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CLASS OF BI-QUADRATIC (BQ) ELECTROMAGNETIC MEDIA

I. V. Lindell

Electromagnetics Group
Department of Radio Science and Engineering
Helsinki University of Technology
Box 3000, Espoo 02015TKK, Finland

Abstract—Electromagnetic fields and media can be compactly represented by applying the four-dimensional differential-form formalism. In particular, classes of linear (bi-anisotropic) media can be defined in terms of the medium dyadic mapping between the electromagnetic two-forms. As a continuation to the process started by medium dyadics satisfying linear and quadratic algebraic equations, the class of bi-quadratic (BQ) media is defined by requiring that the medium dyadics satisfy the bi-quadratic algebraic equation. It is shown that the corresponding four three-dimensional medium dyadics are required to satisfy only two dyadic conditions. After studying general properties of BQ media, a special case is analyzed in detail as an example.

1. INTRODUCTION

Applying differential-form formalism in four-dimensional representation [1–4], the electromagnetic field and medium equations can be shown to take a particularly compact form. In the following, the notation concerning multivectors (elements of the spaces $\mathbb{E}_1 - \mathbb{E}_4$) and multiforms (elements of the spaces $\mathbb{F}_1 - \mathbb{F}_4$) with their different products follows that of [4]. For a brief summary the appendix of [5] can also be consulted.

Expressed for the electromagnetic two-forms $\mathbf{\Phi}, \mathbf{\Psi}$, members of the space $\mathbb{F}_2$, whose expansions in terms of three-dimensional field quantities are

$$\mathbf{\Phi} = \mathbf{B} + \mathbf{E} \wedge \boldsymbol{\varepsilon}_4, \qquad \mathbf{\Psi} = \mathbf{D} - \mathbf{H} \wedge \boldsymbol{\varepsilon}_4, \tag{1}$$

the Maxwell equations can be compactly presented as

$$\mathbf{d} \wedge \mathbf{\Phi} = \boldsymbol{\gamma}_m, \qquad \mathbf{d} \wedge \mathbf{\Psi} = \boldsymbol{\gamma}_e. \tag{2}$$

Here $\boldsymbol{\varepsilon}_4$ denotes the temporal component in the one-form basis $\{\boldsymbol{\varepsilon}_i\} \in \mathbb{F}_1$, $i = 1 \dots 4$. The differential operator has the representation

$$\mathbf{d} = \sum_{i=1}^{4} \boldsymbol{\varepsilon}_i \partial_{x_i} \tag{3}$$

in terms of the spatial coordinates x_1, x_2, x_3 and the normalized temporal coordinate $x_4 = \tau = ct$. The reciprocal vector basis is denoted by $\{\mathbf{e}_i\} \in \mathbb{E}_1$ and it satisfies

$$\mathbf{e}_i | \boldsymbol{\varepsilon}_j = \boldsymbol{\varepsilon}_j | \mathbf{e}_i = \delta_{ij}. \tag{4}$$

The electric and magnetic sources are three-forms $\boldsymbol{\gamma}_e, \boldsymbol{\gamma}_m \in \mathbb{F}_3$ with the three-dimensional expansions

$$\boldsymbol{\gamma}_e = \varrho_e - \mathbf{J}_e \wedge \boldsymbol{\varepsilon}_4, \qquad \boldsymbol{\gamma}_m = \varrho_m - \mathbf{J}_m \wedge \boldsymbol{\varepsilon}_4. \tag{5}$$

They satisfy the conservation equations

$$\mathbf{d} \wedge \boldsymbol{\gamma}_e = 0, \qquad \mathbf{d} \wedge \boldsymbol{\gamma}_m = 0. \tag{6}$$

Any linear electromagnetic medium can be defined in terms of a medium dyadic $\overline{\overline{\mathsf{M}}} \in \mathbb{F}_2\mathbb{E}_2$ mapping two-forms to two-forms as

$$\mathbf{\Psi} = \overline{\overline{\mathsf{M}}} | \mathbf{\Phi}. \tag{7}$$

Because two-forms form a six-dimensional space, the most general medium dyadic $\overline{\overline{\mathsf{M}}}$ involves 36 scalar parameters. Expressed in terms of three-dimensional field quantities, (7) takes the form [4]

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\alpha}} & \overline{\overline{\epsilon}}' \\ \overline{\overline{\mu}}^{-1} & \overline{\overline{\beta}} \end{pmatrix} | \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}, \tag{8}$$

where the four spatial dyadics $\overline{\overline{\alpha}}, \overline{\overline{\epsilon}}', \overline{\overline{\mu}}, \overline{\overline{\beta}}$ are members of different spaces. Expressed in the 'engineering' form

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} | \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{9}$$

the spatial medium dyadics $\overline{\overline{\epsilon}}, \overline{\overline{\xi}}, \overline{\overline{\zeta}}, \overline{\overline{\mu}}$ are members of the same space $\mathbb{F}_2\mathbb{E}_1$.

Different classes of media can be defined by introducing restrictions for the four-dimensional medium dyadic $\overline{\overline{\mathsf{M}}}$. The simplest class of media is obviously defined by $\overline{\overline{\mathsf{M}}}$ satisfying a linear dyadic equation which can be expressed as

$$\overline{\overline{\mathsf{M}}} - M\overline{\overline{\mathsf{I}}}^{(2)T} = 0. \tag{10}$$

The dyadic $\overline{\overline{\mathsf{I}}} \in \mathbb{E}_1\mathbb{F}_1$ denotes the unit dyadic mapping any vector to itself and its transpose $\overline{\overline{\mathsf{I}}}^T \in \mathbb{F}_1\mathbb{E}_1$ maps any one-form to itself. The unit dyadic for bivectors can be expressed as

$$\overline{\overline{\mathsf{I}}}^{(2)} = \frac{1}{2}\overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}} = \mathbf{e}_{12}\boldsymbol{\varepsilon}_{12} + \mathbf{e}_{23}\boldsymbol{\varepsilon}_{23} + \mathbf{e}_{31}\boldsymbol{\varepsilon}_{31} + \mathbf{e}_{14}\boldsymbol{\varepsilon}_{14} + \mathbf{e}_{24}\boldsymbol{\varepsilon}_{24} + \mathbf{e}_{34}\boldsymbol{\varepsilon}_{34}, \tag{11}$$

and its transpose maps any two-form to itself.

The class of media defined by (10) has been called that of axion media [3] or perfect electromagnetic conductor (PEMC) media and it has been previously studied in detail [7]. A medium belonging to this class is defined by one parameter M, the PEMC admittance. Perfect magnetic conductor (PMC) and perfect electric conductor (PEC) are two special cases corresponding to the respective parameter values $M = 0$ and $1/M = 0$.

