE we follow the method presented in the works [9, 11, 12] and represent the density function by a uniformly convergent series

$$\tilde{f}^{1-\nu}(\xi)=(1-\xi^2)^{\nu-1/2}\sum_{n=0}^{\infty}f_n^{\nu}\frac{1}{\nu}C_n^{\nu}(\xi), \quad (13)$$

where $C_n^\nu(\xi)$ denotes Gegenbauer polynomials. This presentation allows to satisfy the edge condition [7] in the following form:

$$\tilde{f}^{1-\nu}(\xi) = O\left((1-\xi^2)^{\nu-1/2}\right), \quad \xi \to \pm 1 \tag{14}$$

Using expressions (13) FT $F^{1-\nu}(\alpha)$ can be presented as [1]

$$F^{1-\nu}(\alpha) = \frac{2\pi}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} (-\imath)^n f_n^\nu \beta_n^\nu \frac{J_{n+\nu}(\epsilon\alpha)}{(2\epsilon\alpha)^\nu} \tag{15}$$

where $\beta_n^\nu = \Gamma(n+2\nu)/\Gamma(n+1)$.

Substituting (15) into IE (10) a system of linear algebraic equations (SLAE) for the definition of unknown coefficients f_n^ν is obtained [1]:

$$\sum_{n=0}^{\infty} (-\imath)^n f_n^\nu \beta_n^\nu C_{kn}^\nu = \gamma_k^\nu, \quad k = 0, 1, 2, \ldots \tag{16}$$

with matrix elements

$$\begin{aligned} C_{kn}^\nu &= \int_{-\infty}^{\infty} J_{n+\nu}(\epsilon\alpha) J_{k+\nu}(\epsilon\alpha)(1-\alpha^2)^{\nu-1/2} \frac{d\alpha}{\alpha^{2\nu}}, \\ \gamma_k^\nu &= -2\Gamma(\nu+1)(2\epsilon)^\nu \imath^{1-\nu}(1-\alpha_0^2)^{\nu/2} \frac{J_{k+\nu}(\epsilon\alpha_0)}{\epsilon\alpha_0} \end{aligned} \tag{17}$$

Since the coefficients f_n^ν are found the potential density function $\tilde{f}^{1-\nu}(\xi)$ and FT $F^{1-\nu}(\alpha)$ can be obtained from Equations (13) and (15), respectively.

For the impedance strip the densities $\tilde{f}_e(\xi)$ and $\tilde{f}_m(\xi)$ are expressed by the series [9]

$$\begin{aligned} \tilde{f}_e(\xi) &= (1-\xi^2)^{-1/2} \sum_{n=0}^{\infty} f_n^{(e)} C_n^0(\xi) \\ &= (1-\xi^2)^{-1/2} \left(f_0^{(e)} + 2\sum_{n=1}^{\infty} \frac{f_n^{(e)}}{n} T_n(\xi) \right) \\ \tilde{f}_m(\xi) &= (1-\xi^2)^{1/2} \sum_{n=0}^{\infty} f_n^{(m)} C_n^1(\xi) \\ &= (1-\xi^2)^{-1/2} \sum_{n=0}^{\infty} f_n^{(m)} U_n(\xi) \end{aligned} \tag{18}$$

where $T_n(\xi)$, $U_n(\xi)$ are Chebyshev polynomials of the first and second kind, respectively. In that case the edge conditions are formulated in the form

$$\tilde{f}_e(\xi) = O\left((1-\xi^2)^{-1/2}\right), \; \tilde{f}_m(\xi) = O\left((1-\xi^2)^{1/2}\right), \quad \xi \to \pm 1 \quad (19)$$

FT $F_e(\alpha)$ and $F_m(\alpha)$ can be expressed via the series

$$\begin{aligned} F_e(\alpha) &= \pi f_0^{(e)} J_0(\epsilon\xi) + 2\pi \sum_{n=1}^{\infty} (-\imath)^n f_n^{(e)} \frac{1}{n} J_n(\epsilon\alpha) \\ F_m(\alpha) &= \pi \sum_{n=0}^{\infty} (-\imath)^n (n+1) f_n^{(m)} \frac{J_{n+1}(\epsilon\alpha)}{\epsilon\alpha} \end{aligned} \quad (20)$$

