

ELECTROMAGNETIC COUPLING THROUGH ARBITRARY APERTURES IN PARALLEL CONDUCTING PLANES

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Abstract—We propose a numerical method to solve the problem of coupling through finite, but otherwise arbitrary apertures in perfectly conducting and vanishingly thin parallel planes. The problem is given a generic formulation using the Method of Moments and the Green's function in the region between the two planes is evaluated using Ewald's method. Numerical applications using Glisson's basis functions to solve the problem are demonstrated and compared with previously published results and the output of FDTD software.

1. INTRODUCTION

The problem of electromagnetic coupling through arbitrary apertures in a perfectly conducting and vanishingly thin plane has been studied by Butler et al. [1, 2] as well as Harrington and Mautz [3] using the equivalence principle to divide the problem into two sub-problems involving the radiation of magnetic sources on a ground plane. Using the continuity of the tangential component of the electric and magnetic fields through the apertures, these authors were able to constrain the solution of these two sub-problems and obtain systems that could be reduced to linear equations thanks to the Method of Moments [4].

This basic approach has later been generalized to a class of situations involving three regions, such as that of half-spaces separated by a cavity in a thick ground plane [5] or by a parallel-plate waveguide [6]. In these works, simplifications as to the shape of the apertures and distribution of the magnetic currents in the apertures are necessary in order to obtain tractable numerical problems. As a result, there is little application of the Method of Moments in the literature

to evaluate of the transmission through arbitrary apertures in parallel plates.

Rahmat-Samii [7], on the other hand, solved the similar problem of coupling through an arbitrary aperture into a two-parallel-plate region without assumptions as to the shape of the magnetic currents by using a Green's function approach. As the author of this last work points out, however, a number of problems arise in the evaluation of the Green's function: the series representing it is poorly convergent in general and diverges when the separation between the plates is a multiple of half the wavelength of the illuminating wave, making it difficult to put the method to general use.

Fortunately, the popularization of Ewald's summation method [8] by Jordan et al. [9] has made it possible to solve a number of problems initially affected with comparable convergence issues, such as the evaluation of the Green's function for periodic point sources [10] or frequency selective surfaces [11]. Furthermore, since the method of images often leads to a formulation of Green's functions in terms of sums of the contribution of periodic images, Ewald's method has allowed for significant improvements in computational efficiency in non periodic problems such as [12], that describes a method for the fast and accurate calculation of the Green's function in a rectangular cavity with six metallic walls.

In this paper, the problem of coupling through arbitrary apertures into a two-parallel-plate region, initially treated by Rahmat-Samii [7], is revisited and the problems associated with the evaluation of the Green's function are solved by application of Ewald's summation techniques. The method is then generalized to three-region problems, to allow for the evaluation of the coupling between two half-spaces separated by a two-plates region. The motivation of this work has been its application to cascaded layers of finite arrays of slot type elements, used as Frequency Selective Surfaces (FSS) for long wavelengths [13].

2. FORMULATION OF THE PROBLEM

2.1. Coupling through an aperture in a two-plates region

The geometry under consideration is shown in Figure 1: the space labelled as Region 1 is bordered by two infinite conductive planes situated at $z = 0$ and $z = d$ and is linked with the half-space labelled as Region 0 by way of a finite but otherwise arbitrary aperture in the plate situated at $z = 0$. Both regions are filled with an homogeneous material and Region 0 is illuminated with an harmonic radiation.