The class of media second in simplicity is defined by medium dyadics satisfying a second-order equation of the general form

$$\overline{\overline{\mathsf{M}}}^2 + A\overline{\overline{\mathsf{M}}} + B\overline{\overline{\mathsf{I}}}^{(2)T} = 0. \tag{12}$$

Because of the second-order equation the class was labeled as that of SD media in [8]. Since the medium dyadics remain invariant in some duality transformations, the class can also be called that of self-dual media [9].

2. BQ MEDIA

Let us now study the class of electromagnetic media defined by medium dyadics satisfying a bi-quadratic equation of the form

$$\overline{\overline{\mathsf{M}}}^4 + A\overline{\overline{\mathsf{M}}}^2 + B\overline{\overline{\mathsf{I}}}^{(2)T} = 0. \tag{13}$$

The class will be labeled as that of BQ media for short. (13) can be written in the equivalent factorized form as

$$(\overline{\overline{\mathsf{M}}} - M_a\bar{\bar{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}} + M_a\bar{\bar{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}} - M_b\bar{\bar{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}} + M_b\bar{\bar{\mathsf{I}}}^{(2)T}) = 0, \quad (14)$$

with parameters related as

$$A = -(M_a^2 + M_b^2), \qquad B = M_a^2 M_b^2. \quad (15)$$

2.1. Eigenvalue Problem

The bracketed dyadics in (14) obviously commute. From its form we can conclude that there are at most four different eigenvalues, for the dyadic eigenvalue equation

$$\overline{\overline{\mathsf{M}}}|\mathbf{\Phi}_i = M_i\mathbf{\Phi}_i. \quad (16)$$

Let us denote $i = a+,\ a-,\ b+,\ b-$ with

$$M_{a\pm} = \pm M_a, \qquad M_{b\pm} = \pm M_b. \quad (17)$$

The eigenvalues can be solved from the bi-quadratic equation

$$(M_i^2)^2 + AM_i^2 + B = 0. \quad (18)$$

Since we know that the general medium dyadic has six eigenvalues, some of the four eigenvalues must be multiple ones, i.e., the eigen-two-forms $\mathbf{\Phi}_i$ corresponding to those eigenvalues must form subspaces in the six-dimensional space $\mathbb{F}_2$ of two-forms.

When the eigenvalues M_i have been solved, the corresponding eigen-two-forms can be found in terms of the following four dyadics:

$$\overline{\overline{\mathsf{P}}}_{a\pm} = \pm\frac{(\overline{\overline{\mathsf{M}}}^2 - M_b^2\bar{\bar{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}} \pm M_a\bar{\bar{\mathsf{I}}}^{(2)T})}{2M_a(M_a^2 - M_b^2)}, \quad (19)$$

$$\overline{\overline{\mathsf{P}}}_{b\pm} = \pm\frac{(\overline{\overline{\mathsf{M}}}^2 - M_a^2\bar{\bar{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}} \pm M_b\bar{\bar{\mathsf{I}}}^{(2)T})}{2M_b(M_b^2 - M_a^2)}, \quad (20)$$

which all commute with the medium dyadic $\overline{\overline{\mathsf{M}}}$. Because of (14) the dyadics satisfy

$$\overline{\overline{\mathsf{P}}}_{a\pm}|(\overline{\overline{\mathsf{M}}} \mp M_a\bar{\bar{\mathsf{I}}}^{(2)T}) = 0 \quad \Rightarrow \quad \overline{\overline{\mathsf{M}}}|\overline{\overline{\mathsf{P}}}_{a\pm} = \pm M_a\overline{\overline{\mathsf{P}}}_{a\pm}, \quad (21)$$

$$\overline{\overline{\mathsf{P}}}_{b\pm}|(\overline{\overline{\mathsf{M}}} \mp M_b \bar{\bar{\mathsf{I}}}^{(2)T}) = 0 \quad \Rightarrow \quad \overline{\overline{\mathsf{M}}}|\overline{\overline{\mathsf{P}}}_{b\pm} = \pm M_b \overline{\overline{\mathsf{P}}}_{b\pm}, \tag{22}$$

whence the eigen-two-forms can be represented as

$$\mathbf{\Phi}_i = \overline{\overline{\mathsf{P}}}_i|\mathbf{\Phi} \tag{23}$$

in terms of any two-form $\mathbf{\Phi}$ producing $\mathbf{\Phi}_i \neq 0$.

The four dyadics $\overline{\overline{\mathsf{P}}}_i$ satisfy

$$\overline{\overline{\mathsf{P}}}_{a+} + \overline{\overline{\mathsf{P}}}_{a-} = \frac{\overline{\overline{\mathsf{M}}}^2 - M_b^2 \bar{\bar{\mathsf{I}}}^{(2)T}}{M_a^2 - M_b^2}, \tag{24}$$

$$\overline{\overline{\mathsf{P}}}_{b+} + \overline{\overline{\mathsf{P}}}_{b-} = \frac{\overline{\overline{\mathsf{M}}}^2 - M_a^2 \bar{\bar{\mathsf{I}}}^{(2)T}}{M_b^2 - M_a^2}, \tag{25}$$

and

$$\overline{\overline{\mathsf{P}}}_{a+} + \overline{\overline{\mathsf{P}}}_{a-} + \overline{\overline{\mathsf{P}}}_{b+} + \overline{\overline{\mathsf{P}}}_{b-} = \bar{\bar{\mathsf{I}}}^{(2)T}. \tag{26}$$

The property (26) allows one to expand any given two-form $\mathbf{\Phi}$ in terms of the eigen-two-forms $\mathbf{\Phi}_i$ as

$$\mathbf{\Phi} = (\overline{\overline{\mathsf{P}}}_{a+} + \overline{\overline{\mathsf{P}}}_{a-} + \overline{\overline{\mathsf{P}}}_{b+} + \overline{\overline{\mathsf{P}}}_{b-})|\mathbf{\Phi} = \mathbf{\Phi}_{a+} + \mathbf{\Phi}_{a-} + \mathbf{\Phi}_{b+} + \mathbf{\Phi}_{b-}. \tag{27}$$

From (14) we obtain the orthogonality conditions

$$\overline{\overline{\mathsf{P}}}_i|\overline{\overline{\mathsf{P}}}_j = 0, \quad i \neq j, \tag{28}$$

while from (26) and orthogonality the property

$$\overline{\overline{\mathsf{P}}}_i^2 = \overline{\overline{\mathsf{P}}}_i, \tag{29}$$

can be seen to follow. This means that the dyadics $\overline{\overline{\mathsf{P}}}_i$ are projection dyadics. The space of two-forms can be split in four subspaces each containing the eigen-two-forms of the dyadic $\overline{\overline{\mathsf{M}}}$ corresponding to the respective eigenvalue.