Finally SLAE for the definition of the unknown coefficients $f_n^{(e)}$, $f_n^{(m)}$ is obtained [9]

$$\begin{aligned} &\sum_{n=0}^{\infty} X_n \left(\eta d_{kn}^{E1} + D_{kn}^{E1}\right) = -\gamma_k^{E1}, \quad k = 0, 1, 2, \ldots \\ &\sum_{n=0}^{\infty} Y_n \left(\frac{1}{\eta} d_{kn}^{E2} + D_{kn}^{E2}\right) = \gamma_k^{E2}, \quad k = 0, 1, 2, \ldots \end{aligned} \quad (21)$$

where

$$\begin{aligned} X_n &= \begin{cases} f_0^{(e)}, & n = 0, \\ 2(-i)^n \frac{1}{n} f_n^{(e)}, & n > 0 \end{cases} \\ Y_n &= (-i)^n (n+1) f_n^{(m)} \end{aligned} \quad (22)$$

And the matrix coefficients are defined as

$$\begin{aligned} d_{kn}^{E1} &= \int_{-\infty}^{\infty} J_k(\epsilon\alpha) J_n(\epsilon\alpha) d\alpha, \\ \gamma_k^{E1} &= 4\imath(-1)^k \frac{J_k(\epsilon\alpha_0)}{\alpha_0} \\ d_{kn}^{E2} &= \int_{-\infty}^{\infty} J_{k+1}(\epsilon\alpha) J_{n+1}(\epsilon\alpha) \frac{d\alpha}{\alpha^2}, \\ \gamma_k^{E2} &= -\frac{4(-1)^k \sqrt{1-\alpha_0^2}}{\alpha_0} J_{k+1}(\epsilon\alpha_0) \\ D_{kn}^{E1} &= \int_{-\infty}^{\infty} J_k(\epsilon\alpha) J_n(\epsilon\alpha) \frac{d\alpha}{(1-\alpha^2)^{1/2}} \end{aligned} \quad (23)$$

$$D_{kn}^{E2} = \int_{-\infty}^{\infty} J_{k+1}(\epsilon\alpha) J_{n+1}(\epsilon\alpha) \frac{(1-\alpha^2)^{1/2} d\alpha}{\alpha^2}$$

As a result of solving SLAE (21) the coefficients $f_n^{(e)}$, $f_n^{(m)}$ are found and the current density functions are found directly from (18). But for the fractional boundary we have different situation with the current densities. Electric and magnetic current densities for the fractional boundary defined as

$$\begin{aligned} j_z^{\nu(e)} &= -(H_x(x,+0) - H_x(x,-0)), \\ j_x^{\nu(m)} &= -(E_z(x,+0) - E_z(x,-0)) \end{aligned} \tag{24}$$

can be obtained after the additional integration of the FT $F^{1-\nu}(\alpha)$

$$\begin{aligned} j_z^{\nu(e)} &= -2\imath \cos(\pi\nu/2) \frac{\imath}{4\pi} \int_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{\imath k\alpha x} (1-\alpha^2)^{\nu/2} d\alpha, \\ j_x^{\nu(m)} &= -2 \sin(\pi\nu/2) \frac{\imath}{4\pi} \int_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{\imath k\alpha x} (1-\alpha^2)^{\nu/2-1/2} d\alpha \end{aligned} \tag{25}$$

