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FRACTIONAL RECTANGULAR CAVITY RESONATOR

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Abstract—Fractional curl operator has been used to derive solutions to the Maxwell equations for fractional rectangular cavity resonator. These solutions to the Maxwell equations may be regarded as fractional dual solutions. Behavior of field lines and surface current density in fractional cavity resonator have been investigated with respect to the fractional parameter. Fractional parameter describes the order of fractional curl operator.

1. INTRODUCTION

Ten years before, interest in exploring the roles and applications of fractional calculus [1] and fractional operators in electromagnetics led to fractionalization of curl operator, an operator which is commonly used in electromagnetics. It is represented by $\text{curl}^\alpha = (\nabla\times)^\alpha$ and is known as fractional curl operator [2]. Generally, the parameter α is noninteger. For $\alpha = 0$, the fractional curl operator becomes an identity operator. Whereas, the fractional curl operator transforms to conventional curl operator when $\alpha = 1$. When α ranges between 0 and 1, the fractional curl operator behaves as intermediate operator between identity operator and conventional/ordinary curl operator.

According to the following relations [2]

$$\mathbf{E}_{fd} = \left[(ik)^{-1}\nabla\times\right]^\alpha \mathbf{E}$$
$$\eta\mathbf{H}_{fd} = \left[(ik)^{-1}\nabla\times\right]^\alpha \eta\mathbf{H}$$

the fractional curl operator generates the fractional dual solution $(\mathbf{E}_{fd}, \eta\mathbf{H}_{fd})$ to the Maxwell equations. In above equations $\mathbf{E}$ and $\mathbf{H}$

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are electric and magnetic fields respectively. Quantities k and η are the wavenumber and impedance of the medium. For $\alpha = 0$, the above relations yield solution $(\mathbf{E}, \eta\mathbf{H})$ while for $\alpha = 1$, above relations yield solution $(\eta\mathbf{H}, -\mathbf{E})$. When α ranges between 0 and 1, above relations yield solutions which may be regarded as intermediate step between solution $(\mathbf{E}, \eta\mathbf{H})$ and solution $(\eta\mathbf{H}, -\mathbf{E})$. That is, intermediate between original solution to Maxwell equations and dual to the original solution to Maxwell equations.

Fractional curl operator has been applied by Naqvi and co-workers on variety of problems, e.g., fractional curl operator in chiral and bi-anisotropic medium, fractional dual solutions in metamaterials, fractional perfect electromagnetic structures, fractional waveguides and transmission lines etc. [3–15]. Valuable contributions on this topic are given by other authors [16–23].

In this paper, we have investigated fractional rectangular cavity resonator. Fractional fields and fractional surface current density in fractional rectangular cavity resonator has been studies.

2. FIELDS IN FRACTIONAL CAVITY RESONATOR

Consider a rectangular cavity resonator, constructed from a waveguide of rectangular cross-section having width a and height $b (a \geq b)$. The waveguide is closed by two perfectly conducting plates located at $z = 0$ and $z = d (d \geq a)$, forming a rectangular parallelepiped or rectangular cavity. Since both TM and TE modes can exist in a rectangular waveguide, we expect TM and TE modes in a rectangular cavity resonator too. For simplicity we choose the z-axis as the reference "direction of propagation". Actually, the existence of conducting walls at $z = 0$ and $z = d$ give rise to multiple reflections and set up standing waves. Therefore, no wave propagates in an enclosed cavity. Note that the longitudinal variation for wave traveling in the $+z$-direction and $-z$-direction are described by propagation factors $e^{-ik_z z}$ and $e^{ik_z z}$ respectively. Consider the TM_{mnp} mode in the rectangular cavity resonator. Where the three symbol $\{mnp\}$ subscript designate a TM or TE standing wave pattern in cavity resonator. Field expressions are [24]

$$\hat{z}E_z(x, y, z) = \hat{z}A_{mnp} \sin(k_x x) \sin(k_y y) \cos(k_z z) \tag{1a}$$

$$\hat{x}E_x(x, y, z) = -\hat{x}\frac{k_z k_x}{k_c^2} A_{mnp} \cos(k_x x) \sin(k_y y) \sin(k_z z) \tag{1b}$$

$$\hat{y}E_y(x, y, z) = -\hat{y}\frac{k_z k_y}{k_c^2} A_{mnp} \sin(k_x x) \cos(k_y y) \sin(k_z z) \tag{1c}$$

$$\hat{x}\eta H_x(x,y,z) = \hat{x}\frac{ikk_y}{k_c^2}A_{mnp}\sin(k_x x)\cos(k_y y)\cos(k_z z) \tag{1d}$$

$$\hat{y}\eta H_y(x,y,z) = -\hat{y}\frac{ikk_x}{k_c^2}A_{mnp}\cos(k_x x)\sin(k_y y)\cos(k_z z) \tag{1e}$$

where $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$ and $k_z = \frac{p\pi}{d}$, $m = n = p = 0, 1, 2\ldots$ and