From the property (26) we can write

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{M}}}|\bar{\bar{\mathsf{I}}}^{(2)T} = M_a \overline{\overline{\mathsf{P}}}_{a+} - M_a \overline{\overline{\mathsf{P}}}_{a-} + M_b \overline{\overline{\mathsf{P}}}_{b+} - M_b \overline{\overline{\mathsf{P}}}_{b-}, \tag{30}$$

and it can be generalized to

$$\overline{\overline{\mathsf{M}}}^m = M_a^m \overline{\overline{\mathsf{P}}}_{a+} + (-M_a)^m \overline{\overline{\mathsf{P}}}_{a-} + M_b^m \overline{\overline{\mathsf{P}}}_{b+} + (-M_b)^m \overline{\overline{\mathsf{P}}}_{b-}, \tag{31}$$

where m may be any integer. Through this formula we can further define non-integer powers of the dyadic $\overline{\overline{\mathsf{M}}}$, and even negative powers provided none of the eigenvalues is zero.

As a check we can substitute the expansions of $\overline{\overline{\mathsf{M}}}^4$, $\overline{\overline{\mathsf{M}}}^3$, $\overline{\overline{\mathsf{M}}}^2$, $\overline{\overline{\mathsf{M}}}$ and $\overline{\overline{\mathsf{I}}}^{(2)T}$ in the left-hand side of (14) and find that the coefficient expressions of each dyadic $\overline{\overline{\mathsf{P}}}_i$ vanish identically.

2.2. Principal BQ Medium

In [3] electromagnetic medium dyadic is decomposed in three components:

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{M}}}_1 + \overline{\overline{\mathsf{M}}}_2 + \overline{\overline{\mathsf{M}}}_3, \tag{32}$$

where $\overline{\overline{\mathsf{M}}}_3 = M_3\overline{\overline{\mathsf{I}}}^{(2)T}$ is called the axion part, $\overline{\overline{\mathsf{M}}}_2$ is called the Skewon part and, $\overline{\overline{\mathsf{M}}}_1$, the principal part. Both $\overline{\overline{\mathsf{M}}}_2$ and $\overline{\overline{\mathsf{M}}}_1$ are trace-free dyadics. They are distinguished by forming the metric dyadics $\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}_1$, $\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}_2 \in \mathbb{E}_2\mathbb{E}_2$, of which the former is assumed to be symmetric and the latter antisymmetric.

The principal part of the medium dyadic carries the information of 'regular' media, while the skewon part corresponds to the physical properties associated with chirality and Faraday rotation [3]. The PEMC medium is defined by vanishing of its principal and skewon parts, $\overline{\overline{\mathsf{M}}}_1 = 0, \overline{\overline{\mathsf{M}}}_2 = 0$.

Considering the principal BQ medium, we first note that from the symmetry condition

$$(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}})^T = \overline{\overline{\mathsf{M}}}^T\rfloor\mathbf{e}_N = \mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}} \tag{33}$$

it follows that also the dyadic $\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}^n$ is symmetric. In fact, because the dyadic

$$\boldsymbol{\varepsilon}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)} = \boldsymbol{\varepsilon}_{12}\boldsymbol{\varepsilon}_{34} + \boldsymbol{\varepsilon}_{23}\boldsymbol{\varepsilon}_{31} + \boldsymbol{\varepsilon}_{31}\boldsymbol{\varepsilon}_{24} + \boldsymbol{\varepsilon}_{14}\boldsymbol{\varepsilon}_{23} + \boldsymbol{\varepsilon}_{24}\boldsymbol{\varepsilon}_{31} + \boldsymbol{\varepsilon}_{34}\boldsymbol{\varepsilon}_{12} \tag{34}$$

is symmetric, we can expand

$$\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}^2 = (\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}})|(\boldsymbol{\varepsilon}_N\lfloor(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}})) = (\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}})|(\boldsymbol{\varepsilon}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)})|(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}), \tag{35}$$

which is obviously of symmetric form. We can proceed similarly for higher powers of $\overline{\overline{\mathsf{M}}}$.

As a conclusion, for principal BQ media the dyadics $\mathbf{e}_N\lfloor\overline{\overline{\mathsf{P}}}_i$ are symmetric. This implies that, defining the 'natural' dot product of two two-forms as [3]

$$\mathbf{\Phi}_1 \cdot \mathbf{\Phi}_2 = \mathbf{\Phi}_1|(\mathbf{e}_N\lfloor\mathbf{\Phi}_2) = \mathbf{e}_N|(\mathbf{\Phi}_1 \wedge \mathbf{\Phi}_2) = \mathbf{\Phi}_2 \cdot \mathbf{\Phi}_1, \tag{36}$$

for principal BQ media the dot product of two eigen-two-forms associated with different eigenvalues is zero. This is seen from the expansion

$$\begin{aligned}\mathbf{\Phi}_i \cdot \mathbf{\Phi}_j &= \mathbf{\Phi}_i|(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{P}}}_j)|\mathbf{\Phi} = (\mathbf{e}_N\lfloor\mathbf{\Phi}_i)|\overline{\overline{\mathsf{P}}}_j|\mathbf{\Phi} = (\mathbf{e}_N\lfloor\overline{\overline{\mathsf{P}}}_i|\mathbf{\Phi})|\overline{\overline{\mathsf{P}}}_j|\mathbf{\Phi} \\ &= \mathbf{\Phi}|(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{P}}}_i)^T|\overline{\overline{\mathsf{P}}}_j|\mathbf{\Phi} = \mathbf{\Phi}|(\mathbf{e}_N\lfloor(\overline{\overline{\mathsf{P}}}_i|\overline{\overline{\mathsf{P}}}_j))|\mathbf{\Phi} = 0,\ i \neq j,\end{aligned} \tag{37}$$

and (28). This can be interpreted as orthogonality of eigenfields in the principal BQ medium.

3. THREE-DIMENSIONAL REPRESENTATION

To be able to study properties of BQ media more closely let us express the medium conditions in terms of more conventional three-dimensional (spatial) medium dyadics.

