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GEOMETRICAL ANALYSIS OF WAVE PROPAGATION IN LEFT-HANDED METAMATERIALS. PART I

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Abstract—Geometrical analyses of basic equations of electromagnetics waves propagation in anisotropic dielectric materials with magnetic isotropy are presented in two complementary papers (Part I and Part II). In the present one, analysis arises from quadrics associated with relative dielectric tensor ($\tilde{\varepsilon}$) compatible with conical surfaces that represent the general plane wave equation (relation of dispersion). This study systematizes rays propagation in left-handed materials (LHMs) exhibiting dielectric anisotropy and magnetic isotropy.

In particular, *indefinite dielectric media* where dielectric permittivities are not all the same sign, have been investigated. Graphical alternative procedures for ray tracing in these media are presented.

1. INTRODUCTION

Forty years ago, the concept of left-handed metamaterials was studied in detail by Veselago [1]. Left-handed metamaterials (LHMs) are media with dielectric permittivity $\varepsilon < 0$ and magnetic permeability $\mu < 0$, exhibiting unique electromagnetic properties, such as negative refraction.

Over the last ten years, LHMs-related research has deserved a great deal of attention since the works of Pendry [2] and Smith et al. [3], who developed the first *experimental* LHM structure with split-ring resonators in the range of microwaves. Since then, a large amount of papers have dealt with this subject [4, 6–17]. Reflection and transmission in these materials with different dispersion relations have been well studied in [18, 19].

Anisotropic character of LHMs is frequently expected. Isotropic LHMs are difficult to prepare for experiments and they are actually anisotropic in nature. For this reason, alternative reinterpretations (that we consider original) of basic equations of wave propagation in anisotropic dielectric materials with magnetic isotropy are presented. The methodology is valid for both positive and negative values of dielectric permittivities (as well in uniaxial as in biaxial media) and magnetic permeability and allows the study of propagation of locally plane electromagnetic waves in LHMs [21].

Reinterpretation of basic equations is made in Section 2, where some features of quadrics associated to the dielectric tensor of these media are outlined. In Section 3, geometrical methodology is applied to ray tracing in RHMs and LHMs with dielectric anisotropy and magnetic isotropy. In particular, the so called *indefinite* media are considered [4, 5]. This term refers to a material for which the eigenvalues of the permittivity and permeability tensor are not all the same sign.

In the complementary part of this paper [21], another kind of geometrical analysis of light propagation in RH and LH media, involving local properties of dielectric permittivity tensor and Möhr's plane graphical construction has been carried out.

2. GEOMETRICAL INTERPRETATION OF PLANE WAVE EQUATION

Let us consider linear media, with dielectric anisotropy and magnetic isotropy. An orthonormal cartesian frame ($\overline{u}_1, \overline{u}_2$ and $\overline{u}_3$) along the main directions of relative dielectric permittivity tensor $\tilde{\varepsilon}$ is used. It is assumed that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$ are the eigenvalues of the relative dielectric tensor, $\tilde{\varepsilon}$, that can be positive or negative.

In dealing with non-conducting media free of currents and charges ($\overline{j} = 0$; $\varrho = 0$), the *relation of dispersion* or *general plane wave equation* can also be written as [22]:

$$\overline{\nu} - (\overline{n} \cdot \overline{\nu})\,\overline{n} = a\,\overline{\varepsilon_\nu} \tag{1}$$

where unit vector $\overline{\nu}$ is defined as $\overline{\nu} = \overline{E}/|\overline{E}|$, bound vector $\overline{\varepsilon_\nu} = \tilde{\varepsilon}\overline{\nu} = \overline{D}/(\varepsilon_0\,|\overline{E}|)$. Unit vector $\overline{n}$ is defined as $\overline{n} = \overline{k}/k$, where $\overline{k}$ is the wave vector whose length is equal to ω/v_p. Parameters ω and v_p are the angular frequency and the phase velocity, respectively. Finally, parameter a is defined as: $a = \mu_r\, v_p^2/c^2$.

Then we have easily shown that when angle θ between extraordinary ray direction and wave-vector is greater than $\pi/2$, **negative refraction occurs** and relative magnetic permeability must be negative ($\mu_r = -1$, LHM). If $\theta \neq \pi$, the phenomenon is known as *imperfect backward-wave* and is a consequence of anisotropy.

Equation (1) is our starting point as the basic plane wave equation for these media. Since we restrict our study to monochromatic waves, dispersion can be disregarded and eigenvalues ε_i are then constants depending only on the material.

When dealing with uniaxial media, one of the plane waves exhibits isotropic behaviour. Obviously, the subject of our study is the other eigen mode. In what follows, we are only concerned with the extraordinary mode: Poynting vector does not coincide with the direction of the wave vector $\overline{k}$.