Note that the currents $j_x^{\nu(m)} = 0$ for $\nu = 0$ and $j_z^{\nu(e)} = 0$ for $\nu = 1$ that agrees with the properties of FBC mentioned before. In the general case $0 < \nu < 1$ the E-polarized plane wave incident on a "fractional strip" excites two surface currents — electric and magnetic, magnetic current directs along the axis z and electric current is along the axis x. Similar current distributions are observed in the diffraction of the E-polarized plane wave on an impedance strip. However, the solution of the impedance problem is presented via two functions $f_e(x)$ and $f_m(x)$ while the solution of the fractional problem is expressed through one function $f^{1-\nu}(x)$. As a result the fractional problem requires less computational efforts to solve IE and SLAE than the impedance problem. From the other hand for the impedance problem the unknown functions $f_e(x)$ and $f_m(x)$ are the physical surface current densities, but for the fractional problem we have to apply an additional integration (25) to $f^{1-\nu}(x)$ to find the physical currents. For the special cases of $\nu = 0$ and $\nu = 1$ the expressions (25) become simpler and the function $f^{1-\nu}(x)$ describes electric and magnetic current density, respectively.

Analyzing IE (10) for "fractional strip" it is seen that for the special case of $\nu = 0.5$ the kernel becomes $\frac{\sin\epsilon(\alpha-\beta)}{\alpha-\beta}$ and the IE can be solved analytically for any value of $\epsilon = ka$:

$$\tilde{f}^{0.5}(\xi) = -2\imath\epsilon(1-\alpha_0^2)^{1/4} e^{-\imath\epsilon\alpha_0\xi + \imath\pi/4}, \tag{26}$$

$$F^{0.5}(\alpha) = -4\imath(1-\alpha_0^2)^{1/4} e^{\imath\pi/4} \frac{\sin\epsilon(\alpha+\alpha_0)}{\alpha+\alpha_0} \tag{27}$$

3. RELATION BETWEEN FRACTIONAL AND IMPEDANCE STRIP

In this section we compare FBC (1) and IBC (5). Consider plane wave reflection from the infinite fractional and impedance boundaries. Having fractional boundary with certain value of the fractional order ν we can find an equivalent impedance boundary with $\eta = \eta(\nu)$ as a function of ν and input parameters of the problem. Equivalence of the fractional and impedance boundaries is treated in such sense that the reflected plane waves from both boundaries are equal. Indeed for the incident E-polarized plane wave $\vec{E}^i(x,y) = \vec{z}e^{-\imath k(x\cos\theta_0 + y\sin\theta_0)}$ comparing the reflected fields for the fractional and impedance boundaries the following relation between the impedance and the fractional order is obtained [1, 5, 13, 14]:

$$\nu = \frac{1}{\imath\pi} ln \frac{1-\eta\sin\theta_0}{1+\eta\sin\theta_0}, \quad \eta = -\imath\frac{1}{\sin\theta_0}\tan(\frac{\pi\nu}{2}) \tag{28}$$

The value $\nu = 0$ corresponds to the impedance $\eta = 0$ (PEC) and $\nu = 1$ corresponds to $\eta = -\imath\infty$ (PMC). For the intermediate values $0 < \nu < 1$ the impedance has pure imaginary values between 0 and $-\imath\infty$. Both boundaries support electric and magnetic surface currents which are perpendicular to each other. It is known that for the impedance boundary the value of the impedance is defined as the ratio of components of the currents

$$\eta = -\frac{j_x^{(m)}}{j_z^{(e)}} \tag{29}$$

or equivalently as ratio of the field components

$$\eta = \frac{E_z}{H_x}\Big|_{y\to\pm 0} \tag{30}$$

We introduce similar ratio for the fractional solution:

$$\zeta(x) := \frac{E_z(x)}{H_x(x)} = \frac{j_x^{\nu(m)}}{j_z^{\nu(e)}} = \imath\tan\left(\frac{\pi\nu}{2}\right)\frac{A_\nu(x)}{B_\nu(x)}, \quad x\in(-a,a) \tag{31}$$

where

$$\begin{aligned} A_\nu(x) &= \int_{-\infty}^{\infty} F^{1-\nu}(\alpha)e^{\imath k\alpha x}(1-\alpha^2)^{(\nu-1)/2}d\alpha, \\ B_\nu(x) &= \int_{-\infty}^{\infty} F^{1-\nu}(\alpha)e^{\imath k\alpha x}(1-\alpha^2)^{\nu/2}d\alpha, \end{aligned} \tag{32}$$