Following [5], we replace this problem by two equivalent sub-problems by substituting the apertures on the plane at $z = 0$ by

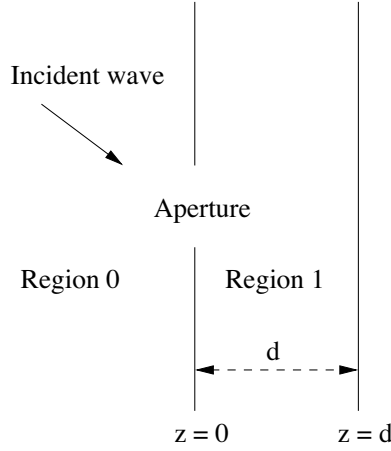


Figure 1. Setup and notations for problem of coupling through an aperture in a two-parallel-plate region.

a magnetic current distribution on the unslotted planes: due to the continuity of the tangential component of the electric field, the aperture is replaced by a by a magnetic current $-\vec{M}$ as viewed from Region 0, and by a magnetic current $\vec{M}$ as viewed from Region 1.

As a result, the total magnetic fields $\vec{H}^{(0)}$ and $\vec{H}^{(1)}$ in Regions 0 and 1 can be written as

$$\vec{H}^{(0)} = \vec{H}^{(sc)} - \vec{H}^{(0)}(\vec{M}) \quad (1)$$

$$\vec{H}^{(1)} = \vec{H}^{(1)}(\vec{M}) \quad (2)$$

where $\vec{H}^{(sc)}$ is total the magnetic field in Region 0 due to the illumination of the electromagnetic wave on the unslotted plate and $\vec{H}^{(i)}(\vec{M})$ is the magnetic field in Region i due to the magnetic current $\vec{M}$.

Since the tangential component of the magnetic field is continuous through the aperture, we can write, in the aperture

$$[\vec{H}^{(1)}(\vec{M}) + \vec{H}^{(0)}(\vec{M})] \times \vec{u}_z = \vec{H}^{(sc)} \times \vec{u}_z \quad (3)$$

As $\vec{H}^{(i)}$ is linear with regards to $\vec{M}$, we can use the Method of Moments in order to obtain a linear system to solve these equations: Let us first decompose the magnetic currents into the a linear

combination of suitable basis functions:

$$\vec{M}(r) = \sum_{n \in \mathbb{N}} I_n \vec{f}_n(r) \quad (4)$$

Replacing the magnetic currents by this expression in Equation (3) and testing the results against $\vec{f}_p \times \vec{u}_z$ yields

$$Y \cdot I = V \quad (5)$$

with the load V of the system defined by

$$V_p = \langle \vec{H}^{(sc)}, \vec{f}_p \rangle \quad (6)$$

and the generalized admittance Y of the system defined by

$$Y_{p,n} = \langle \vec{H}^{(1)}(\vec{f}_n) + \vec{H}^{(0)}(\vec{f}_n), \vec{f}_p \rangle \quad (7)$$

with

$$\langle \vec{\alpha}, \vec{\beta} \rangle = \int_A \vec{\alpha}(r') \cdot \vec{\beta}(r') dr' \quad (8)$$

the inner product for the aperture A .

The evaluation of such a linear system depends on the basis functions used and on the availability of suitable Green's functions for all regions. The terms of the generalized admittance matrix are then obtained by using (see for example [2, 14])

$$\vec{H}^{(i)}(\vec{M}) = -j\omega\epsilon\vec{F}^{(i)} - \nabla\Phi^{(i)} \quad (9)$$

with $F^{(i)}$ the vector electric potential and $\Phi^{(i)}$ the scalar magnetic potential in Region i contributed by $\vec{M}$, defined by

$$\vec{F}^{(i)}(r) = \int_A g^{(i)}(r, r') \vec{M}(r') dr' \quad (10)$$

and

$$\Phi^{(i)}(r) = -\frac{1}{j\omega\mu} \int_A g^{(i)}(r, r') \left(\vec{\nabla}_S \cdot \vec{M} \right) dr' \quad (11)$$

with $g^{(i)}$ the Green's function for the magnetic problem for Region i . The evaluation of $g^{(i)}$, for $i = 0, 1$, is treated in Section 3.

