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APPLICATION OF THE ENERGY BALANCE METHOD FOR STRONGLY NONLINEAR OSCILLATORS

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Abstract—In this paper, He's energy balance method is applied to nonlinear oscillators. We illustrate that the energy balance is very effective and convenient and does not require linearization or small perturbation. Contrary to the conventional methods, in energy balance, only one iteration leads to high accuracy of the solutions. It is predicted that the energy balance method can be found wide application in engineering problems.

1. INTRODUCTION

To illustrate its basic concepts of the VIM, we consider the following differential equation [1]:

$$u'' + \omega_0^2 u + \varepsilon f(u) = 0 \tag{1}$$

With the initial condition

$$u(0) = A, \quad u'(0) = 0 \tag{2}$$

where f is a nonlinear function of u'', u', u, in this preliminary report, we limit ourselves to the simplest case, i.e., f depends upon only the function of u.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Recently, considerable attention has been directed towards the analytical solutions for nonlinear equations without possible small parameters. The traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the

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shortcomings, many new techniques have appeared in open literature, for example, d-perturbation method [3, 4], variational iteration method (VIM) [5–10], homotopy perturbation method [11–18], bookkeeping parameter.

Perturbation method [19], just to name a few, a review on some recently developed nonlinear analytical methods can be found in detail in [20–23].

In energy balance method, a variational principle for the nonlinear oscillation is established, then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones. Some examples reveal that even the lowest order approximations are of high accuracy [1].

2. BASIC IDEA

First we consider the Duffing equation [1]:

$$u'' + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{3}$$

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 \right\} d\tau \tag{4}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 = \frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4 \tag{5}$$

Or:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 - \frac{1}{2}A^2 - \frac{1}{4}\varepsilon A^4 = 0 \tag{6}$$

In Eqs. (5) and (6) the kinetic energy (E) and potential energy (T) can be respectively expressed as $u'^2/2$, $u^2/2 + \varepsilon u^4/4$ throughout the oscillation, it holds that $H = E + T$ constant.

We use the following trial function to determine the angular frequency ω.

$$u = A\cos\omega t \tag{7}$$

Substituting (7) into (6), we obtain the following residual equation:

$$R(t) = \omega^2 \sin^2\omega t + \cos^2\omega t + \frac{1}{2}\varepsilon A^2 \cos^4\omega t - 1 - \frac{1}{2}\varepsilon A^2 \tag{8}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by

appropriate choice of ω. Since Eq. (7) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives:

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2} \tag{9}$$

We can apply various other techniques, for examples, least square method, Galerkin method, to identify the constant ω.

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\varepsilon A^2}} \tag{10}$$

The approximate period obtained by the traditional perturbation method reads (Nayfeh, 1985).

$$T_{pert} = 2\pi\left(1 - \frac{3}{8}\varepsilon A^2\right) \tag{11}$$

So our theory, in case $\varepsilon \ll 1$, gives exactly the same result with those obtained by perturbation method.

What is rather surprising about the remarkable range of validity of (10) is that the actual asymptotic period as $\varepsilon \to \infty$ is also of high accuracy.

$$\lim_{\varepsilon\to\infty} \frac{T_{ex}}{T} = \frac{2\sqrt{3/4}}{\pi}\int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = 0.9294 \tag{12}$$

The lowest order approximation given by (10) is actually within 7.6% of the exact frequency regardless of the magnitude of εA^2.

If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly [1].

3. APPLICATIONS

In order to assess the advantages and the accuracy of the energy balance method, we will consider the following three examples:

3.1. Example 1

We consider the following nonlinear oscillator [2]:

$$u'' + u^3 = 0 \tag{13}$$

With the boundary condition of:

$$u(0) = A, \quad u'(0) = 0 \tag{14}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{4}u^4 - \frac{1}{4}A^4 = 0 \tag{15}$$

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{4}A^4\cos^4\omega t - \frac{1}{4}A^4 = 0 \tag{16}$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \sqrt{\frac{3}{4}A^2} \tag{17}$$

Its period can be written in the form:

$$T = \frac{2\pi}{\sqrt{\frac{3}{4}A^2}} = \frac{7.2554}{A} \tag{18}$$

The exact period [23] is $T = 7.4163A^{-1}$. Therefore, it can be easily proved that the maximal relative error is less than 2.17%.

We can obtain the following approximate solution:

$$u = A\cos\sqrt{\frac{3}{4}A^2 t} \tag{19}$$

3.2. Example 2

We consider the following nonlinear oscillator [2]:

$$u'' + u + u^{\frac{1}{3}} = 0, \tag{20}$$

With the boundary condition of:

$$u(0) = A, \quad u'(0) = 0 \tag{21}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{3}{4}u^{\frac{4}{3}} - \frac{1}{2}A^2 - \frac{3}{4}A^{\frac{4}{3}} = 0 \tag{22}$$

Table 1. Comparison of energy balance frequency with parameter-expanding frequency.