The ratio (31) may depend on the coordinate x $(-a < x < a)$ while the ratio (29) for impedance is a constant η by definition. However, for one special value $\nu = 0.5$ the function $\zeta(x)$ is a constant for all values of ϵ

$$\zeta(x)|_{\nu=0.5} = \frac{1}{\imath \sin\theta_0} \tag{33}$$

In general for diffraction problems on a strip the ratio $\zeta(x)$ can be evaluated numerically. For the physical optics (PO) approximation $(\epsilon \to \infty)$ we can use asymptotic formulas [1, 9] for the integrals

$$F^{1-\nu}(\beta) \approx -4\imath^{\nu}\frac{(1-\alpha_0^2)^{(1-\nu)/2}}{(1-\beta^2)^{(1-2\nu)/2}}\frac{\sin\epsilon(\beta-\alpha_0)}{\beta-\alpha_0},\ \epsilon \to \infty \tag{34}$$

$$A_{\nu}(x) \approx -4\pi\imath^{\nu+1}e^{-\imath k\alpha_0 x}, B_{\nu}(x) \approx -4\pi\imath^{\nu+1}(1-\alpha_0^2)^{1/2}e^{-\imath k\alpha_0 x}, \epsilon \to \infty \tag{35}$$

In this case the ratio $\zeta(x)$ is expressed analytically (31)

$$\zeta(x) \approx \frac{1}{\imath \sin\theta_0}\tan\left(\frac{\pi\nu}{2}\right), \quad \epsilon \to \infty \tag{36}$$

For finite boundaries in the case of PO approximation we have exactly the same relation between the fractional order and the impedance (28), (36) as for infinite boundaries. For other values of ϵ the expressions for currents will be found solving SLAE using reduction method [1].

As mentioned earlier for the special case $\nu = 0.5$ IE (10) can be solved analytically (26), (27) for all values of ϵ. However, IE for impedance strip for the value $\eta = -\frac{\imath}{\sin\theta_0}$ corresponding to $\nu = 0.5$ does not allow similar analytical solution.

For all values of ϵ the impedance for the fractional strip can be introduced as the ratio (31) which can be obtained numerically by solving the diffraction problem. The closer $\zeta(x)$ to constant (28) for $-a < x < a$ the more the fractional boundary behaves as impedance boundary.

4. NUMERICAL RESULTS

In this section we compare the solutions of diffraction problems by the "fractional strip" and the impedance strip by analyzing physical characteristics of the scattered fields. We focus on such physical characteristics as radiation pattern (RP), monostatic and bistatic radar cross sections (RCS), and densities of the surface currents.

Using the method of the stationary phase in the far-zone $kr \to \infty$ the scattered field $E_z^s(x, y)$ can be presented in the cylindrical coordinate system (r, ϕ), $x = rcos\phi, y = rsin\phi$ as [1, 9]

$$E_z^s(r,\phi) \approx A(kr)\Phi(\phi), \quad kr \to \infty \tag{37}$$

where

$$A(kr) = \sqrt{\frac{2}{\pi kr}} e^{\imath kr - \imath\pi/4} \tag{38}$$

The function $\Phi(\phi)$ denotes the RP of the scattered field. For the "fractional strip" RP is expressed as [1]

$$\begin{aligned}\Phi^\nu(\phi) &= -\frac{\imath}{4} e^{\pm\imath\nu\pi/2} F^{1-\nu}(\cos\phi)\sin^\nu\phi \\ &= \frac{\imath\pi e^{\pm\imath\nu\pi/2}}{2\Gamma(\nu+1)} \tan^\nu\phi \sum_{n=0}^{\infty} (-\imath)^\nu f_n^\nu \beta_n^\nu \frac{J_{n+\nu}(\epsilon\cos\phi)}{(2\epsilon)^\nu}\end{aligned} \tag{39}$$

where the upper sign is chosen for the values $\phi \in [0, \pi]$, and the bottom sign when $\phi \in [\pi, 2\pi]$. And for the impedance strip the RP is [9]:

$$\begin{aligned}&\Phi^{imp}(\phi) = \Phi_e^{imp}(\phi) + \Phi_m^{imp}(\phi) \\ &\Phi_e^{imp}(\phi) = \frac{1}{4} F_e(\cos\phi) = \frac{\pi}{4} \sum_{n=0}^{\infty} X_n J_n(\epsilon\cos\phi) \\ &\Phi_m^{imp}(\phi) = \frac{1}{4}\epsilon\sin\phi F_m(\cos\phi) = \frac{\pi}{4}\epsilon\sin\phi \sum_{n=0}^{\infty} Y_n \frac{J_n(\epsilon\cos\phi)}{\epsilon\cos\phi}\end{aligned} \tag{40}$$

For the special case $\nu = 0.5$ and all values of $\epsilon = ka$ we have exact analytical expression

$$\Phi^\nu(\phi) = e^{\mp ipi3/4} e^{ipi/4} \sqrt{\sin\phi\sin\theta_0} \, \frac{\sin\epsilon(\cos\phi + \cos\theta_0)}{\cos\phi + \cos\theta_0} \tag{41}$$

Bistatic RCS $\frac{\sigma_{2d}}{\lambda}$ and monostatic RCS σ_{2d} $(monostatic)$ are defined via RP $\Phi(\phi)$:

$$\frac{\sigma_{2d}}{\lambda}(\phi) = \frac{2}{\pi}|\Phi(\phi)|^2, \quad \sigma_{2d}(monostatic) = \frac{\sigma_{2d}}{\lambda}(\theta_0) \tag{42}$$

We used the reduction method to solve the SLAEs for both the "fractional strip" and impedance strip and calculated the values of the coefficients f_n^ν, $f_n^{(e)}$, $f_n^{(m)}$. Then RP, monostatic RCS, bistatic RCS and the fractional potential density $\tilde{f}^{1-\nu}(\xi)$ are evaluated.

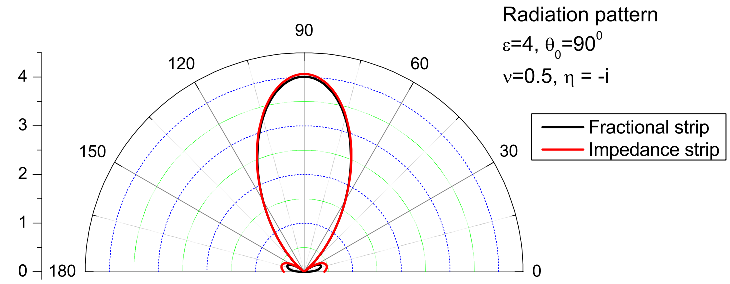

Figure 1. Radiation pattern for the "fractional" and impedance strips for frequency parameter $\epsilon = 4$ and the incidence angle $\theta_0 = 90°$.

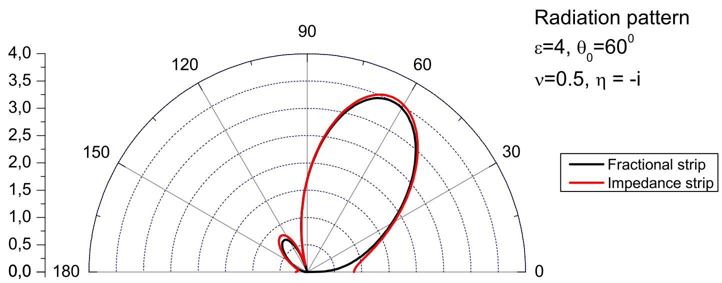

Figure 2. Radiation pattern for the frequency parameter $\epsilon = 4$ and the incidence angle $\theta_0 = 60°$.

Figures 1 and 2 show RP for the "fractional strip" and impedance strip for the frequency parameter $\epsilon = 4$ and the incidence angles $\theta_0 = 90°$ and $\theta_0 = 60°$, respectively. On Figures 1 and 2 the fractional parameter is chosen to be $\nu = 0.5$ and the impedance obtained from the formula (28) equals to $\eta = -\imath$. For the wide range of observation angle ϕ it shows good agreement.