From the dot product of Eq. (1) and vector $\overline{\varepsilon_\nu}$, two important relations are obtained:

$$\sigma_\nu = a\,|\overline{\varepsilon_\nu}|^2; \quad \cos\theta = a\,|\overline{\varepsilon_\nu}| \tag{2}$$

where θ is the angle between $\overline{E}$ and $\overline{D}$ and $\sigma_\nu = \overline{\varepsilon_\nu} \cdot \overline{\nu}$.

On the other hand, in [22] the authors propose an alternative expression of Fresnel's equation of wave normals in terms of intrinsic components (τ_n, σ_n) of bound vector $\overline{\varepsilon_n}$. This equation reads:

$$\tau_n^2 + \sigma_n^2 - \sigma_n\left(I - \frac{1}{a}\right) + a\,\Delta = 0 \tag{3}$$

where I and Δ are the trace and the determinant of tensor $\tilde{\varepsilon}$, respectively. Eq. (3) describes the locus of the end of $\overline{\varepsilon_n}$ in the plane cartesian coordinate system (σ, τ). This locus is a circumference centered at point $((I - 1/a)/2, 0)$ of radius R given by $R = \sqrt{((I - 1/a)/2)^2 - a\,\Delta}$.

If $\overline{t}$ is an unit vector in the same direction as Poynting vector $\mathbf{S}$, the dot product of $\overline{t}$ and $\overline{n}$ gives the angle between the wave vector $\overline{k}$ and the ray direction that, as expected, is also θ:

$$\overline{t} \cdot \overline{n} = a\,|\overline{\varepsilon_\nu}| = \frac{\mu_r\, v_p^2}{c^2}\,|\overline{\varepsilon_\nu}| \tag{4}$$

Then, values of $\theta > \pi/2$ occur with $\mu_r < 0$. Similarly, if θ is greater than $\pi/2$, the normal component σ_ν of vector $\overline{\varepsilon_\nu}$ is also **negative** and one can see from (2), that this only occurs for $\mu_r < 0$.

Let us give a new geometrical interpretation of Fresnel's equation of wave normals, given by Eq. (3). Let $\overline{n}(\alpha_1, \alpha_2, \alpha_3)$ be an unit vector along any direction of propagation given by $\overline{n} = \overline{k}/k$. Intrinsic components of vector $\overline{\varepsilon_n}$ bound to $\overline{n}$ verify the following three equations [22]:

$$\tau_n^2 + \sigma_n^2 = \varepsilon_1^2\,\alpha_1^2 + \varepsilon_2^2\,\alpha_2^2 + \varepsilon_3^2\,\alpha_3^2 \tag{5}$$

$$\sigma_n = \varepsilon_1\,\alpha_1^2 + \varepsilon_2\,\alpha_2^2 + \varepsilon_3\,\alpha_3^2 \tag{6}$$

$$1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \tag{7}$$

If Eq. (5) is multiplied by the unit, Eq. (6) by $\frac{1}{a} - I$, and Eq. (7) by $a\,\Delta$ and if addition is made, Eq. (3) is reobtained, because this sum vanishes.

Thus, the sum of the three right-hand sides of Eqs. (5), (6) and (7) gives:

$$\left(\varepsilon_1^2 + \varepsilon_1 \left(\frac{1}{a} - I\right) + a\,\Delta\right) \alpha_1^2 + \left(\varepsilon_2^2 + \varepsilon_2 \left(\frac{1}{a} - I\right) + a\,\Delta\right) \alpha_2^2 + \left(\varepsilon_3^2 + \varepsilon_3 \left(\frac{1}{a} - I\right) + a\,\Delta\right) \alpha_3^2 = 0 \implies \tag{8}$$

$$\implies \lambda_1\,\alpha_1^2 + \lambda_2\,\alpha_2^2 + \lambda_3\,\alpha_3^2 = 0 \tag{9}$$

with $\lambda_i = \varepsilon_i^2 + \varepsilon_i \left(\frac{1}{a} - I\right) + a\,\Delta$.

Property 1. Coefficients λ_i from Eq. (9) are equal to the power of point $(\varepsilon_i, 0)$ belonging to σ axis with respect to the circumference given by (3).

Proof: It is well known that the power $P(\,,\,)$ of any point with respect to a circumference is obtained by substituting the point coordinates into the equation of the circumference. Substitution of coordinates $(\varepsilon_i, 0)$ into (3) gives:

$$P(\varepsilon_i, 0) = \varepsilon_i^2 - \varepsilon_i \left(I - \frac{1}{a}\right) + a\,\Delta = \lambda_i$$

that coincides, in fact, with λ_i.