$$k_c = \sqrt{k^2 - k_x^2 - k_y^2 - k_z^2}.$$

We may develop TM_{mnp} in cavity resonator as the linear combination of $+z$-direction and $-z$-direction traveling TM rectangular waveguide mode, and is given by

$$\hat{z}E_z^+(x,y,z) = \hat{z}\frac{A_{mn}}{2}\sin(k_x x)\sin(k_y y)\exp(-ik_z z) \tag{2a}$$

$$\hat{x}E_x^+(x,y,z) = -\hat{x}\frac{ik_z k_x}{2k_c^2}A_{mn}\cos(k_x x)\sin(k_y y)\exp(-ik_z z) \tag{2b}$$

$$\hat{y}E_y^+(x,y,z) = -\hat{y}\frac{ik_z k_y}{2k_c^2}A_{mn}\sin(k_x x)\cos(k_y y)\exp(-ik_z z) \tag{2c}$$

$$\hat{x}\eta H_x^+(x,y,z) = \hat{x}\frac{ikk_y}{2k_c^2}A_{mn}\sin(k_x x)\cos(k_y y)\exp(-ik_z z) \tag{2d}$$

$$\hat{y}\eta H_y^+(x,y,z) = -\hat{y}\frac{ikk_x}{2k_c^2}A_{mn}\cos(k_x x)\sin(k_y y)\exp(-ik_z z) \tag{2e}$$

and

$$\hat{z}E_z^-(x,y,z) = \hat{z}\frac{A_{mn}}{2}\sin(k_x x)\sin(k_y y)\exp(ik_z z) \tag{3a}$$

$$\hat{x}E_x^-(x,y,z) = \hat{x}\frac{ik_z k_x}{2k_c^2}A_{mn}\cos(k_x x)\sin(k_y y)\exp(ik_z z) \tag{3b}$$

$$\hat{y}E_y^-(x,y,z) = \hat{y}\frac{ik_z k_y}{2k_c^2}A_{mn}\sin(k_x x)\cos(k_y y)\exp(ik_z z) \tag{3c}$$

$$\hat{x}\eta H_x^-(x,y,z) = \hat{x}\frac{ikk_y}{2k_c^2}A_{mn}\sin(k_x x)\cos(k_y y)\exp(ik_z z) \tag{3d}$$

$$\hat{y}\eta H_y^-(x,y,z) = -\hat{y}\frac{ikk_x}{2k_c^2}A_{mn}\cos(k_x x)\sin(k_y y)\exp(ik_z z) \tag{3e}$$

For $+z$-directed wave, the corresponding fractional field $(\mathbf{E}_{fd}^+, \eta\mathbf{H}_{fd}^+)$ are given by [using (11a) and (11b) in 2] are

$$\mathbf{E}_{fd}^+ = \hat{x}E_{xfd}^+ + \hat{y}E_{yfd}^+ + \hat{z}E_{zfd}^+ \tag{4a}$$

$$\eta\mathbf{H}_{fd}^+ = \hat{x}\eta H_{xfd}^+ + \hat{y}\eta H_{yfd}^+ + \hat{z}\eta H_{zfd}^+ \tag{4b}$$

where

$$E_{xfd}^{+}=-i\frac{A_{mn}}{2k_c^2}\left\{k_zk_x\cos\left(\alpha\frac{\pi}{2}\right)+kk_y\sin\left(\alpha\frac{\pi}{2}\right)\right\}$$
$$\times\cos\left(k_xx-\alpha\frac{\pi}{2}\right)\sin\left(k_yy-\alpha\frac{\pi}{2}\right)\exp\left(-ik_zz\right) \tag{5a}$$

$$E_{yfd}^{+}=-i\frac{A_{mn}}{2k_c^2}\left\{k_zk_y\cos\left(\alpha\frac{\pi}{2}\right)-kk_x\sin\left(\alpha\frac{\pi}{2}\right)\right\}$$
$$\times\sin\left(k_xx-\alpha\frac{\pi}{2}\right)\cos\left(k_yy-\alpha\frac{\pi}{2}\right)\exp\left(-ik_zz\right) \tag{5b}$$

$$E_{zfd}^{+}=\frac{A_{mn}}{2}\cos\left(\alpha\frac{\pi}{2}\right)\sin\left(k_xx-\alpha\frac{\pi}{2}\right)\sin\left(k_yy-\alpha\frac{\pi}{2}\right)\exp\left(-ik_zz\right) \tag{5c}$$

and

$$\eta H_{xfd}^{+} = i\frac{A_{mn}}{2k_c^2}\left\{kk_y\cos\left(\alpha\frac{\pi}{2}\right)-k_zk_x\sin\left(\alpha\frac{\pi}{2}\right)\right\}$$
$$\times\sin\left(k_xx-\alpha\frac{\pi}{2}\right)\cos\left(k_yy-\alpha\frac{\pi}{2}\right)\exp\left(-ik_zz\right) \tag{5d}$$