3.1. BQ Conditions

Applying the expansion of the medium dyadic [4]

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\alpha}} + \overline{\overline{\epsilon}}' \wedge \mathbf{e}_4 + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mu}}^{-1} + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\beta}} \wedge \mathbf{e}_4, \tag{38}$$

where $\overline{\overline{\alpha}}, \overline{\overline{\epsilon}}', \overline{\overline{\mu}}^{-1}, \overline{\overline{\beta}}$ are 3D spatial medium dyadics, and denoting

$$\overline{\overline{\mathsf{A}}} = \overline{\overline{\alpha}}^2 - \overline{\overline{\epsilon}}'|\overline{\overline{\mu}}^{-1}, \tag{39}$$

$$\overline{\overline{\mathsf{B}}} = \overline{\overline{\alpha}}|\overline{\overline{\epsilon}}' - \overline{\overline{\epsilon}}'|\overline{\overline{\beta}}, \tag{40}$$

$$\overline{\overline{\mathsf{C}}} = \overline{\overline{\mu}}^{-1}|\overline{\overline{\alpha}} - \overline{\overline{\beta}}|\overline{\overline{\mu}}^{-1}, \tag{41}$$

$$\overline{\overline{\mathsf{D}}} = \overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}' - \overline{\overline{\beta}}^2, \tag{42}$$

we can write

$$\overline{\overline{\mathsf{M}}}^2 = \overline{\overline{\mathsf{A}}} + \overline{\overline{\mathsf{B}}} \wedge \mathbf{e}_4 + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mathsf{C}}} + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mathsf{D}}} \wedge \mathbf{e}_4. \tag{43}$$

Inserting this in (14) or the equivalent equation

$$(\overline{\overline{\mathsf{M}}}^2 - M_a^2\overline{\overline{\mathsf{I}}}^{(2)T})|(\overline{\overline{\mathsf{M}}}^2 - M_b^2\overline{\overline{\mathsf{I}}}^{(2)T}) = 0, \tag{44}$$

and separating spatial and temporal components, the BQ-medium conditions take the form of four 3D equations

$$(\overline{\overline{\mathsf{A}}} - M_a^2\overline{\overline{\mathsf{I}}}_s^{(2)T})|(\overline{\overline{\mathsf{A}}} - M_b^2\overline{\overline{\mathsf{I}}}_s^{(2)T}) = \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{C}}}, \tag{45}$$

$$(\overline{\overline{\mathsf{A}}} - M_a^2\overline{\overline{\mathsf{I}}}_s^{(2)T})|\overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{B}}}|(\overline{\overline{\mathsf{D}}} + M_b^2\overline{\overline{\mathsf{I}}}_s^{T}), \tag{46}$$

$$\overline{\overline{\mathsf{C}}}|(\overline{\overline{\mathsf{A}}} - M_b^2\overline{\overline{\mathsf{I}}}_s^{(2)T}) = (\overline{\overline{\mathsf{D}}} + M_a^2\overline{\overline{\mathsf{I}}}_s^{T})|\overline{\overline{\mathsf{C}}}, \tag{47}$$

$$(\overline{\overline{\mathsf{D}}} + M_a^2\overline{\overline{\mathsf{I}}}_s^{T})|(\overline{\overline{\mathsf{D}}} + M_b^2\overline{\overline{\mathsf{I}}}_s^{T}) = \overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{B}}}. \tag{48}$$

These equations are also valid if we swap the subscripts a and b. The equations are not independent in the general case. For example, if the inverse dyadic $\overline{\overline{\mathsf{B}}}^{-1}$ exists, one can show that (45) and (46) imply (47) and (48), whence the latter ones can be omitted.

In engineering electromagnetics a second set of 3D medium dyadics defined by [11]

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} | \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{49}$$

is used instead of the previous one. The two sets obey the relations [4]

$$\overline{\overline{\epsilon}}' = \overline{\overline{\epsilon}} - \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}}, \quad \overline{\overline{\alpha}} = \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}, \quad \overline{\overline{\beta}} = -\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}}. \tag{50}$$

Thus, we can also write

$$\overline{\overline{\mathsf{A}}} = \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}} + \overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1} - \overline{\overline{\epsilon}}|\overline{\overline{\mu}}^{-1}, \tag{51}$$

$$\overline{\overline{\mathsf{B}}} = \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} - \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}} + \overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}} + \overline{\overline{\epsilon}}|\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}}, \tag{52}$$

$$\overline{\overline{\mathsf{C}}} = \overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}} + \overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1}, \tag{53}$$

$$\overline{\overline{\mathsf{D}}} = \overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} - \overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}} + \overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}}. \tag{54}$$

When substituted in (45)–(48), BQ medium conditions are obtained for the medium dyadics $\overline{\overline{\epsilon}}, \overline{\overline{\xi}}, \overline{\overline{\zeta}}, \overline{\overline{\mu}}$. This is quite an involved task and the details are shown in Appendix A. As a result, it is shown that, as pointed out above, there is a double redundancy in the conditions (45)–(48). Actually, two of the dyadic conditions are sufficient to define a BQ medium, provided the dyadic $\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}$ has an inverse which is assumed in the sequel. The conditions can be reduced to the form (A23) and (A24), rewritten here for convenience as

$$\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}|\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}} + \overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} = M_a^2 M_b^2 \overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}} + \overline{\overline{\zeta}}), \tag{55}$$

$$\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}|(\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}} + \overline{\overline{\zeta}}))^2 = (\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} + M_a^2\overline{\overline{\mathsf{I}}}_s^{T})|(\overline{\overline{\mu}}^{-1}\overline{\overline{\epsilon}} + M_b^2\overline{\overline{\mathsf{I}}}_s^{T}). \tag{56}$$

It is worth noting that, in these conditions, the four medium dyadics appear through just two dyadics, $\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}}+\overline{\overline{\zeta}})$. This means that, for example, the dyadic $\overline{\overline{\xi}}-\overline{\overline{\zeta}}$ is not restricted by the BQ-medium conditions.