Property 2. The locus described by the Eq. (9):

$$\lambda_1\,\alpha_1^2 + \lambda_2\,\alpha_2^2 + \lambda_3\,\alpha_3^2 = 0$$

describes a cone in the space of wave vector directions.

Proof: Power of a point with respect to a circumference C_1 is positive when the point lies outside C_1 and negative when the point lies inside. The power vanishes when the point belongs to C_1.

In the present case, when dealing with Eq. (9), one of the $(\varepsilon_i, 0)$ $(i = 1, 2, 3)$ points lies in a different region than the others (see Fig. 1); if one of the points lies outside the circumference given by (3), the other two points lie inside. Then two of the λ_i coefficients in Eq. (9) are the same sign and the sign of the other coefficient is opposite to them, consequently the locus described by Fresnel's equation of wave normals in the space of wave vector directions is a **cone**. In what follows, we refer to this cone as Fresnel's cone.

Property 3: The locus described by the first equation in (2):

$$\sigma_\nu = \mu_r\,\frac{v_p^2}{c^2}\,|\overline{\varepsilon_\nu}|^2 \tag{10}$$

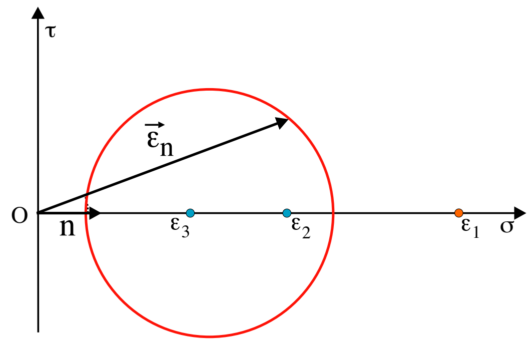

Figure 1. Power of points ε_i with respect to the Fresnel's circumference (locus of the ends of bound vector $\boldsymbol{\varepsilon_n}$). Power of point ε_1 with respect to the circumference is positive, whereas powers of ε_2 and ε_3 are negative.

is also a cone in the space of wave vector directions.

Proof: If $\overline{\nu}(\alpha_1, \alpha_2, \alpha_3)$ is an unit vector along the direction of vector $\overline{E}$, intrinsic components of vector $\overline{\varepsilon_\nu}$ (parallel to $\overline{D}$) verify the following equations:

$$\tau_\nu^2 + \sigma_\nu^2 = \varepsilon_1^2\,\alpha_1^2 + \varepsilon_2^2\,\alpha_2^2 + \varepsilon_3^2\,\alpha_3^2 \tag{11}$$

$$\sigma_\nu = \varepsilon_1\,\alpha_1^2 + \varepsilon_2\,\alpha_2^2 + \varepsilon_3\,\alpha_3^2 \tag{12}$$

Multiplying Eq. (11) by the unit, Eq. (12) by $-\frac{1}{a}$ and adding them, Eq. (10) is reobtained, because the sum vanishes. Therefore

$$\tau_\nu^2 + \sigma_\nu^2 - \frac{1}{a}\,\sigma_\nu = 0 \implies$$

$$\varepsilon_1\left(\varepsilon_1 - \frac{1}{a}\right)\,\alpha_1^2 + \varepsilon_2\left(\varepsilon_2 - \frac{1}{a}\right)\,\alpha_2^2 + \varepsilon_3\left(\varepsilon_3 - \frac{1}{a}\right)\,\alpha_3^2 = 0 \tag{13}$$

It is immediately seen that Eq. (13) also describes a cone. We call this cone, *vector* $\overline{D}$ *cone.*

A further analysis involving Möhr's plane construction and other geometrical analysis of Eqs. (2) and (3) is carried out in Part II of this work.

Finally, let us assume an unit vector $\overline{\nu}\,(\alpha_1, \alpha_2, \alpha_3)$, along any direction in space $\mathcal{R}_3$ (that can be the direction of propagation, direction of electric field $\overline{E}$, and so on), the dot product of $\overline{\nu}$ and the bound vector $\overline{\varepsilon_\nu}$ to this direction is given by:

$$\sigma_\nu = \overline{\varepsilon_\nu}\cdot\overline{\nu} = \varepsilon_1\,\alpha_1^2 + \varepsilon_2\,\alpha_2^2 + \varepsilon_3\,\alpha_3^2 \implies \pm 1 = \varepsilon_1\,X_1^2 + \varepsilon_2\,X_2^2 + \varepsilon_3\,X_3^2 \tag{14}$$

where $X_i = \frac{\alpha_i}{\sqrt{\sigma_\nu}}$. Last equation represents **Cauchy's quadric** (See Fig. 2). An important property of Cauchy's quadric [24] is that its normal at every point is collinear with bound vector $\overline{\varepsilon_\nu}$. This property will be used next.