$$\eta H_{yfd}^{+} = -i\frac{A_{mn}}{2k_c^2}\left\{kk_x\cos\left(\alpha\frac{\pi}{2}\right)+k_zk_y\sin\left(\alpha\frac{\pi}{2}\right)\right\}$$
$$\times\cos\left(k_xx-\alpha\frac{\pi}{2}\right)\sin\left(k_yy-\alpha\frac{\pi}{2}\right)\exp\left(-ik_zz\right) \tag{5e}$$

$$\eta H_{zfd}^{+} = -\frac{A_{mn}}{2}\sin\left(\alpha\frac{\pi}{2}\right)\cos\left(k_xx-\alpha\frac{\pi}{2}\right)\cos\left(k_yy-\alpha\frac{\pi}{2}\right)\exp(-ik_zz) \tag{5f}$$

Similarly, for $-z$-directed wave, the corresponding fractional field $(\mathbf{E}_{fd}^{-}, \eta\mathbf{H}_{fd}^{-})$ becomes

$$\mathbf{E}_{fd}^{-} = \hat{x}E_{xfd}^{-}+\hat{y}E_{yfd}^{-}+\hat{z}E_{zfd}^{-} \tag{6a}$$
$$\eta\mathbf{H}_{fd}^{-} = \hat{x}\eta H_{xfd}^{-}+\hat{y}\eta H_{yfd}^{-}+\hat{z}\eta H_{zfd}^{-} \tag{6b}$$

where

$$E_{xfd}^{-}=-i\frac{A_{mn}}{2k_c^2}\left\{-k_zk_x\cos\left(\alpha\frac{\pi}{2}\right)+kk_y\sin\left(\alpha\frac{\pi}{2}\right)\right\}$$
$$\times\cos\left(k_xx-\alpha\frac{\pi}{2}\right)\sin\left(k_yy-\alpha\frac{\pi}{2}\right)\exp\left(ik_zz\right) \tag{7a}$$

$$E_{yfd}^{-}=-i\frac{A_{mn}}{2k_c^2}\left\{-k_zk_y\cos\left(\alpha\frac{\pi}{2}\right)-kk_x\sin\left(\alpha\frac{\pi}{2}\right)\right\}$$
$$\times\sin\left(k_xx-\alpha\frac{\pi}{2}\right)\cos\left(k_yy-\alpha\frac{\pi}{2}\right)\exp\left(ik_zz\right) \tag{7b}$$

$$E_{zfd}^{-} = \frac{A_{mn}}{2} \cos\left(\alpha \frac{\pi}{2}\right) \sin\left(k_x x - \alpha \frac{\pi}{2}\right) \sin\left(k_y y - \alpha \frac{\pi}{2}\right) \exp\left(i k_z z\right) \tag{7c}$$

and

$$\begin{aligned} \eta H_{xfd}^{-} &= i \frac{A_{mn}}{2k_c^2} \left\{ k k_y \cos\left(\alpha \frac{\pi}{2}\right) + k_z k_x \sin\left(\alpha \frac{\pi}{2}\right) \right\} \\ &\quad \times \sin\left(k_x x - \alpha \frac{\pi}{2}\right) \cos\left(k_y y - \alpha \frac{\pi}{2}\right) \exp\left(i k_z z\right) \end{aligned} \tag{7d}$$

$$\begin{aligned} \eta H_{yfd}^{-} &= -i \frac{A_{mn}}{2k_c^2} \left\{ k k_x \cos\left(\alpha \frac{\pi}{2}\right) - k_z k_y \sin\left(\alpha \frac{\pi}{2}\right) \right\} \\ &\quad \times \cos\left(k_x x - \alpha \frac{\pi}{2}\right) \sin\left(k_y y - \alpha \frac{\pi}{2}\right) \exp\left(i k_z z\right) \end{aligned} \tag{7e}$$

$$\eta H_{zfd}^{-} = -\frac{A_{mn}}{2} \sin\left(\alpha \frac{\pi}{2}\right) \cos\left(k_x x - \alpha \frac{\pi}{2}\right) \cos\left(k_y y - \alpha \frac{\pi}{2}\right) \exp\left(i k_z z\right) \tag{7f}$$