3.2. Eigenproblem

The eigenvalue Equation (16) can be expanded in 3D components as

$$\overline{\overline{\alpha}}|\mathbf{B}_i+\overline{\overline{\epsilon}}'|\mathbf{E}_i = M_i\mathbf{B}_i, \tag{57}$$

$$\overline{\overline{\mu}}^{-1}|\mathbf{B}_i+\overline{\overline{\beta}}|\mathbf{E}_i = -M_i\mathbf{E}_i, \tag{58}$$

whence, substituting

$$\mathbf{B}_i=-(\overline{\overline{\mu}}|\overline{\overline{\beta}}+M_i\overline{\overline{\mu}})|\mathbf{E}_i \tag{59}$$

to the former, the equation

$$(\overline{\overline{\epsilon}}'-\overline{\overline{\alpha}}|\overline{\overline{\mu}}|\overline{\overline{\beta}}-M_i(\overline{\overline{\alpha}}|\overline{\overline{\mu}}-\overline{\overline{\mu}}|\overline{\overline{\beta}})+M_i^2\overline{\overline{\mu}})|\mathbf{E}_i=0 \tag{60}$$

is obtained. In terms of the second set of 3D medium dyadics the same equations read

$$\mathbf{B}_i=(\overline{\overline{\zeta}}-M_i\overline{\overline{\mu}})|\mathbf{E}_i \tag{61}$$

$$(\overline{\overline{\epsilon}}-M_i(\overline{\overline{\zeta}}+\overline{\overline{\xi}})+M_i^2\overline{\overline{\mu}})|\mathbf{E}_i=0. \tag{62}$$

After solving for the eigenvalues M_i from the bi-quadratic Equation (18) the corresponding eigenvectors $\mathbf{E}_i$ can be found from (62), after which the $\mathbf{B}_i$ are obtained from (61).

3.3. Principal BQ Medium

When the medium dyadic consists only of the principal part, the 3D medium dyadics satisfy certain additional conditions. They are obtained when the expansion (38) is inserted in the condition (33):

$$\mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}' = (\mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}')^T, \tag{63}$$

$$\mathbf{e}_{123}\lfloor\overline{\overline{\mu}}^{-1} = (\mathbf{e}_{123}\lfloor\overline{\overline{\mu}}^{-1})^T, \tag{64}$$

$$\mathbf{e}_{123}\lfloor\overline{\overline{\alpha}} = -(\mathbf{e}_{123}\lfloor\overline{\overline{\beta}})^T. \tag{65}$$

After some algebraic steps these can be shown to equal the following condition for the second set of 3D medium dyadics:

$$\begin{pmatrix} \mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}} & \mathbf{e}_{123}\lfloor\overline{\overline{\xi}} \\ \mathbf{e}_{123}\lfloor\overline{\overline{\zeta}} & \mathbf{e}_{123}\lfloor\overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}} & \mathbf{e}_{123}\lfloor\overline{\overline{\xi}} \\ \mathbf{e}_{123}\lfloor\overline{\overline{\zeta}} & \mathbf{e}_{123}\lfloor\overline{\overline{\mu}} \end{pmatrix}^T. \tag{66}$$

4. SPECIAL CASE

Since it appears difficult to proceed with the general BQ-medium conditions (55), (56), let us concentrate on a special case defined by

$$\overline{\overline{\xi}} + \overline{\overline{\zeta}} = 0. \tag{67}$$

For such a medium the medium dyadic (38) can be expressed as

$$\overline{\overline{\mathsf{M}}} = \overline{\overline{\epsilon}} \wedge \mathbf{e}_4 + (\overline{\overline{\xi}} + \boldsymbol{\varepsilon}_2 \wedge \overline{\overline{\mathsf{I}}}{}_s^T)|\overline{\overline{\mu}}^{-1}|(\overline{\overline{\mathsf{I}}}{}_s^{(2)T} + \overline{\overline{\xi}} \wedge \mathbf{e}_4). \tag{68}$$

The condition (55) is now automatically satisfied while (56) is reduced to

$$(\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} + M_a^2 \overline{\overline{\mathsf{I}}}{}_s^T)|(\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} + M_b^2 \overline{\overline{\mathsf{I}}}{}_s^T) = 0. \tag{69}$$

There is no restriction for the dyadic $\overline{\overline{\xi}} = -\overline{\overline{\zeta}}$.

From the conclusion given in Appendix B it follows that one of the bracketed dyadics in (69) must be of the simple form $\boldsymbol{\beta}\mathbf{a}$ for some one-form $\boldsymbol{\beta}$ and vector $\mathbf{a}$. Thus, for a BQ medium of this special kind, the dyadics $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ must satisfy a relation of the form

$$\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} = -M^2 \overline{\overline{\mathsf{I}}}{}_s^T + \boldsymbol{\beta}\mathbf{a}, \tag{70}$$

for some scalar M^2. Another form for the condition is

$$\overline{\overline{\epsilon}} + M^2 \overline{\overline{\mu}} = (\overline{\overline{\mu}}|\boldsymbol{\beta})\mathbf{a}. \tag{71}$$

Of the two possibilities let us denote

$$M_a^2 = M^2. \tag{72}$$

Changing the order of the bracketed dyadics in (69) we have

$$(\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} + M_b^2 \overline{\overline{\mathsf{I}}}{}_s^T)|\boldsymbol{\beta} = 0, \tag{73}$$

which shows us that $\boldsymbol{\beta}$ is an eigen-one-form of the dyadic $\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}$ corresponding to the eigenvalue $-M_b^2$. Comparing with (70) we obtain the relation

$$M_b^2 = M^2 - \mathbf{a}|\boldsymbol{\beta}. \tag{74}$$

Thus, the four eigenvalues of the dyadic $\overline{\overline{\mathsf{M}}}$ in this special case are

$$\begin{aligned} M_{a+} &= M, \qquad M_{a-} = -M, \\ M_{b+} &= \sqrt{M^2 - \mathbf{a}|\boldsymbol{\beta}}, \qquad M_{b-} = -\sqrt{M^2 - \mathbf{a}|\boldsymbol{\beta}}. \end{aligned} \tag{75}$$

The 3D components of the eigenvalue equation $\overline{\overline{\mathsf{M}}}|\mathbf{\Phi}_i = M_i\mathbf{\Phi}_i$ are

$$(\overline{\epsilon} + \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}|\overline{\overline{\xi}})|\mathbf{E}_i + \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}|\mathbf{B}_i \;=\; M_i\mathbf{B}_i, \tag{76}$$

$$\mathbf{B}_i + \overline{\overline{\xi}}|\mathbf{E}_i \;=\; -M_i\overline{\overline{\mu}}|\mathbf{E}_i. \tag{77}$$

Substituting $\mathbf{B}_i$ from the latter into the former yields the eigenequation

$$\overline{\overline{\mu}}^{-1}|\overline{\epsilon}|\mathbf{E}_i = -M_i^2\mathbf{E}_i, \tag{78}$$

or, because of (70),

$$\boldsymbol{\beta}\mathbf{a}|\mathbf{E}_i = (M^2 - M_i^2)\mathbf{E}_i. \tag{79}$$

This has two sets of solutions. For $M_{a\pm} = \pm M$ the eigen-one-forms satisfy

$$\mathbf{a}|\mathbf{E}_{a\pm} = 0, \tag{80}$$

while for $M_{b\pm}^2 \neq M^2$ we have

$$\mathbf{E}_{b\pm} = \lambda_\pm\boldsymbol{\beta}, \tag{81}$$

where $\lambda_\pm$ are any scalars. Since $\mathbf{E}_{b\pm}$ are both multiples of $\boldsymbol{\beta}$, they define a 1D subspace, while the possible eigen-one-forms $\mathbf{E}_{a\pm}$, span a 2D subspace.