2.2. Transmission through Apertures in a Two-Plates Region

The extension of the previous problem to that of the transmission through a two-plates region, represented in Figure 2, follows easily from the previous section: the aperture in Plane 1, as seen from Region 0, is substituted by a magnetic current $-\vec{M}_1$, and by $\vec{M}_1$ as seen from Region 1. Likewise, the aperture in Plane 2 is replaced by the magnetic currents $-\vec{M}_2$ and $\vec{M}_2$ as seen from Region 1 and Region 2 respectively.

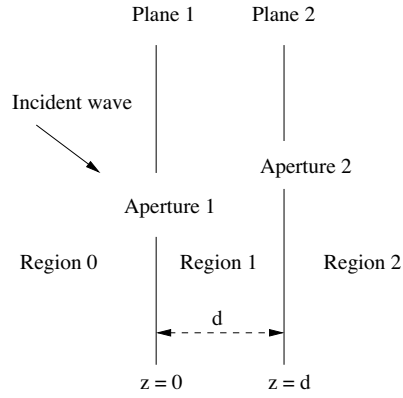


Figure 2. Setup and notations for problem of transmission through apertures through a two-parallel-plate region.

The total magnetic fields $\vec{H}^{(0)}$, $\vec{H}^{(1)}$ and $\vec{H}^{(2)}$ in the Regions 0, 1 and 2 can be written as

$$\vec{H}^{(0)} = \vec{H}^{(sc)} - \vec{H}^{(0)}(\vec{M}_1) \quad (12)$$

$$\vec{H}^{(1)} = \vec{H}^{(1)}(\vec{M}_1) - \vec{H}^{(1)}(\vec{M}_2) \quad (13)$$

$$\vec{H}^{(1)} = \vec{H}^{(1)}(\vec{M}_2) \quad (14)$$

and due to the continuity of the tangential part of the magnetic field through both apertures, we can write: for Aperture 1:

$$\left[\vec{H}^{(1)}(\vec{M}_1) + \vec{H}^{(0)}(\vec{M}_1) \right] \times \vec{u}_z = \vec{H}^{(sc)} \times \vec{u}_z \quad (15)$$

and for Aperture 2:

$$\left[\vec{H}^{(1)}(\vec{M}_1) - \vec{H}^{(1)}(\vec{M}_2) \right] \times \vec{u}_z = \vec{0} \quad (16)$$

The magnetic currents are projected on an appropriate set of basis functions

$$\vec{M}_1(r) = \sum_{n \in \mathbb{N}} I_{1,n} \vec{f}_{1,n}(r) \quad r \in \text{Aperture 1} \quad (17)$$

$$\vec{M}_2(r) = \sum_{m \in \mathbb{N}} I_{2,m} \vec{f}_{2,m}(r) \quad r \in \text{Aperture 2} \quad (18)$$

and these expressions are replaced in Equations (15) and (16) respectively. Testing the results against $\vec{f}_{1,p} \times \vec{u}_z$ and $\vec{f}_{2,p} \times \vec{u}_z$ yields

$$\begin{pmatrix} Y^{(1,1)} & Y^{(1,2)} \\ Y^{(2,1)} & Y^{(2,2)} \end{pmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} \quad (19)$$

with the load V of the system defined by

$$V_p = \langle \vec{H}^{(sc)}, \vec{f}_{1,p} \rangle_1 \quad (20)$$

and the generalized admittances of the system defined by

$$\begin{aligned} Y_{p,n}^{(1,1)} &= \langle \vec{H}^{(1)}(\vec{f}_{1,n}) + \vec{H}^{(0)}(\vec{f}_{1,n}), \vec{f}_{1,p} \rangle_1 \\ Y_{p,m}^{(1,2)} &= - \langle \vec{H}^{(1)}(\vec{f}_{2,m}), \vec{f}_{1,p} \rangle_1 \\ Y_{p,n}^{(2,1)} &= - \langle \vec{H}^{(1)}(\vec{f}_{1,n}), \vec{f}_{2,p} \rangle_2 \\ Y_{p,m}^{(2,2)} &= \langle \vec{H}^{(1)}(\vec{f}_{2,m}) + \vec{H}^{(2)}(\vec{f}_{2,m}), \vec{f}_{2,p} \rangle_2 \end{aligned} \quad (21)$$

with $\langle \vec{\alpha}, \vec{\beta} \rangle_i$ the inner product for Aperture i .