The total fractional dual solution of TM_{mnp} mode in the fractional rectangular cavity resonator becomes

$$\mathbf{E}_{fd} = \hat{x} E_{xfd} + \hat{y} E_{yfd} + \hat{z} E_{zfd} \tag{8a}$$

$$\eta \mathbf{H}_{fd} = \hat{x} \eta H_{xfd} + \hat{y} \eta H_{yfd} + \hat{z} \eta H_{zfd} \tag{8b}$$

where

$$E_{xfd} = E_{xfd}^{+} + E_{xfd}^{-} \tag{9a}$$

$$E_{yfd} = E_{yfd}^{+} + E_{yfd}^{-} \tag{9b}$$

$$E_{zfd} = E_{zfd}^{+} + E_{zfd}^{-} \tag{9c}$$

$$\eta H_{xfd} = \eta H_{xfd}^{+} + \eta H_{xfd}^{-} \tag{9d}$$

$$\eta H_{yfd} = \eta H_{yfd}^{+} + \eta H_{yfd}^{-} \tag{9e}$$

$$\eta H_{zfd} = \eta H_{zfd}^{+} + \eta H_{zfd}^{-} \tag{9f}$$

For $\alpha = 0$,

$$E_{zfd} = A_{mnp} \sin\left(k_x x\right) \sin\left(k_y y\right) \cos\left(k_z z\right) = E_z \tag{10a}$$

$$E_{xfd} = -\frac{k_z k_x}{k_c^2} A_{mnp} \cos\left(k_x x\right) \sin\left(k_y y\right) \sin\left(k_z z\right) = E_x \tag{10b}$$

$$E_{yfd} = -\frac{k_z k_y}{k_c^2} A_{mnp} \sin\left(k_x x\right) \cos\left(k_y y\right) \sin\left(k_z z\right) = E_y \tag{10c}$$

$$\eta H_{xfd} = i\frac{kk_y}{k_c^2}A_{mnp}\sin(k_x x)\cos(k_y y)\cos(k_z z) = \eta H_x \quad (10d)$$

$$\eta H_{yfd} = -i\frac{kk_x}{k_c^2}A_{mnp}\cos(k_x x)\sin(k_y y)\cos(k_z z) = \eta H_y \quad (10e)$$

$$\eta H_{zfd} = 0 = \eta H_z \quad (10f)$$

gives the original field solution of TM mode in rectangular cavity resonator with PEC walls. For $\alpha = 1$, the field behavior changes from TM mode to TE mode in cavity resonator with PMC walls. In other words we can interpret the solution as the dual of the original solution that satisfies the Maxwell's equations and is given below

$$E_{zfd} = 0 = \eta H_z \quad (11a)$$

$$E_{xfd} = \frac{ikk_y}{k_c^2}A_{mnp}\sin(k_x x)\cos(k_y y)\cos(k_z z) = \eta H_x \quad (11b)$$

$$E_{yfd} = -\frac{ikk_x}{k_c^2}A_{mnp}\cos(k_x x)\sin(k_y y)\cos(k_z z) = \eta H_y \quad (11c)$$

$$\eta H_{xfd} = \frac{k_z k_x}{k_c^2}A_{mnp}\cos(k_x x)\sin(k_y y)\sin(k_z z) = -E_x \quad (11d)$$

$$\eta H_{yfd} = \frac{k_z k_y}{k_c^2}A_{mnp}\sin(k_x x)\cos(k_y y)\sin(k_z z) = -E_y \quad (11e)$$

$$\eta H_{zfd} = A_{mnp}\sin(k_x x)\sin(k_y y)\cos(k_z z) = -E_z \quad (11f)$$

For $0 < \alpha < 1$, the fields given by (9) describe the fractional dual solution between two solutions given by (10) and (11). Which 'in other sense' replicates the intermediate fractional behavior between PEC and PMC cavities.

3. NUMERICAL ANALYSIS OF FRACTIONAL FIELDS

To study the behavior of fractional TM_{mnpfd} fields in dielectric filled rectangular cavity resonator of relative permittivity $\epsilon_r = 2$ and relative permeability $\mu_r = 1$, we have carried out the numerical simulation of $(2 \times 1 \times 3)$ cm rectangular cavity resonator at a frequency 35 GHz. We have considered only TM_{111fd} mode because of its dominance in cavity resonator. In case of cavity, triply infinite number of resonant frequencies correspond to different field distributions. The resonant frequencies of different mode in cavity can be measured using the following relation

$$f_{mnp} = \frac{c_0}{2\sqrt{\mu_r \epsilon_r}} \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2 \right]^{\frac{1}{2}}$$