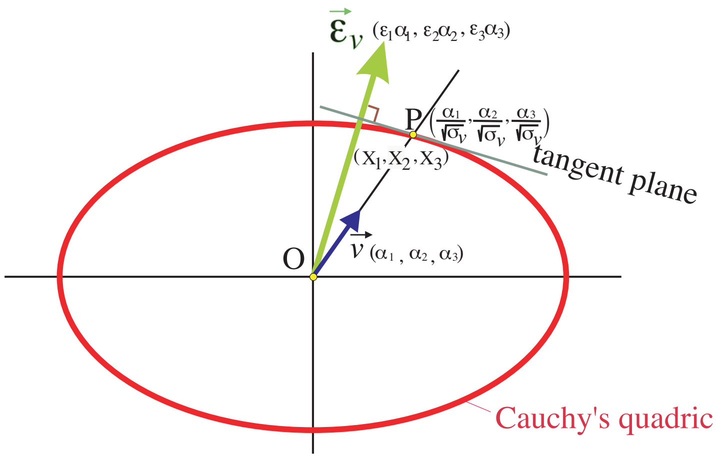

Figure 2. Cauchy's quadric for a medium with dielectric anisotropy, where $\boldsymbol{\nu}$ is an unit vector referred to space $\mathcal{R}_3$. In this case, eigenvalues of $\tilde{\varepsilon}$ are all the same sign.

If the eigenvalues ε_i are not all the same sign, we have hyperboloids (one-sheeted and two-sheeted) depending on the sign of σ_ν. In fact, the asymptotic cone ($\varepsilon_1 X_1^2 + \varepsilon_2 X_2^2 + \varepsilon_3 X_3^2 = 0$), divides the space into two regions, one for $\sigma_\nu > 0$ (one-sheeted hyperboloid) and the other one for $\sigma_\nu < 0$ (two-sheeted hyperboloid) (see Fig. 3).

It must be pointed out that the directions of vector $\overline{\nu}$ that lie inside the asymptotic cone provide angles between $\overline{\nu}$ and $\overline{\varepsilon_\nu}$ less than $\pi/2$. The directions of $\overline{\nu}$ outside the cone give angles greater than $\pi/2$.

In order to state the correspondence between the proposed methods and those used by other authors, uniaxial media are considered. If methodology proposed in [23–26] is applied, one has that the section of wave vector surface for the extraordinary propagation for $\varepsilon_1 = \varepsilon_2 > 0$, $\varepsilon_3 < 0$ and $\mu_r = -1$ is given by:

$$-\frac{X_1^2}{|\varepsilon_3|} + \frac{X_3^2}{|\varepsilon_1|} = -1 \tag{15}$$

where $X_i = (k_i\, c)/\omega$. If ϕ and ψ are the angles that $\overline{n}$ and $\overline{t}$ form with

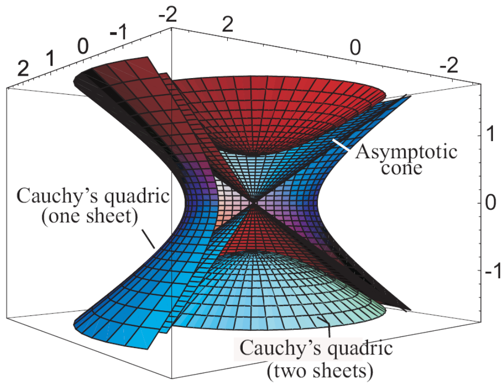

Figure 3. Cauchy's quadric for an *indefinite dielectric media*, where eigenvalues of $\tilde{\varepsilon}$ are not all the same sign. The quadric consists of a two-sheeted hyperboloid and an one-sheeted one. Asymptotic cone, that divides the space into two regions, is also shown.

$\overline{u}_1$ axis, respectively, it is easily shown that:

$$\tan\psi = -\frac{|\varepsilon_3|}{\varepsilon_1}\tan\phi \tag{16}$$

Our proposed alternative method attains the same result (16), by using Cauchy's quadric section given by:

$$\varepsilon_1\, x_1^2 - |\varepsilon_3|\, x_3^2 = -1$$

3. RAY TRACING IN RHM AND LHM INDEFINITE MEDIA WITH DIELECTRIC ANISOTROPY

In [22], it was shown that in uniaxial right-handed media, vector $\overline{\varepsilon_n}$ is collinear with the extraordinary ray direction.