The corresponding eigen-two-forms become

$$\mathbf{\Phi}_{a\pm} \;=\; \mathbf{B}_{a\pm} + \mathbf{E}_{a\pm}\wedge\boldsymbol{\varepsilon}_4 = -(\overline{\overline{\xi}} \pm M\overline{\overline{\mu}} - \boldsymbol{\varepsilon}_4\wedge\overline{\overline{\mathsf{I}}}{}_s^T)|\mathbf{E}_a, \tag{82}$$

$$\mathbf{\Phi}_{b\pm} \;=\; \mathbf{B}_{b\pm} + \mathbf{E}_{b\pm}\wedge\boldsymbol{\varepsilon}_4 = -(\overline{\overline{\xi}} \pm \sqrt{M^2 - \mathbf{a}|\boldsymbol{\beta}}\;\overline{\overline{\mu}} - \boldsymbol{\varepsilon}_4\wedge\overline{\overline{\mathsf{I}}}{}_s^T)|\mathbf{E}_b, \tag{83}$$

where $\mathbf{E}_a$ satisfies $\mathbf{a}|\mathbf{E}_a = 0$ and $\mathbf{E}_b = \lambda\boldsymbol{\beta}$.

A suitable decomposition of any given one-form $\mathbf{E}$ into eigen-one-forms $\mathbf{E}_a, \mathbf{E}_b$ as

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b, \tag{84}$$

can be made through the bac-cab formula [4] which for the present quantities takes the form

$$\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\mathbf{E}) = \boldsymbol{\beta}(\mathbf{a}|\mathbf{E}) - (\mathbf{a}|\boldsymbol{\beta})\mathbf{E}. \tag{85}$$

In fact, we can identify the two eigen-one-forms as

$$\mathbf{E}_a = -\frac{\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\mathbf{E})}{\mathbf{a}|\boldsymbol{\beta}}, \qquad \mathbf{E}_b = \frac{\mathbf{a}|\mathbf{E}}{\mathbf{a}|\boldsymbol{\beta}}\boldsymbol{\beta}. \tag{86}$$

As a check we can form the square of (68) as

$$\overline{\overline{\mathsf{M}}}^2 = -\overline{\overline{\epsilon}}|\overline{\overline{\mu}}^{-1}|(\overline{\overline{\mathsf{I}}}_s^{(2)T} + \overline{\overline{\xi}} \wedge \mathbf{e}_4) + (\overline{\overline{\xi}} + \boldsymbol{\varepsilon}_2 \wedge \overline{\overline{\mathsf{I}}}_s^T)|\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} \wedge \mathbf{e}_4, \qquad (87)$$

where we yet have to substitute $\overline{\overline{\epsilon}}$ from (71). Inserting this expression and (72), (74) in the left-hand side of (44), after some simple algebraic steps it can be shown to vanish, which means that (68) really satisfies the bi-quadratic equation.

If the medium is required to have only the principal component, there is an additional restriction to the medium parameters. First, the dyadics $\mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}$ and $\mathbf{e}_{123}\lfloor\overline{\overline{\mu}}$ must be symmetric. From (71) it follows that $\mathbf{a}$ and $\boldsymbol{\beta}$ must have the relation

$$\mathbf{a} = \lambda \mathbf{e}_{123}\lfloor\overline{\overline{\mu}}|\boldsymbol{\beta} \qquad (88)$$

for some scalar λ. From (66) it follows that the dyadic $\mathbf{e}_{123}\lfloor\overline{\overline{\xi}}$ must be antisymmetric.

The class of SD media was defined by the medium dyadic satisfying a second-order algebraic Equation [8]. It was shown that for the most general of such media the conditions

$$\overline{\overline{\epsilon}} + M^2\overline{\overline{\mu}} = 0, \qquad \overline{\overline{\xi}} + \overline{\overline{\zeta}} = \lambda\overline{\overline{\mu}} \qquad (89)$$

must be satisfied by the three-dimensional medium dyadics. Comparing these with the condition (71) corresponding to the special case 2 of the BQ medium it is seen that the latter is more general because of the additional term $\boldsymbol{\beta}\mathbf{a}$. However, the latter is also less general because of the assumption $\overline{\overline{\xi}} + \overline{\overline{\zeta}} = 0$. A comparison with the general BQ medium cannot be made because explicit conditions for the three-dimensional medium dyadics are still missing.

5. CONCLUSION

The process of defining classes of linear electromagnetic media in terms of algebraic conditions satisfied by the four-dimensional medium dyadic $\overline{\overline{\mathsf{M}}}$ was continued by considering medium dyadics satisfying the bi-quadratic equation. General relations for the eigensolutions and conditions for the three-dimensional medium dyadics were derived. It was shown that the four dyadic conditions could be replaced by just two for two dyadic medium quantities. As a concrete example, a medium satisfying $\overline{\overline{\xi}} + \overline{\overline{\zeta}} = 0$ was considered. In this case the BQ-medium conditions require that the permittivity and permeability dyadics satisfy a linear condition of the form (70). When comparing with a recent review paper on classification of bi-anisotropic media [10], one can see that, unlike the class of SD media satisfying the second-order equation, the class considered here lies outside of its classification map.

Due to a request by a referee, items [12–17] have been added to the list of references, showing the progress in applying differential-form formalism to the analysis of different classes of electromagnetic media in four-dimensional formalism.

APPENDIX A.