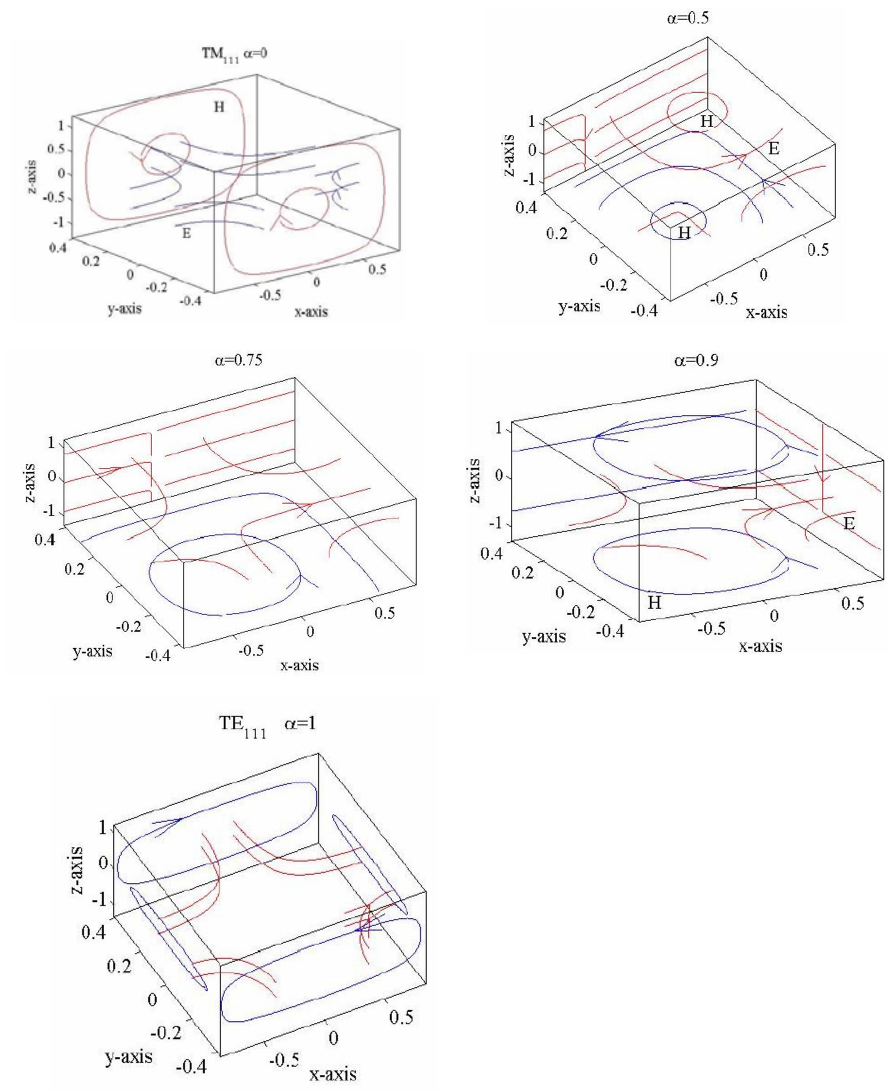
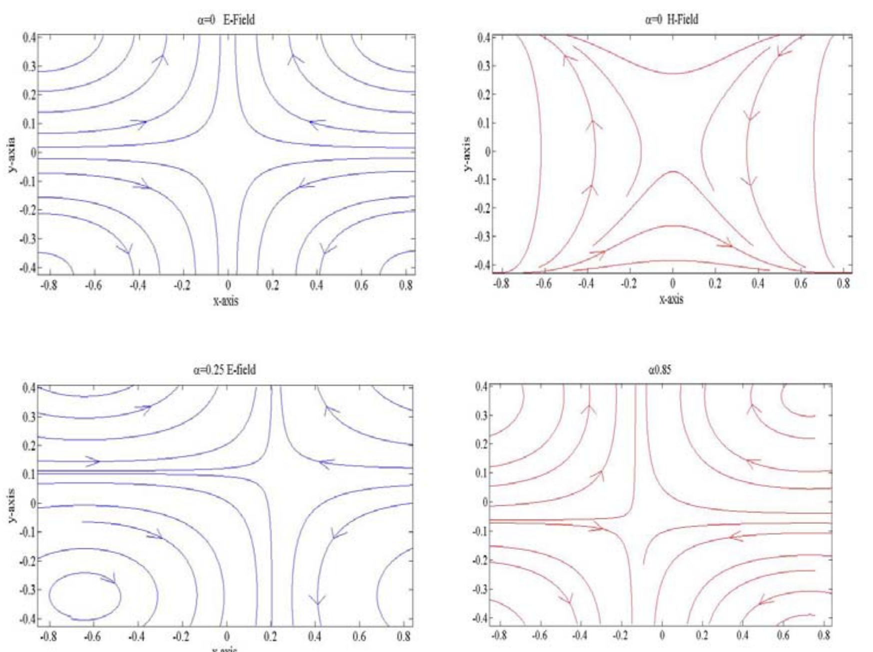


Figure 1. 3-D fractional TM_{111} standing wave field patterns, for $\alpha = 0$, 0.5, 0.75, 0.85, 0.9, 1.

where c_0 is the velocity of light. The indices m, n, p in resonance frequency relation refer to the number of variations in the standing-wave pattern in the x, y and z axes, respectively. When $\alpha = 0$, the simulated result yield the original field pattern of TM mode in PEC walls cavity as shown in Fig. 1. The plot reports that the field forming loops in xz-plane is tangential magnetic field (red coloured), whereas the electric field (blue coloured) lies in xy-plane, is normal to the adjoining planes. For $\alpha = 1$, the whole situation changes, in such a way that the TM mode in PEC walls cavity, transform to TE mode in PMC walls cavity. Besides this, the field pattern rotates in counter-clockwise direction by $\alpha\pi/2$. So that the normal electric field (blue coloured) transform to tangential electric field in form of loops in xz-plane and the tangential magnetic field (red coloured) to normal magnetic field, which is the property of PMC material. For $0 < \alpha < 1$, we get intermediate effects between the above mentioned results.