A	Energy balance frequency	Parameter-Expanding frequency
0.1	2.427031	2.526334
0.2	2.020101	2.095394
0.3	1.830600	1.894088
0.4	1.714872	1.770783
0.5	1.634784	1.685450
1	1.433046	1.469656
5	1.166333	1.181768
10	1.107698	1.117957
50	1.038090	1.041844
100	1.024161	1.026559

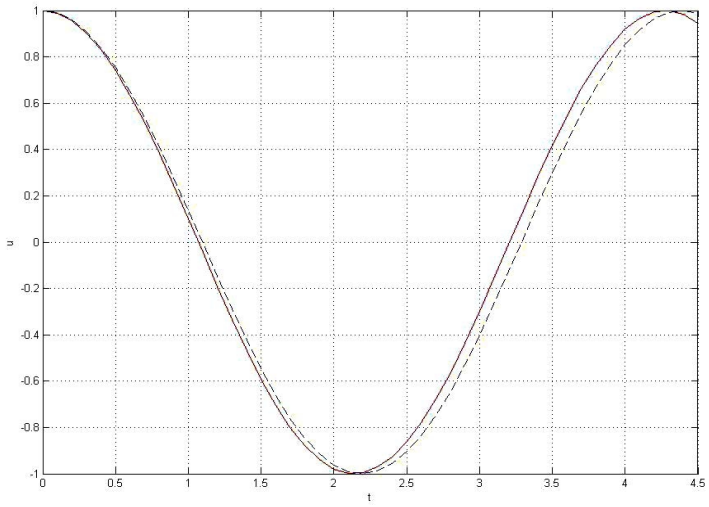

Figure 1. Comparison of the energy balance solution with the parameter-expanding solution: dashed line: parameter-expanding and solid line: the energy balance solution ($A = 1$).

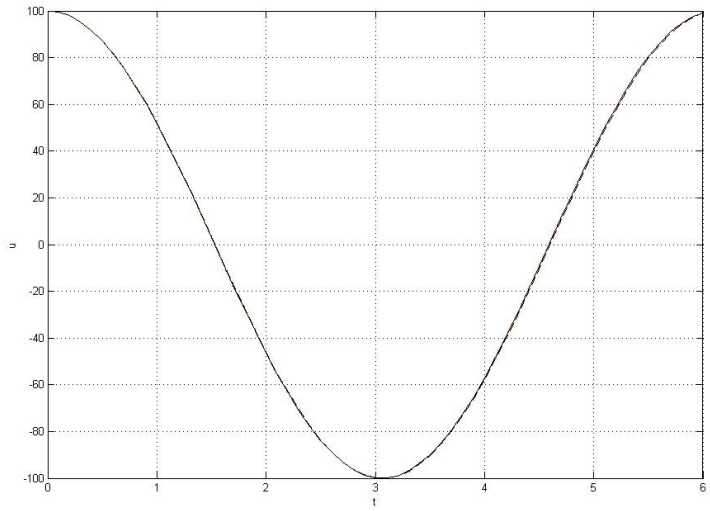

Figure 2. Comparison of the energy balance solution with the parameter-expanding solution: dashed line: parameter-expanding and solid line: the energy balance solution ($A = 100$).

Table 2. Comparison of energy balance frequency with parameter-expanding frequency.

A	Energy balance frequency	Parameter-Expanding frequency
0.1	2.271610	2.321608
0.2	1.809974	1.849507
0.3	1.595201	1.629439
0.4	1.471345	1.501996
0.5	1.396318	1.424164
1	1.363862	1.381881
5	4.373746	4.375680
10	8.674051	8.674666
50	43.302215	43.302256
100	86.602838	86.602851

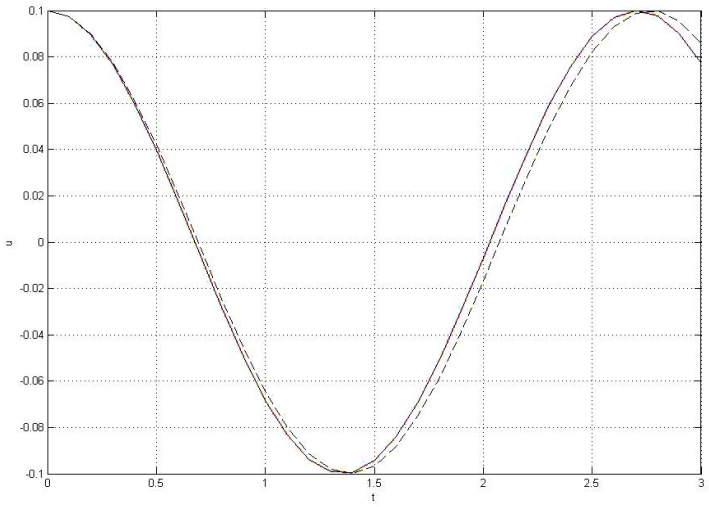

Figure 3. Comparison of the energy balance solution with the parameter-expanding solution: dashed line: parameter-expanding and solid line: the energy balance solution ($A = 0.1$).