In uniaxial left-handed media, $\overline{\varepsilon_n}$ also lies in the same plane that $\overline{\nu}$, $\overline{\varepsilon_\nu}$ and $\overline{n}$. Since the angle between $\overline{E}$ and $\overline{D}$ is the same as the angle between $\overline{t}$ and $\overline{n}$ and Cauchy's theorem [22] states that $\overline{\varepsilon_n}\cdot\overline{\nu}$ $(= \overline{n}\cdot\overline{\varepsilon_\nu})$ also vanishes, vector $\overline{\varepsilon_n}$ is collinear with $\overline{t}$, but may be antiparallel.

Our method is applied to four media with dielectric anisotropy and magnetic isotropy [4] with the following material parameters:

This kind of media are named *indefinite* because the eigenvalues of their permittivity tensors are not all the same sign.

Figure 3 shows the plot of the two shells (double-sheeted and one-sheeted hyperboloids) of Cauchy's quadric for dielectric parameters of

Medium	Class	ε_1	ε_2	ε_3	μ_r
"1"	RHM	1	1	-2	1
"2"	LHM	2	-1	-1	-1
"3"	RHM	2	-1	-1	1
"4"	LHM	1	1	-2	-1

media "1" and "4". The asymptotic cone that divides the space into two regions is also shown.

Let us consider three cases that deserve attention in the study of propagation of plane waves in these media.

3.1. Direction of Electric Field Is Known

This means that unit vector $\overline{\nu}$ is given. Then, allowed directions of $\overline{\nu}$ can only intercept the positive (or negative) shell of Cauchy's quadric, depending on the sign of μ_r. Asymptotic cone can be regarded as the boundary of allowed directions of $\overline{\nu}$ for RH and LH media. In this case, the graphical procedure is as follows:

1. Positive shell (if $\mu_r > 0$), or negative one (if $\mu_r < 0$) of Cauchy's quadric and unit vector $\overline{\nu}$ are drawn.
2. Let P be the intersection point of $\overline{\nu}$ direction with Cauchy's quadric. The gradient of Cauchy's quadric at P gives the direction of $\overline{\varepsilon_\nu}$. The angle between $\overline{\nu}$ and $\overline{\varepsilon_\nu}$ is θ.
3. To obtain phase velocity v_p or parameter a, it suffices to substitute coordinates of P into the equation of vector $\overline{D}$ cone.

Figure 4 shows the intersection between the one-sheeted surface of Cauchy's quadric and the vector $\overline{D}$ cone for a RH medium (medium "1") and for $v_p^2 = c^2/2$.

Because of the symmetry of uniaxial media, a two dimensional representation can also be used. Without loss of generality, medium "4" is considered. Meridian sections of the negative shell of Cauchy's quadric and direction of given vector $\overline{\nu}$ are depicted in Fig. 5.

3.2. Direction of Propagation ($\overline{n}$) Is Known

Let us outline the procedure in a 2-D plot.

1. Meridian sections of positive and negative shells of Cauchy's quadric are drawn.

2. Let Q be the intersection point of $\overline{n}$ direction with Cauchy's quadric. The gradient of Cauchy's quadric at Q gives the direction of $\overline{\varepsilon_n}$. Vector $\overline{t}$ is along $\overline{\varepsilon_n}$ or $-\overline{\varepsilon_n}$, depending on the sign of μ_r, as Eq. (2) states.

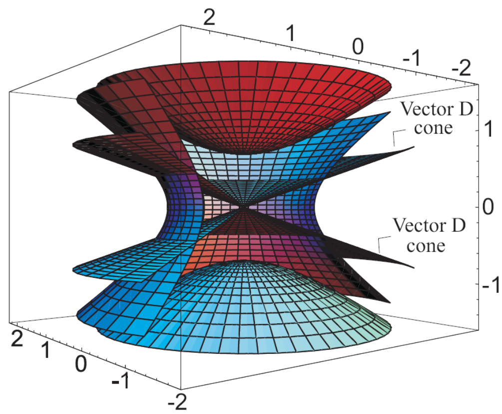

Figure 4. Intersection between the one-sheeted surface of Cauchy's quadric and the vector **D** cone for a RH medium (medium "1") and for $v_p^2 = c^2/2$.

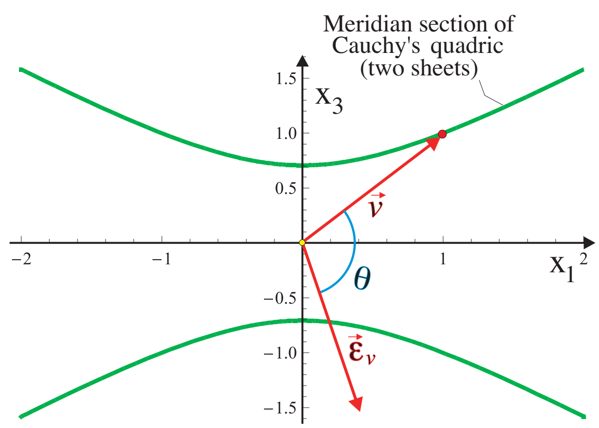

Figure 5. Plane graphical determination of angle θ between $\boldsymbol{\nu}$ (parallel to **E**) and $\boldsymbol{\varepsilon_\nu}$ (parallel to **D**), or between direction of propagation, **n** and extraordinary ray direction **t**, for a left-handed indefinite dielectric medium (uniaxial medium "4"), when $\boldsymbol{\nu}$ is given. Meridian section of Cauchy's quadric (two-sheeted) is shown.