Results for $0 < \alpha < 1$, are not very much clear. To highlight the behavior of fractional dual results, we have carried out the numerical results in 2-D plane as shown in Fig. 2 and intermediate steps can be visualized very easily.



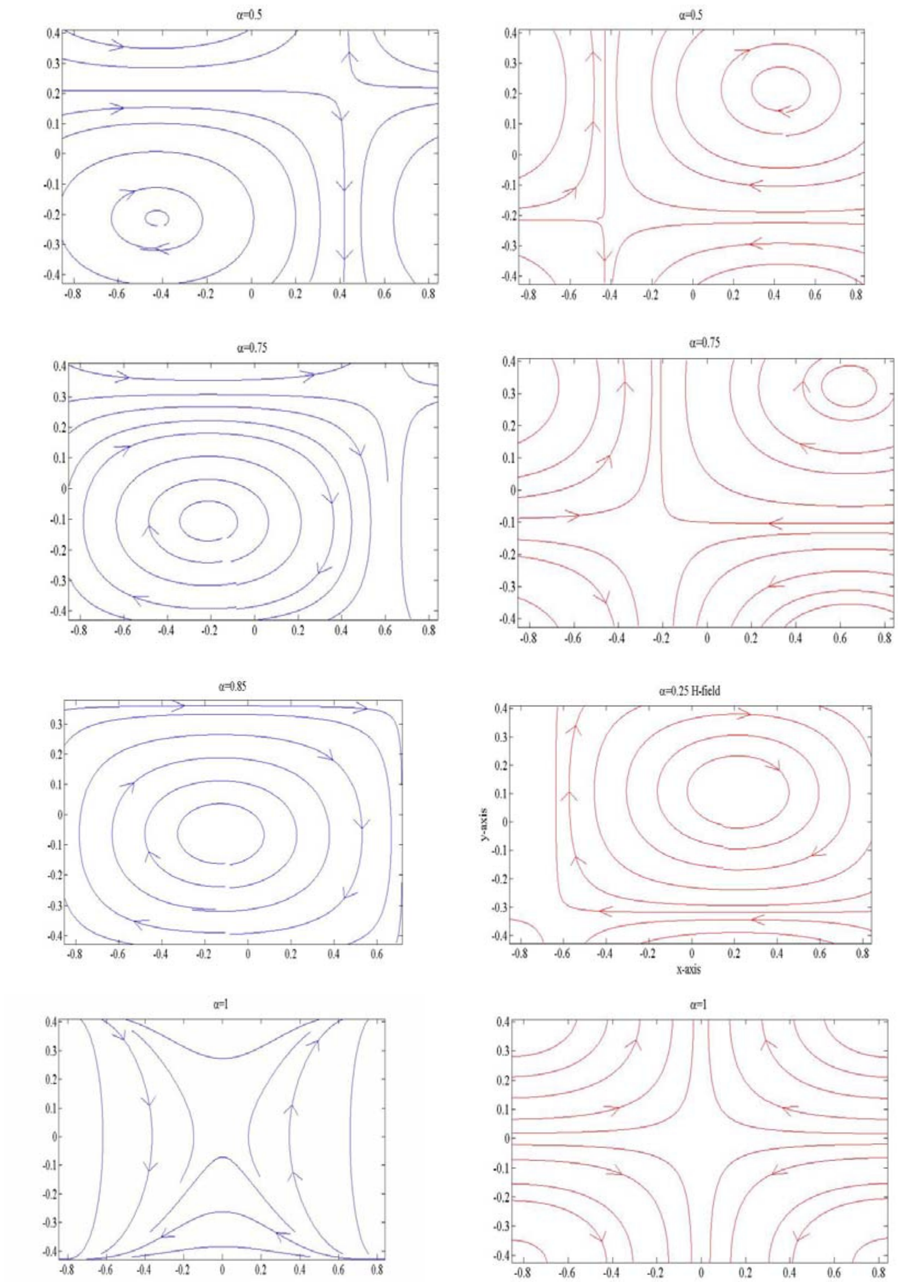


Figure 2. Fractional TM_{111} standing wave field patterns in 2-D at $z =$ constant plane for various values of fractional order α.