3. EVALUATION OF THE GREEN'S FUNCTION

In Region 0, $g^{(0)}$ is readily obtained by application of the method of images: as seen from the region of interest, a magnetic current source placed on a ground plane is equivalent to a magnetic source of doubled intensity in free space, hence

$$g^{(0)}(r, r') = \frac{e^{-jk|r-r'|}}{2\pi|r-r'|} \quad (22)$$

In Region 1, the Green's function can be obtained by using image theory [15]: a magnetic current $\vec{M}$ on the plane $z = 0$ generates a problem equivalent to that of a series of magnetic images of twice its magnitude and situated at the coordinates $z_n = 2nd$ for $n \in \mathbb{N}$. As

a consequence, the Green's function in the two-plates region can be written as

$$g^{(1)}(r, r') = \sum_{n \in \mathbb{Z}} \frac{e^{-jkR_n}}{2\pi R_n} \quad (23)$$

where r' is a point situated at $z = 0$ and

$$R_n(r, r') = \sqrt{(x - x')^2 + (y - y')^2 + (z - 2nd)^2} \quad (24)$$

As pointed out in [7], the summation in Equation (23) is not well behaved for a lossless medium between the two ground planes: its precise evaluation requires the summation of a large number of terms, making its use impractical, if possible at all. As soon as losses are introduced, however, all series involved in what follows become absolutely convergent, making it possible to easily manipulate and reorder terms of the series: following Harrington [15], we study the resolution of the problem by assuming infinitesimal losses and later studying the limits of the solution as losses tend to zero.

Following [9, 12], we use Ewald's method [8] to accelerate the convergence of this series and start by noting that

$$\frac{e^{-jkR}}{R} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-R^2 s^2 + \frac{k^2}{4s^2}} ds \quad (25)$$

for a suitably chosen path of integration. Note that a review and further information about this method is available in [16].

Let us choose a splitting parameter L and note $g = g_1 + g_2$ with

$$g_1(r, r') = \frac{1}{\pi\sqrt{\pi}} \sum_{n \in \mathbb{N}} \int_0^L e^{-R^2 s^2 + \frac{k^2}{4s^2}} ds \quad (26)$$

$$g_2(r, r') = \frac{1}{\pi\sqrt{\pi}} \sum_{n \in \mathbb{N}} \int_L^\infty e^{-R^2 s^2 + \frac{k^2}{4s^2}} ds \quad (27)$$

Using the identity

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \int_L^\infty e^{-R_n^2 s^2 + \frac{k^2}{4s^2}} ds &= \frac{1}{2R_n} \left[e^{jkR_n} \operatorname{erfc} \left(R_n L + \frac{jk}{2L} \right) \right. \\ &\quad \left. + e^{-jkR_n} \operatorname{erfc} \left(R_n L - \frac{jk}{2L} \right) \right] \end{aligned} \quad (28)$$

with erfc the complementary error function, g_2 can be evaluated

readily:

$$g_2(r, r') = \frac{1}{4\pi} \sum_{n \in \mathbb{Z}} \frac{1}{R_n} \left[e^{jkR_n} \operatorname{erfc} \left(R_n L + \frac{jk}{2L} \right) + e^{-jkR_n} \operatorname{erfc} \left(R_n L - \frac{jk}{2L} \right) \right] \quad (29)$$