Figure 6 shows the construction for medium “4” and a given direction of $\overline{n}$.

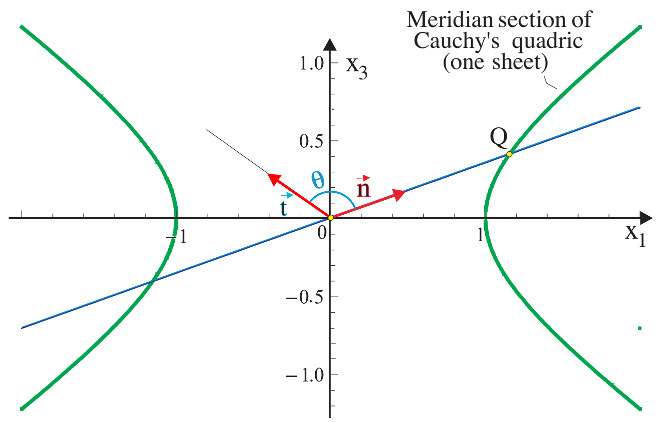

Figure 6. Plane graphical determination of angle θ from the knowledge of the unit wave-vector, $\mathbf{n}$, for a left-handed indefinite dielectric medium (uniaxial medium “4”). Meridian section of Cauchy’s quadric (one-sheeted) is shown. The normal line to Cauchy’s quadric at Q gives the direction of $\boldsymbol{\varepsilon_n}$. Unit vector $\mathbf{t}$ is along the extraordinary ray.

3.3. Inverse Problem: Phase Velocity Is Known

If v_p is known, parameter a is given. There are two procedures to find the angle between $\overline{t}$ and $\overline{n}$.

First procedure

1. Let us trace the Fresnel’s cone for this value of a (in this case symmetry of revolution allows to work in the meridian plane and, in this plane, the cone becomes a straight line).
2. Let A be the intersection of the Fresnel’s cone with the Cauchy’s quadric. The straight line OA is the direction of vector $\overline{n}$.
3. The gradient to the Cauchy’s quadric at point A is collinear with $\overline{\varepsilon_n}$. Sign of a determines which sense of $\overline{\varepsilon_n}$ corresponds to energy flow (ray direction).

Figure 7 describes the procedure for medium “1” and for $a = 1/3$.

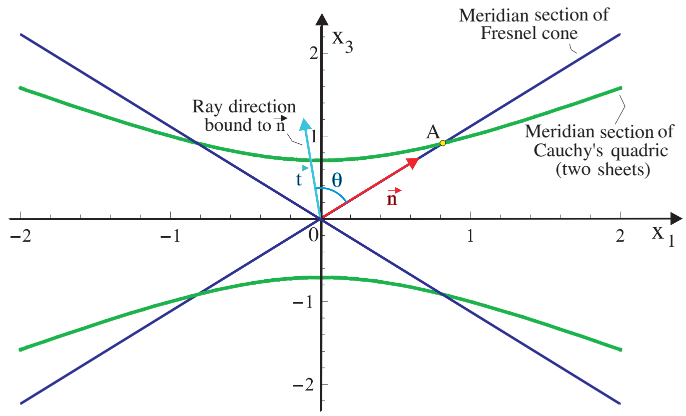

Figure 7. First graphical determination of angle θ from the knowledge of phase velocity, v_p, for a RH uniaxial indefinite medium (medium "1" with $v_p^2 = c^2/3$). Meridian sections of Fresnel's cone and Cauchy's quadric (two sheets) are shown.

Second procedure

1. Let us trace the vector $\overline{D}$ cone for the given value of a (As above, symmetry of revolution allows to work in the meridian plane and, in this plane, the cone becomes a straight line).
2. Let B be the intersection of the straight line with the shell of Cauchy's quadric corresponding to the sign of a. The straight line OB is the direction of vector $\boldsymbol{\nu}$.
3. The gradient to the Cauchy's quadric at point B is collinear with $\boldsymbol{\varepsilon_\nu}$. The angle between $\overline{\nu}$ and $\overline{\varepsilon_\nu}$ is θ.