Graphics 1 and 2 for RP show comparison of solutions for one fixed value of the incidence angle, while graphics for monostatic RCS are plotted for all range of the incidence angle from 0 to 90 degrees.

Monostatic RCSs for the "fractional strip" for $\nu = 0.5$ and for the impedance strip with corresponding $\eta = -\imath$ are presented on Figure 3 as well as analytical solution for the "fractional strip". It is seen that the curve for the analytical fractional solution has deep resonance while other two curves obtained numerically have less precise minimum. As seen from Figure 3 the monostatic RCS for the "fractional strip" with $\nu = 0.5$ obtained numerically coincides with the monostatic RCS expressed analytically for $\nu = 0.5$.

Monostatic RCS for the intermediate values of ν are shown on

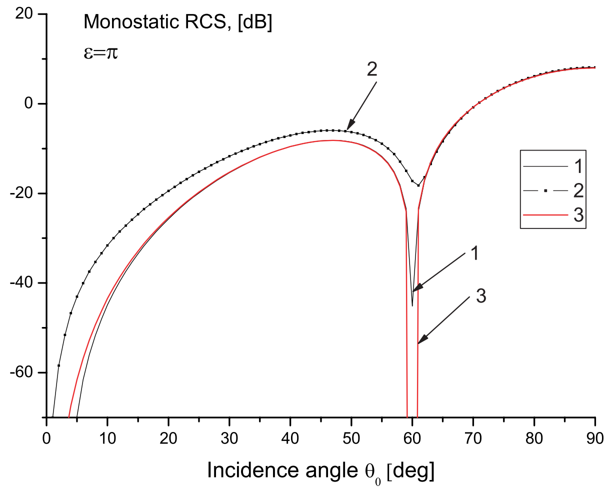

Figure 3. Monostatic RCS versus the incidence angle for $\epsilon = \pi$. (1) fractional strip with $\nu = 0.5$, calculated numerically; (2) impedance strip with $\eta = -\imath/\sin(\theta_0)$; (3) fractional strip with $\nu = 0.5$ calculated analytically.

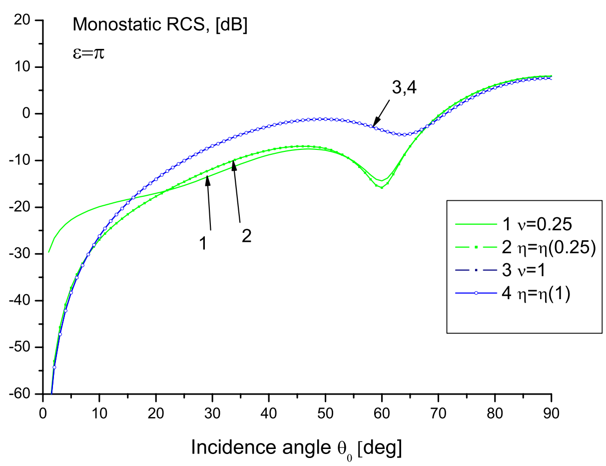

Figure 4. Monostatic RCS versus the incidence angle for $\epsilon = \pi$. (1) fractional strip $\nu = 0.25$; (2) impedance strip with impedance defined for $\nu = 0.25$, $\epsilon = \pi$; (3) $\nu = 1$; (4) impedance corresponding to $\nu = 1$.