In order to achieve an acceptable convergence rate, Poisson's summation is used in the evaluation of g_1 : defining $\rho = \sqrt{(x - x')^2 + (y - y')^2}$, $\beta = z$ and

$$f(x) = \int_0^L e^{-[\rho^2 + (\beta - x)^2]s^2 + \frac{k^2}{4s^2}} ds \quad (30)$$

we have

$$g_1(r, r') = \frac{1}{2d\pi\sqrt{\pi}} \sum_{n \in \mathbb{Z}} \tilde{f} \left(\frac{n\pi}{d} \right) \quad (31)$$

with $\tilde{f}$ the Fourier transform of f . Following a very similar course of calculation in [10], $\tilde{f}$ can be evaluated as

$$\tilde{f}(\alpha) = \frac{\sqrt{\pi}}{2} e^{-j\beta\alpha} \sum_{p \geq 0} \frac{(-1)^p}{p!} (\rho L)^{2p} E_{p+1} \left(\frac{\alpha^2 - k^2}{4L^2} \right) \quad (32)$$

with E_p the p^{th} exponential integral.

Further details on these calculations and a discussion on the evaluation of individual terms are available in Appendix A.

The number of terms needed to achieve convergence of the series (at a truncation of 10^{-6} , for example), is strongly dependant on the value of the splitting parameter L , as shown in Figure 3. In the cases presented, numerical erosion occurs for low values of L and a total of only a few hundred terms, including the evaluation of $\tilde{f}$, are needed to reach convergence. This contrasts with the evaluation of the summation of the Green's function in its natural form where, at best, the number the number of terms to take into account is proportional to the inverse of the truncation error required.

4. NUMERICAL RESULTS

4.1. Coupling through a Square Aperture into a Two-Parallel-Plate Region

Let us consider a geometry of the form shown in Figure 1, where the aperture between Region 0 and Region 1 is a square slot of width $\lambda/2$

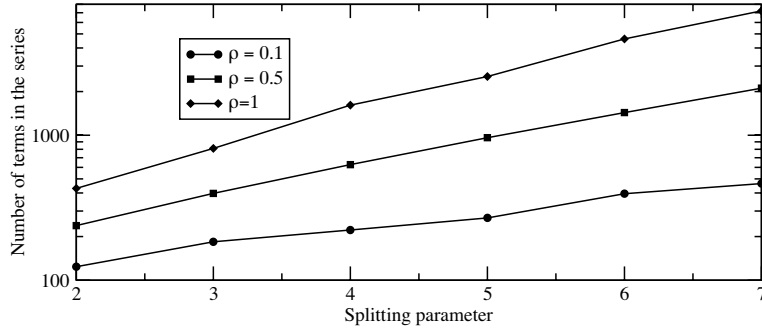


Figure 3. Number of terms in the series to reach a truncation error of 10^{-6} , for $\rho = 0.1$ m, $\rho = 0.5$ m and $\rho = 1$ m, with $d = 2.8\lambda$ and $\lambda = 0.5$ m.

centered at $(0,0)$, aligned with the axis $(\vec{u}_x, \vec{u}_y)$ and illuminated by a plane wave of wavelength λ propagating in the direction $\vec{u}_z$ and of polarization $\vec{E} = \vec{u}_x$. Following [7], we choose the separation between the two plates as $d = 2.8\lambda$.

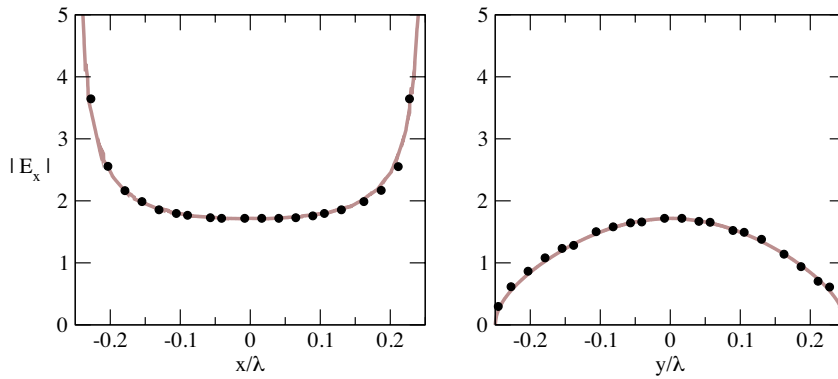


Figure 4. Magnitude of E_x in the $\lambda/2 \times \lambda/2$ aperture, sampled along the $(O, \vec{u}_x)$ and $(O, \vec{u}_y)$ axis. Dots represent results from the present method and the thick line those calculated with FDTD software.