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{2}A^2\cos^2\omega t + \frac{3}{4}(A\cos\omega t)^{\frac{4}{3}} - \frac{1}{2}A^2 - \frac{3}{4}A^{\frac{4}{3}} = 0 \quad (23)$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \sqrt{1 + 1.0536216A^{-\frac{2}{3}}}, \quad T = \frac{2\pi}{\omega} \quad (24)$$

In order to compare with parameter-expanding solution, we write Lan Xu's result [2]:

$$\omega = \sqrt{1 + 1.15959527A^{-\frac{2}{3}}}, \quad T = \frac{2\pi}{\omega} \quad (25)$$

We can obtain the following approximate solution:

$$u = A\cos\sqrt{1 + 1.0536216A^{-\frac{2}{3}}}t \quad (26)$$

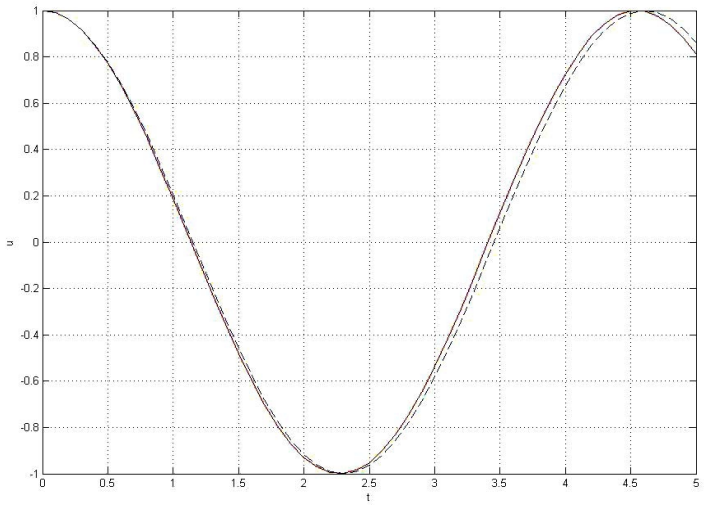

Figure 4. Comparison of the energy balance solution with the parameter-expanding solution: dashed line: parameter-expanding and solid line: the energy balance solution ($A = 1.0$).

4. EXAMPLE 3

We consider the following nonlinear oscillator [2]:

$$u'' + u^3 + u^{\frac{1}{3}} = 0, \tag{27}$$

With the boundary condition of:

$$u(0) = A, \quad u'(0) = 0 \tag{28}$$

Its Hamiltonian, therefore, can be written in the form:

$$H = \frac{1}{2}u'^2 + \frac{1}{4}u^4 + \frac{3}{4}u^{\frac{4}{3}} - \frac{1}{4}A^4 - \frac{3}{4}A^{\frac{4}{3}} = 0 \tag{29}$$

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{4}A^4\cos^4\omega t + \frac{3}{4}(A\cos\omega t)^{\frac{4}{3}} - \frac{1}{4}A^4 - \frac{3}{4}A^{\frac{4}{3}} = 0 \tag{30}$$

If we collocate at $\omega t = \pi/4$, we obtain:

$$\omega = \sqrt{\frac{3}{4}A^2 + 1.1101184A^{-\frac{2}{3}}}, \quad T = \frac{2\pi}{\omega} \tag{31}$$

In order to compare with parameter-expanding solution, we write Lan Xu's result [2]:

$$\omega = \sqrt{\frac{3}{4}A^2 + 1.15959527A^{-\frac{2}{3}}}, \quad T = \frac{2\pi}{\omega} \tag{32}$$

We can obtain the following approximate solution:

$$u = A\cos\sqrt{\frac{3}{4}A^2 + 1.1101184A^{-\frac{2}{3}}}t \tag{33}$$

5. CONCLUSIONS

In this paper, the Energy balance method has been successfully used to study the nonlinear oscillators. The method, which is proved to be a powerful mathematical tool to study of nonlinear oscillators, can be easily extended to any nonlinear equation. We demonstrated the accuracy and efficiency of the method by solving some examples.

We showed that the obtained solutions are valid for the whole domain. The examples show that even the lowest order approximations obtained by the present theory are actually of high accuracy.

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