Figures 4 and 5. It is seen that the lines for the "fractional strip" and corresponding lines of impedance solution have similar behavior and

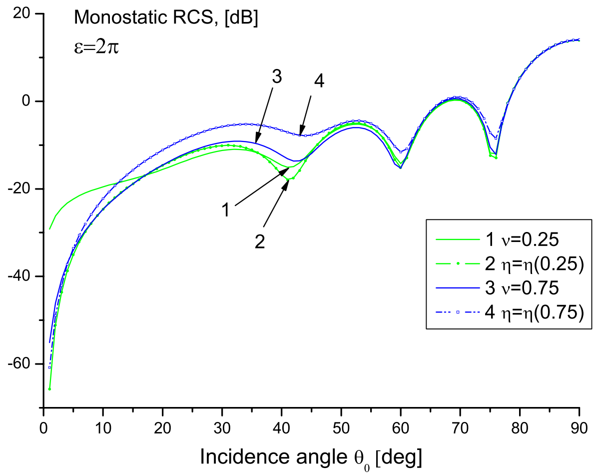

Figure 5. Monostatic RCS versus the incidence angle for $\epsilon = 2\pi$. (1) fractional strip $\nu = 0.25$; (2) impedance strip with impedance defined for $\nu = 0.25$; (3) $\nu = 0.75$; (4) impedance corresponding to $\nu = 0.75$.

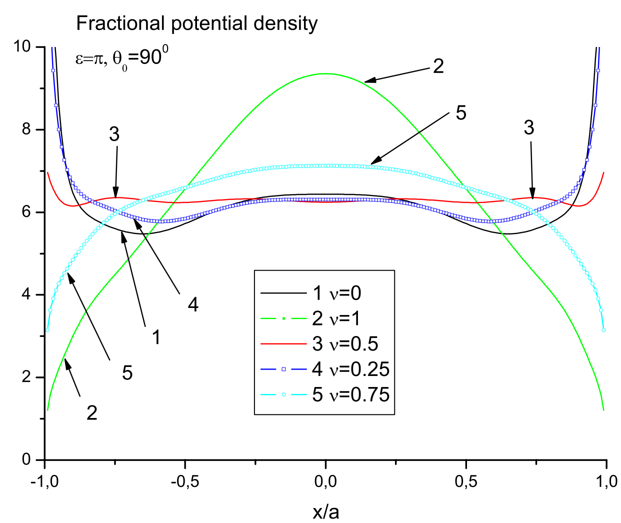

Figure 6. The fractional potential density $|\tilde{f}^{1-\nu}(\xi)|$ for $\epsilon = \pi$, $\theta_0 = 90°$.

have minimums at the same values of incidence angle. For a fixed value of ν and ϵ the impedance has different values for different incidence angles.

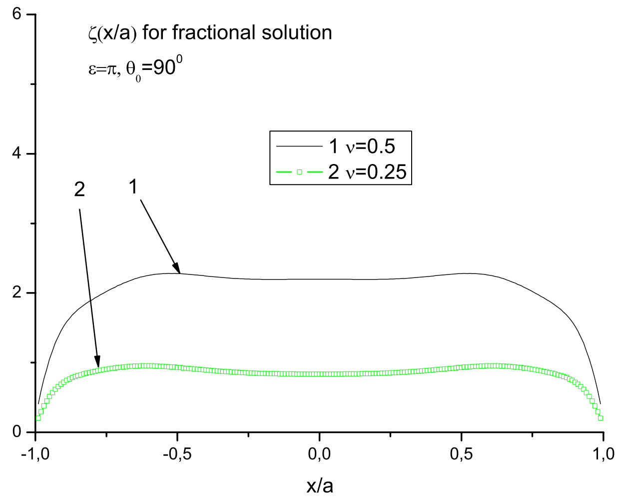

Figure 7. The ratio $\zeta(x)$ for the fractional strip for $\epsilon = \pi$, $\theta_0 = 90°$.

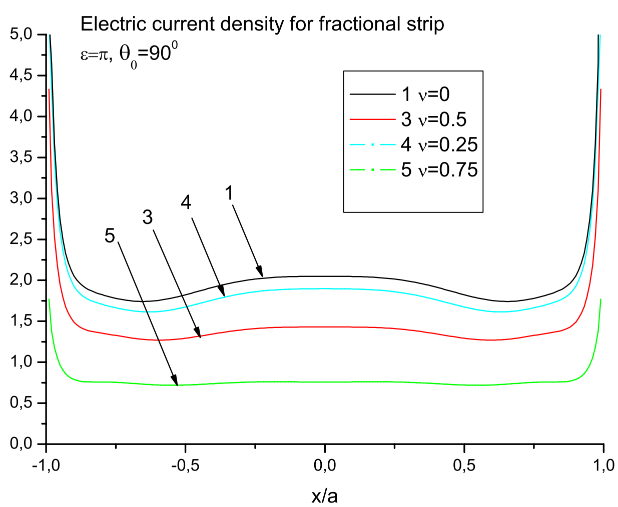

Figure 8. Electric current density for the fractional strip for the values of parameters as on Figure 6.