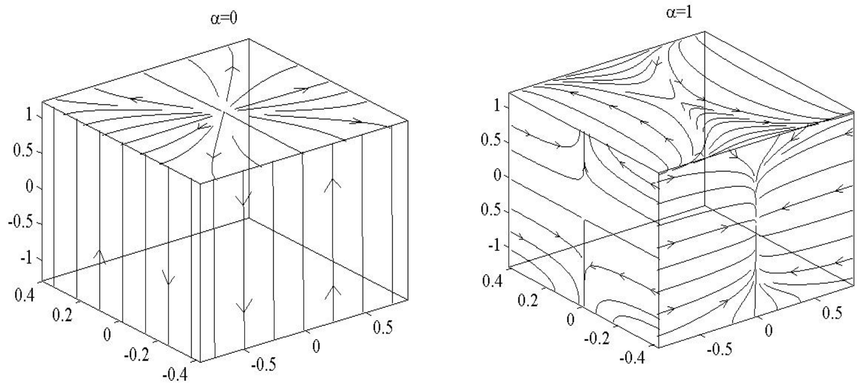


Figure 3. 3-D fractional surface current density wave patterns for two different values of α. i.e., $\alpha = 0$ and $\alpha = 1$, represent electric and magnetic surface current densities.

4. FRACTIONAL SURFACE CURRENT DENSITY

Surface current density on walls of the fractional cavity resonator is obtained using fractional TM_{111} fields inside the cavity resonator. Following relation has been used to obtained the fractional surface current density $\mathbf{J_{sfd}}$

$$\mathbf{J_{sfd}} = \hat{\mathbf{n}} \times \mathbf{H_{fd}}$$

where $\hat{\mathbf{n}}$ is the outward normal to the walls of cavity and $\mathbf{H_{fd}}$ is the fractional magnetic field intensity on the walls. In components form, we can write

$$J_{sfd}(x=0) = -\hat{y}H_{zfd} + \hat{z}H_{yfd} = -J_{sfd}(x=a) \quad (12a)$$
$$J_{sfd}(y=0) = \hat{x}H_{zfd} - \hat{z}H_{zfd} = -J_{sfd}(y=b) \quad (12b)$$
$$J_{sfd}(z=0) = -\hat{x}H_{yfd} + \hat{y}H_{xfd} = -J_{sfd}(z=d) \quad (12c)$$

In the above relations we have assumed $\cos(k_\xi \xi - \frac{\alpha\pi}{2}) \approx 1$ for each surface. Where ξ is x or y or z. To elaborate a clear picture of fractional surface current density, the results are represented in 3-dimensional and 2-dimensional graphs as shown in Figs. 3 to 6. For $\alpha = 0$ and $\alpha = 1$, the results are explicitly given in 3-D. These results report that electric surface current density transform to magnetic surface current density when α changes from 0 to 1. Which also reveal the change of boundary condition, from $\mathbf{J}_{sfd} = \hat{\mathbf{n}} \times \mathbf{H}_{fd} = \hat{\mathbf{n}} \times \mathbf{H} = \mathbf{J}_s^e$

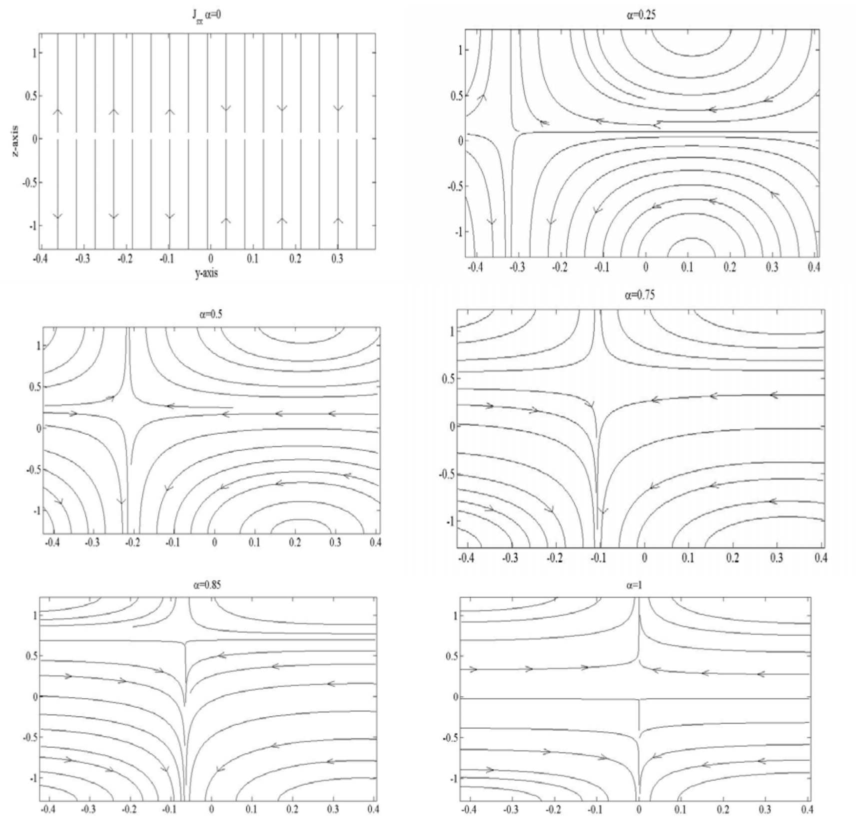
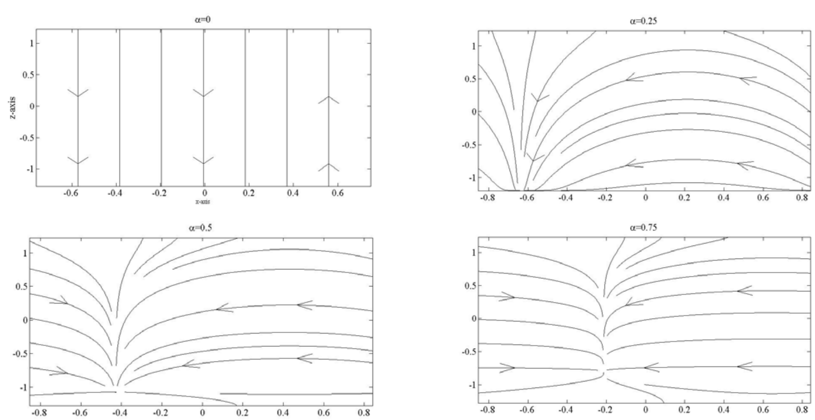