The description of the procedure for the same medium ("1") and for the same value of a ($a = 1/3$) is shown in Fig. 8.

3.4. "Opposite" Media

One can observe that parameters of medium "2" and "3" are the opposite of those corresponding to "1" and "4". Thus an interesting property raises:

Property 4: For media with reversed values of ε_i and μ_r, values of θ (angle between $\overline{t}$ and $\overline{n}$) become a pair of supplementary angles for the same direction of propagation.

Proof: Because of the symmetry of uniaxial media, direction of propagation can be measured from the angle β that $\overline{n}$ forms with

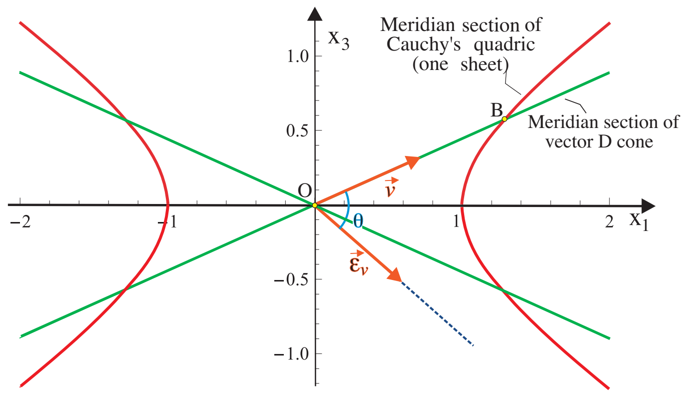

Figure 8. Second graphical procedure to find angle θ from the knowledge of phase velocity, v_p for a RH uniaxial indefinite medium (medium "1" with $v_p^2 = c^2/3$). Meridian sections of vector **D** cone and Cauchy's quadric (one sheet) are shown.

the optic axis. The optic axis is along the principal direction, which corresponds to the different eigenvalue.

Let us consider an indefinite material with these parameters: $\varepsilon_1 = \varepsilon_2 > 0$, $\varepsilon_3 < 0$, and $\mu_r = -1$. Then, the optic axis is along $\overline{u}_3$, and $\overline{n}$ can be written as $\overline{n} = \sin\beta\,\overline{u}_1 + \cos\beta\,\overline{u}_3$. Cauchy's quadric for this medium is the surface:

$$\varepsilon_1\, x_1^2 - |\varepsilon_3|\, x_3^2 = -1 \tag{17}$$

An unit vector normal to the Cauchy's quadric can be expressed as:

$$\overline{u}_G = \frac{2\,x_1\,\varepsilon_1\,\overline{u}_1 - 2\,x_3\,|\varepsilon_3|\overline{u}_3}{\sqrt{4\,\varepsilon_1^2\,x_1^2 + 4\,|\varepsilon_3|^2\,x_3^2}}$$

At the intersection point with the direction of propagation, $\overline{u}_G$ can be written as:

$$\overline{u}_G = \frac{2\,\sin\,\beta\,\varepsilon_1\,\overline{u}_1 - 2\,\cos\,\beta\,|\varepsilon_3|\overline{u}_3}{\sqrt{4\,\varepsilon_1^2\,\sin^2\,\beta + 4\,|\varepsilon_3|^2\,\cos^2\,\beta}}$$

and the dot product of $\overline{n}$ and $\overline{u}_G$ is equal to:

$$\overline{n}\cdot\overline{u}_G = \frac{2\,\sin^2\,\beta\,\varepsilon_1 - 2\,\cos^2\,\beta\,|\varepsilon_3|}{\sqrt{4\,\varepsilon_1^2\,\sin^2\,\beta + 4\,|\varepsilon_3|^2\,\cos^2\,\beta}}$$

On the other hand, for the *"opposite"* medium, new $\varepsilon_1' = |\varepsilon_3|$, $\varepsilon_2' = \varepsilon_3' = -\varepsilon_1$. But, in this case, the new optic axis is along $\overline{u}_1$, and direction of propagation is expressed as $\mathbf{n}' = \cos\beta\,\overline{u}_1 + \sin\beta\,\overline{u}_3$. Cauchy's quadric may be expressed in the form:

$$\varepsilon_1'\, x_1^2 - |\varepsilon_3'|\, x_3^2 = 1 \tag{18}$$

and an unit vector normal to this Cauchy's quadric at the intersection

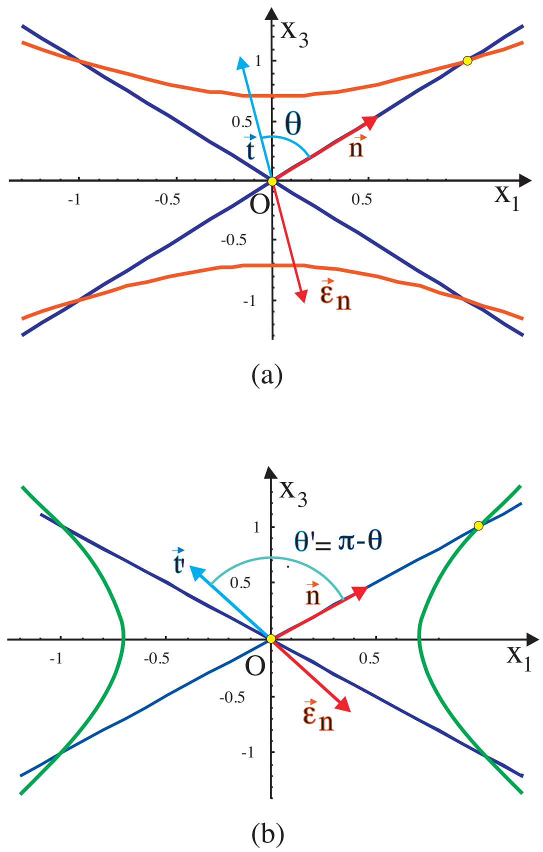

Figure 9. Ray tracing in *"opposite media"*. *"Opposite" media* are media with opposite values of their dielectric and magnetic parameters (eigenvalues ε_i of $\tilde{\varepsilon}$ and μ_r). For a given $\mathbf{n} = \mathbf{k}/k$, the angle θ between $\mathbf{n}$ and the extraordinary ray in medium "1", and the angle θ' between $\mathbf{n}$ and the extraordinary ray in medium "2" become a pair of supplementary angles ($\theta' = \pi - \theta$).

point becomes:

$$\mathbf{u}'_G = \frac{2\,\cos\,\beta\,|\varepsilon_3|\,\overline{u}_1 - 2\,\sin\,\beta\,|\varepsilon_1|\,\overline{u}_3}{\sqrt{4\,\varepsilon_3^2\,\cos^2\,\beta + 4\,\varepsilon_1^2\,\sin^2\,\beta}}$$

The dot product of $\overline{n}\cdot\overline{u}_G$ is then equal to $-\overline{n'}\cdot\overline{u'}_G$, and the corresponding angles become a pair of supplementary angles. Then, the respective rays for these media have opposite directions.

Figure 9 shows this property, for media "1" and "2".

4. CONCLUSIONS

An apparently new geometrical interpretation of basic equations of wave propagation in a medium with dielectric anisotropy and magnetic isotropy has been performed in terms of components of an unit vector (parallel to wave vector $\mathbf{k}$) referred to space $\mathcal{R}_3$, eigenvalues ε_i of $\tilde{\varepsilon}$, relative magnetic permeability μ_r, and phase velocity v_p.

We have shown that the locus described by Fresnel's equation of wave normals in the space of the director cosines of a direction is a cone. We have called it Fresnel cone, which is the locus of the directions of propagation of plane waves that are compatible with a given velocity of propagation in any given medium.

It must be pointed out that not all the directions of propagation along the physical medium are allowed, only those directions that lie on the Fresnel conical surface are physically possible.

Then it is found that the locus of another main equation describing the projection of vector $\mathbf{D}$ onto $\mathbf{E}$ is also a cone (named vector $\mathbf{D}$ cone).

Although the proposed methodology is obviously valid for biaxial media, in this paper we are only concerned with uniaxial media dealing with conics instead of quadrics in order to better visualize some properties. Moreover, this methodology is also valid for negative values of dielectric permittivities and magnetic permeability, it allows the study of propagation of locally plane electromagnetic waves in LHMs. Therefore, characteristic phenomena of LHMs, like *imperfect backward-wave* propagation have been easily explained. The correspondence with other analogous works has been also stated.

A detailed discussion on wave propagation along *indefinite dielectric media* has been carried out. As an example, four indefinite dielectric uniaxial media (RHMs and LHMs) with given parameters have been investigated.

Graphical constructions to obtain the extraordinary ray direction $\overline{t}$ from the knowledge of an unit wave-vector $\overline{n} = \overline{k}/k$ has been proposed in three interesting cases: when the direction of $\overline{E}$ is known, when $\overline{n}$ is known and when phase velocity, v_p is given. As a marked advantage, proposed graphical procedures allow a joint study of materials having the same eigenvalues of $\tilde{\varepsilon}$, but opposite values of μ_r. In this case, the associated Cauchy quadric has two shells (one-sheeted hyperboloid and a two-sheeted one).

Finally, the opposite sense of ray propagation in the so called *"opposite media"* (media with opposite values of ε_i and μ_r) has been also shown.

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