Figure 4 shows the magnitude of E_x in the aperture sampled along the axis $(O, \vec{u}_x)$ and $(O, \vec{u}_y)$ respectively, as obtained with the present method and FDTD software. We observe a good agreement between the FDTD and the present method. These results are also consistent with those of Rahmat-Samii [7].

4.2. Transmission through Square Apertures in Parallel Plates

Let us now consider a geometry of the form shown in Figure 2. We choose the two apertures to be squares of width $\lambda/2$ centered on the original and aligned with the axis, with the same illumination as in the previous section. We choose the separation between the two plates as $d = 0.3\lambda$.

A comparison between the results of an FDTD method and the present method is presented in Figure 5 and shows a good agreement on the profile of the tangential part of the electric field on both apertures.

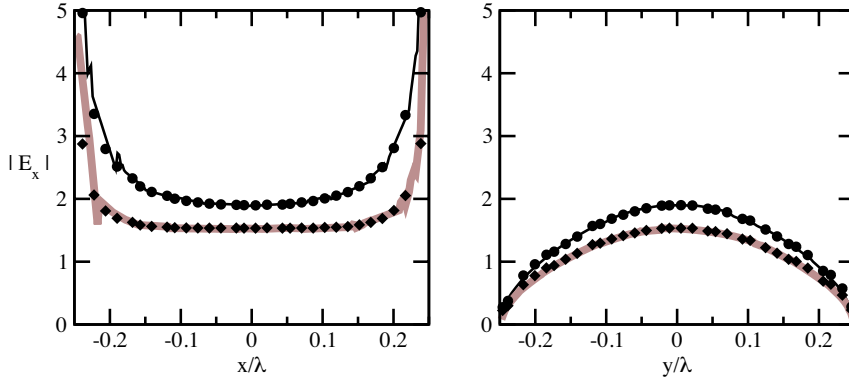


Figure 5. Magnitude of E_x in the $\lambda/2 \times \lambda/2$ apertures, sampled along the axis. Continuous lines represent the output of FDTD software in the left (thin black) and right (thick grey) apertures, the squares and diamonds represent the output of the present method in the left and right apertures respectively.

5. CONCLUSION AND FUTURE WORK

A method has been presented that allows for the robust evaluation of the transmission through apertures in a two-plates region. Due to its use of Ewald's summation, the method is immune to problems and breakdowns commonly associated with the infinite summation of free space Green's functions, and terms of the linear system can be evaluated relatively rapidly and with good accuracy. Its use with flexible boundary elements such as Glisson's makes it possible to treat apertures of essentially arbitrary shapes.

As with other applications of the Method of Moments, the calculation of the admittance central to the resolution depends only on the geometry of the problem and the frequency of the illumination: variations in the nature of the illumination (i.e., plane or spherical wave), its angle of incidence or polarization have no consequence in the assembly of the associated linear system and only require its resolution with a new load, resulting, for example, in significant savings in the evaluation of angular sweeps.

The free splitting parameter L present in Ewald's summation has a significant influence on the amount of calculations needed to evaluate the Green's function, as seen in Figure 3. Values of this parameter that are either too low or too large respectively lead to numerical erosion and slower convergence, a behavior that evidently hints at the existence of some optimal value of the parameter based on the wavelength of operation, the separation between the two parallel planes and the variable ρ . As the quality and speed of implementation of the method are expected to be strongly related to an optimal choice of this parameter, future works should include parametric or analytical studies of this behavior. A discussion on a number of these issues and such an analysis are available in [16, 17] for different applications of Ewald's method.