Figure 6 shows the fractional potential densities $\tilde{f}^{1-\nu}(\xi)$ for $\epsilon = \pi$, $\theta_0 = 90°$ and different values of ν from 0 to 1. The currents for the values $\nu = 0$ and $\nu = 1$ correspond to the electric current $j_z^{0(e)}$ existing on PEC strip and the magnetic current $j_x^{1(m)}$ on PMC strip, respectively. It is interesting to note that the fractional density for

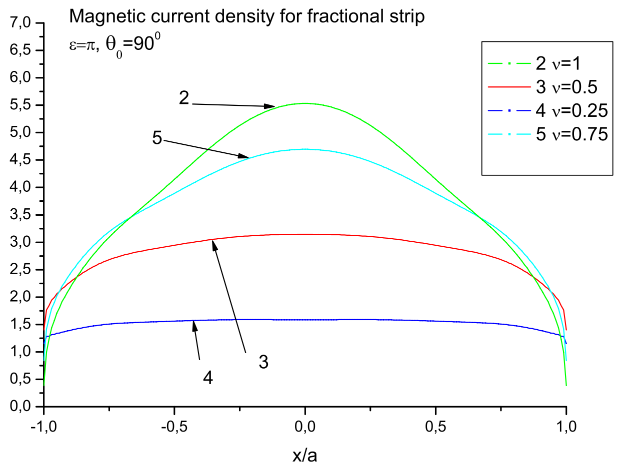

Figure 9. Magnetic current density for the fractional strip for the values of parameters as on Figure 6.

the intermediate case $\nu = 0.5$ is close to the value obtained from the analytical formula (33), i.e., to the constant $|\tilde{f}^{0.5}|_{\theta_0=90^\circ} = 2\epsilon$.

Electric and magnetic surface current densities existing on the "fractional strip" are plotted on Figures 8 and 9. These currents are found after integration of the fractional density $\tilde{f}^{1-\nu}$ (25). Electric current $j_z^{\nu(e)}$ has a singularity at $\xi = \pm 1$ while magnetic current $j_x^{\nu(m)}$ becomes zero for $\xi = \pm 1$. This fact is in good agreement with the behavior of the currents near the edges of the impedance strip (19). Similar graphics for electric and magnetic currents on the impedance strip were presented in [9].

Figure 7 presents the graphic of the ratio $\zeta(x)$ for the "fractional strip". For the wide range of the coordinate x the function $\zeta(x)$ is close to the value of impedance (28). It means that approximately a "fractional strip" can be treated as an impedance strip with pure imaginary impedance.

5. CONCLUSION

E-polarized plane wave diffraction by the "fractional strip" and the impedance strip has been analyzed. Following the previous works to solve considered diffraction problems field presentations, surface currents, integral equations and physical characteristics are compared. It is shown that the solution for a "fractional strip" can be expressed through one potential density function, but for an impedance strip the solution is defined via two current densities functions. Moreover the integral equations for the "fractional strip" can be solved analytically for the special intermediate value of the fractional order $\nu = 0.5$. The relation between the fractional order and the impedance is derived. Fractional boundary conditions result in existence of both electric and magnetic surface currents on the strip. Relation for the surface currents on the "fractional strip" proves that the fractional boundary conditions are similar to the impedance boundary conditions with pure imaginary impedance and in the physical optics approximation the ratio of the surface currents is the same as for the impedance strip. Numerical results are presented showing the comparison of the physical characteristics of the fractional and impedance strips such as radiation pattern, monostatic radar cross section and surface current densities.

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