Let us study the BQ-medium conditions (45)–(48) more closely. To streamline the analysis dyadics are denoted by

$$E=\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}},\quad X=\overline{\overline{\mu}}^{-1}|\overline{\overline{\xi}},\quad Z=\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}},\quad I=\overline{\overline{\mu}}^{-1}|I_s^{(2)T}|\overline{\overline{\mu}}=\overline{\overline{\mathsf{I}}}_s^T. \quad \text{(A1)}$$

Omitting the | multiplication sign we can express (51)–(54) as

$$\overline{\overline{\mu}}^{-1}|\overline{\overline{\mathsf{A}}}|\overline{\overline{\mu}} = X(X+Z)-E, \quad \text{(A2)}$$

$$\overline{\overline{\mu}}^{-1}|\overline{\overline{\mathsf{B}}} = XE+EZ-X(X+Z)Z, \quad \text{(A3)}$$

$$\overline{\overline{\mathsf{C}}}|\overline{\overline{\mu}} = X+Z, \quad \text{(A4)}$$

$$\overline{\overline{\mathsf{D}}} = E-(X+Z)Z. \quad \text{(A5)}$$

Substituting (A2)–(A5) in the BQ-medium conditions (45)–(48) yields

$$(X(X+Z)-E-M_a^2I)(X(X+Z)-E-M_b^2I)$$
$$= (XE+EZ-X(X+Z)Z)(X+Z), \quad \text{(A6)}$$

$$(X(X+Z)-E-M_a^2I)(XE+EZ-X(X+Z)Z)$$
$$= (XE+EZ-X(X+Z)Z)(E+M_b^2I-(X+Z)Z), \quad \text{(A7)}$$

$$(X+Z)(X(X+Z)-E-M_b^2I)$$
$$= (E+M_a^2I-(X+Z)Z)(X+Z), \quad \text{(A8)}$$

$$(E+M_a^2I-(X+Z)Z)(E+M_b^2I-(X+Z)Z)$$
$$= (X+Z)(XE+EZ-X(X+Z)Z). \quad \text{(A9)}$$

These four conditions can be respectively rewritten as

$$\begin{aligned} & X(X+Z)^3-(E+M_a^2I)X(X+Z)-X(X+Z)(E+M_b^2I) \\ & +(E+M_a^2I)(E+M_b^2I))-(XE+EZ)(X+Z)=0, \qquad \text{(A10)} \\ & -X(X+Z)^3Z+X(X+Z)(XE+EZ)-(E+M_a^2I)(XE+EZ) \\ & +(E+M_a^2I)X(X+Z)Z-(XE+EZ)(E+M_b^2I) \\ & +X(X+Z)Z(E+M_b^2I)+(XE+EZ)(X+Z)Z=0, \qquad \text{(A11)} \\ & (X+Z)^3-(E+M_a^2I)(X+Z)-(X+Z)(E+M_b^2I)=0. \qquad \text{(A12)} \\ & (X+Z)^3Z-(E+M_a^2I)(X+Z)Z-(X+Z)Z(E+M_b^2I) \\ & +(E+M_a^2I)(E+M_b^2I)-(X+Z)(XE+EZ)=0. \qquad \text{(A13)} \end{aligned}$$

Multiplying (A12) by X from the left and subtracting from (A10) leaves us with the condition

$$E(X+Z)^2=(E+M_a^2I)(E+M_b^2I). \qquad \text{(A14)}$$

Multiplying (A12) by Z from the right and subtracting from (A13) leaves us with the condition

$$(X+Z)^2E=(E+M_a^2I)(E+M_b^2I). \qquad \text{(A15)}$$

Assuming that the inverse E^{-1} exists, (A14) and (A15) actually represent the same equation and they imply the relation

$$E(X+Z)^2=(X+Z)^2E. \qquad \text{(A16)}$$

Multiplying (A12) by X from the left and by Z from the right and subtracting from (A11) leaves us with

$$\begin{aligned} & E(X+Z)^2Z+X(X+Z)^2E \\ = \; & (E+M_a^2I)(XE+EZ)+(XE+EZ)(E+M_b^2I). \qquad \text{(A17)} \end{aligned}$$

Combining this with (A14) and (A15) yields

$$E(X+Z)E=M_a^2M_b^2(X+Z). \qquad \text{(A18)}$$

From this analysis it is clear that, assuming the existence of E^{-1}, the BQ-medium conditions (A6)–(A9) are equivalent with three conditions: (A12), (A18) and either (A14) or (A15). It is seen that X-Z is not restricted by these conditions and can be freely chosen.

Now one can further show that only two of the above conditions, for example (A18) and (A14), are actually sufficient. In fact, adding

$M_b^2E(X+Z)+E(E+M_a^2I)(X+Z)$ on both sides of (A18), the left-hand side becomes

$$\begin{aligned} & E(X+Z)E+M_b^2E(X+Z)+E(E+M_a^2I)(X+Z) \\ = \; & E(X+Z)(E+M_b^2I)+E(E+M_a^2I)(X+Z), \end{aligned} \tag{A19}$$

while the right-hand side becomes

$$\begin{aligned} & M_a^2M_b^2(X+Z)+M_b^2E(X+Z)+E(E+M_a^2I)(X+Z) \\ = \; & (E+M_b^2I)(E+M_a^2I)(X+Z). \end{aligned} \tag{A20}$$

Applying here (A14), we have

$$(E+M_b^2I)(E+M_a^2I)(X+Z) = E(X+Z)^2(X+Z) = E(X+Z)^3. \tag{A21}$$

Equating (A19) and (A21) yields

$$E(X+Z)(E+M_b^2I)+E(E+M_a^2I)(X+Z) = E(X+Z)^3, \tag{A22}$$

whence assuming that E^{-1} exists, (A12) is seen to follow.

The sufficient BQ-medium conditions (A18) and (A14) can be deciphered in terms of the original 3D medium dyadics as

$$\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}|\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}}+\overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}} = M_a^2M_b^2\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}}+\overline{\overline{\zeta}}), \tag{A23}$$

$$\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}|(\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}}+\overline{\overline{\zeta}}))^2 = (\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}+M_a^2\overline{\overline{\mathsf{I}}}_s^T)|(\overline{\overline{\mu}}^{-1}|\overline{\overline{\epsilon}}+M_b^2\overline{\overline{\mathsf{I}}}_s^T). \tag{A24}$$

These conditions can be given a more symmetric form:

$$\begin{aligned} \overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}}+\overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1} &= M_a^2M_b^2\overline{\overline{\epsilon}}^{-1}|(\overline{\overline{\xi}}+\overline{\overline{\zeta}})|\overline{\overline{\epsilon}}^{-1} && \text{(A25)} \\ (\overline{\overline{\xi}}+\overline{\overline{\zeta}})|\overline{\overline{\mu}}^{-1}|(\overline{\overline{\xi}}+\overline{\overline{\zeta}}) &= (\overline{\overline{\epsilon}}+M_a^2\overline{\overline{\mu}})|\overline{\overline{\epsilon}}^{-1}|(\overline{\overline{\epsilon}}+M_b^2\overline{\overline{\mu}}). && \text{(A26)} \end{aligned}$$

APPENDIX B.