Recent applications of the aperture problem include experimental work on finite slot Frequency Selective Surfaces at mobile communication wavebands in connection with the electromagnetic architecture of buildings [13]. It is expected that the present method, along with techniques for cavities based on the results of [12], will allow for a better modelling of these situations.

ACKNOWLEDGMENT

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APPENDIX A. DETAILS OF THE CALCULATIONS

Calculations presented here are derived from similar ones found in [10, 16].

In Equation (26), g_1 is defined by

$$g_1(r, r') = \frac{1}{\pi\sqrt{\pi}} \sum_{n \in \mathbb{Z}} \int_0^L e^{-R^2 s^2 + \frac{k^2}{4s^2}} ds \quad (\text{A1})$$

With $\rho = \sqrt{x^2 + y^2}$ and $\beta = z$, we can write

$$g_1(r, r') = \frac{1}{\pi\sqrt{\pi}} \sum_{n \in \mathbb{Z}} f(2nd) \quad (\text{A2})$$

with

$$f(x) = \int_0^L e^{-(\rho^2 + (\beta - x)^2)s^2 + \frac{k^2}{4s^2}} ds \quad (\text{A3})$$

and by application of Poisson's summation

$$g_1(r, r') = \frac{1}{2d\pi\sqrt{\pi}} \sum_{n \in \mathbb{Z}} \tilde{f}\left(\frac{n\pi}{d}\right) \quad (\text{A4})$$

The Fourier transform $\tilde{f}$ of f is defined by

$$\tilde{f}(\alpha) = \int_{-\infty}^{+\infty} f(x) e^{-j\alpha x} dx \quad (\text{A5})$$

and can be developped into

$$\tilde{f}(\alpha) = \int_{s=0}^L e^{-\rho^2 s^2 + \frac{k^2}{4s^2} - \beta s^2} \left[\int_{x=-\infty}^{+\infty} e^{-x^2 s^2 + (2\beta s^2 - j\alpha)x} dx \right] ds \quad (\text{A6})$$

As pointed out in [10], $\int_{-\infty}^{+\infty} e^{-ax^2 + bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$, hence

$$\tilde{f}(\alpha) = \sqrt{\pi} e^{-j\alpha\beta} \int_{s=0}^L \frac{1}{s} e^{-\rho^2 s^2 + \frac{k^2 - \alpha^2}{4s^2}} ds \quad (\text{A7})$$

By using $s = \frac{1}{4u^2}$ as the integration variable, noting that $e^{-\frac{\rho^2}{4u}} = \sum_{p \geq 0} \frac{(-1)^p (\rho/2)^{2p}}{p! u^p}$ and later using $v = 4L^2 u$ as the integration variable, we find

$$\tilde{f}(\alpha) = \frac{\sqrt{\pi}}{2} e^{-j\beta\alpha} \sum_{p \geq 0} \frac{(-1)^p}{p!} (\rho L)^{2p} E_{p+1} \left(\frac{\alpha^2 - k^2}{4L^2} \right) \quad (\text{A8})$$

with E_p the p^{th} integral equation, defined by

$$E_p(x) = \int_1^\infty \frac{e^{-xt}}{t^p} dt \quad (\text{A9})$$

E_p can be calculated with the recurrence formula $E_{p+1}(x) = \frac{1}{p} [e^{-x} - xE_p(x)]$ with $E_1(x) = -E_I(-x) - j\pi$ for x such as $\text{Re}(x) < 0$ and $\text{Im}(x) \ll 1$. $E_1(x)$ and $E_I(x)$ can be evaluated using off-the-shelf mathematical packages.

It can be noted that $E_p(x)$ is monotone decreasing with regards to p . Likewise, $(\rho L)^{2p}/p!$ is monotone decreasing as soon as $p > (\rho L)^2$, hence the series conforms to Leibniz criterion for alternating series as soon as this condition is satisfied, making it possible to evaluate the distance to convergence of the summation with an arbitrary precision.

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