Let us study the dyadic equation

$$\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}} = 0, \tag{B1}$$

where $\overline{\overline{\mathsf{A}}}, \overline{\overline{\mathsf{B}}} \in \mathbb{E}_1\mathbb{F}_1$ are two 3D dyadics mapping vectors to vectors and they are assumed to be nonzero. Because $\det\overline{\overline{\mathsf{A}}} \neq 0$ would imply existence of $\overline{\overline{\mathsf{A}}}^{-1}$ and, hence, $\overline{\overline{\mathsf{B}}} = 0$ and, similarly, $\det\overline{\overline{\mathsf{B}}} = 0$ would imply $\overline{\overline{\mathsf{A}}} = 0$, we must have $\det\overline{\overline{\mathsf{A}}} = \det\overline{\overline{\mathsf{B}}} = 0$. This being the case, both dyadics can be expressed in two-term expansions as

$$\overline{\overline{\mathsf{A}}} = \mathbf{a}_1\boldsymbol{\alpha}_1 + \mathbf{a}_2\boldsymbol{\alpha}_2, \qquad \overline{\overline{\mathsf{B}}} = \mathbf{b}_1\boldsymbol{\beta}_1 + \mathbf{b}_2\boldsymbol{\beta}_2. \tag{B2}$$

Let us now assume that

$$\overline{\overline{\mathsf{A}}}^{(2)} = (\mathbf{a}_1 \wedge \mathbf{a}_2)(\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2) \neq 0, \tag{B3}$$

whence $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are linearly independent. Choosing a third one-form $\boldsymbol{\alpha}_3$ and a 3D trivector $\mathbf{k} \in \mathbb{E}_3$ so that they satisfy $\mathbf{k}|(\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \boldsymbol{\alpha}_3) = 1$, the one-forms $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ make a basis whose reciprocal vector basis is [4]

$$\mathbf{k}\lfloor(\boldsymbol{\alpha}_2 \wedge \boldsymbol{\alpha}_3), \quad \mathbf{k}\lfloor(\boldsymbol{\alpha}_3 \wedge \boldsymbol{\alpha}_1), \quad \mathbf{k}\lfloor(\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2). \tag{B4}$$

Because $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent, (B1) implies

$$\boldsymbol{\alpha}_1|\overline{\overline{\mathsf{B}}} = 0, \qquad \boldsymbol{\alpha}_2|\overline{\overline{\mathsf{B}}} = 0. \tag{B5}$$

Expanding $\mathbf{b}_1$ and $\mathbf{b}_2$ in the basis (B4), shows us that the dyadic $\overline{\overline{\mathsf{B}}}$ must actually be of the form

$$\overline{\overline{\mathsf{B}}} = (\mathbf{k}\lfloor(\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2))\boldsymbol{\beta}, \tag{B6}$$

for some one-form $\boldsymbol{\beta}$. This means that $\overline{\overline{\mathsf{B}}}^{(2)} = 0$. Starting originally from the assumption $\overline{\overline{\mathsf{B}}}^{(2)} \neq 0$, the result would have been $\overline{\overline{\mathsf{A}}}^{(2)} = 0$.

To summarize, when the dyadics $\overline{\overline{\mathsf{A}}}$ and $\overline{\overline{\mathsf{B}}}$ satisfy (B1) and neither of them vanishes, we must have either $\overline{\overline{\mathsf{B}}}^{(2)} = 0$ or $\overline{\overline{\mathsf{A}}}^{(2)} = 0$, i.e., either the dyadic $\overline{\overline{\mathsf{B}}}$ is of the simple form $\overline{\overline{\mathsf{B}}} = \mathbf{b}\boldsymbol{\beta}$ or $\overline{\overline{\mathsf{A}}}$ is of the form $\overline{\overline{\mathsf{A}}} = \mathbf{a}\boldsymbol{\alpha}$.

REFERENCES

1. Flanders, H., *Differential Forms*, Academic Press, New York, 1963.
2. Deschamps, G. A., "Electromagnetics and differential forms," *Proc. IEEE*, Vol. 69, 676–696, 1981.
3. Hehl, F. W. and N. Yu. Obukhov, *Foundations of Classical Electrodynamics*, Boston, Birkhäuser, 2003.
4. Lindell, I. V., *Differential Forms in Electromagnetics*, Wiley and IEEE Press, New York, 2004.

5. Lindell, I. V. and K. H. Wallén, "Wave equations for bi-anisotropic media in differential forms," *Journal of Electromagnetic Waves and Applications*, Vol. 16, 1615–1635, 2002.
6. Lindell, I. V., "Differential forms and bi-anisotropic media," *Electromagnetics*, Vol. 26, 191–201, 2006.
7. Lindell, I. V. and A. H. Sihvola, "Perfect electromagnetic conductor," *Journal of Electromagnetic Waves and Applications*, Vol. 19, 861–869, 2005.
8. Lindell, I. V., "Class of electromagnetic SD media," *Metamaterials*, 2008,
9. Lindell, I. V., "Electromagnetic fields in self-dual media in differential-form representation," *Progress In Electromagnetics Research*, PIER 58, 319–333, 2006.
10. Olyslager, F. and I. V. Lindell, "Electromagnetics and exotic media: A quest for the Holy Grail," *IEEE Ant. Propag. Mag.*, Vol. 44, No. 2, 48–58, April 2002.
11. Lindell, I. V., A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media*, Artech House, Boston, 1994.
12. Lindell, I. V. and K. H. Wallén, "Differential-form electromagnetics and bi-anisotropic Q-media," *Journal of Electromagnetic Waves and Applications*, Vol. 18, 957–968, 2004.
13. Lindell, I. V. and K. H. Wallén, "Generalized Q-media and field decomposition in differential-form approach," *Journal of Electromagnetic Waves and Applications*, Vol. 18, 1045–1056, 2004.
14. Lindell, I. V., "Affine transformations and bi-anisotropic media in differential-form approach," *Journal of Electromagnetic Waves and Applications*, Vol. 18, 1259–1273, 2004.
15. Lindell, I. V., "Electromagnetic wave equation in differential-form representation," *Progress In Electromagnetics Research*, PIER 54, 321–333, 2005.
16. Lindell, I. V., "The class of electromagnetic IB-media," *Progress In Electromagnetics Research*, PIER 57, 1–18, 2006.
17. Lindell, I. V., "Inverse for the skewon medium dyadics," *Progress In Electromagnetics Research*, PIER 63, 21–32, 2006.