Figure 4. 2-D fractional surface current density wave patterns in yz-plane for various values of fractional order α.



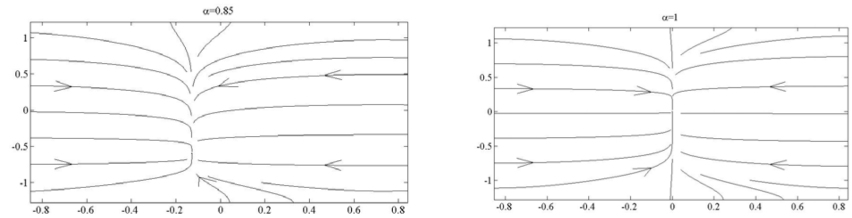


Figure 5. 2-D fractional surface current density wave patterns in xz-plane for various values of fractional order α.

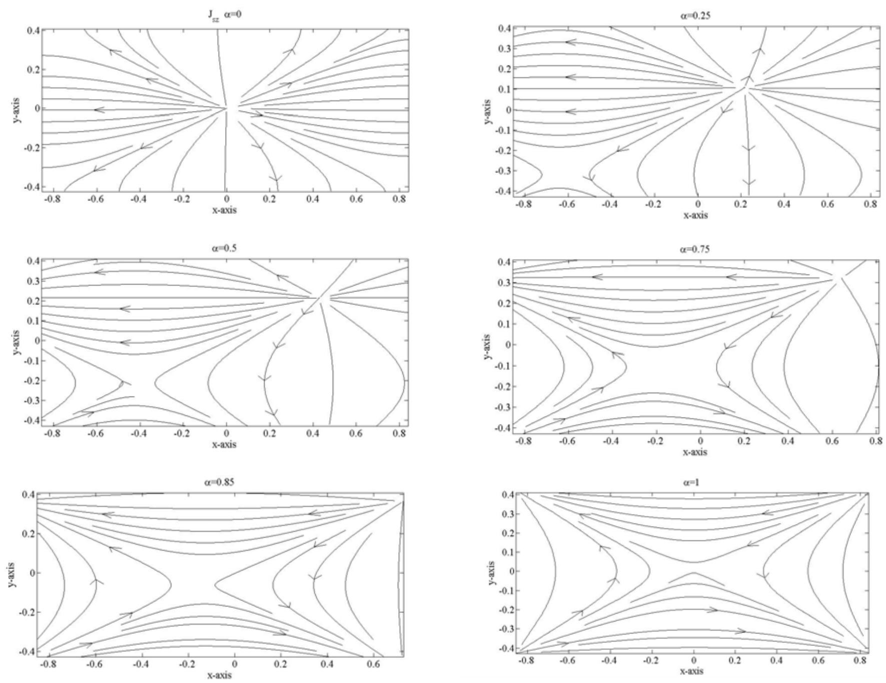


Figure 6. 2-D fractional surface current density in xy-plane for various values of fractional order α.

to $\mathbf{J}_{sfd} = \hat{\mathbf{n}} \times \mathbf{H}_{fd} = -\hat{n} \times \mathbf{E} = \mathbf{J}_s^m$. For $0 < \alpha < 1$, the plots are reported in 2-D as shown in Fig. 4, Fig. 5 and Fig. 6, respectively, which corresponds to intermediate steps between the electric current density $\mathbf{J}_s^e$ and magnetic current density $\mathbf{J}_s^m$.

5. CONCLUSIONS

In this paper we have discussed the field behavior as well as behavior of surface current density in fractional rectangular cavity resonator. When $\alpha = 0$, we get the original TM field and electric surface current density in PEC walls Cavity resonator. For $\alpha = 1$ the TM field and electric surface current density behavior in PEC walls cavity resonator change to TE field and magnetic surface current density in PMC walls cavity, respectively. For $0 < \alpha < 1$, we get intermediate steps between the two canonical cases.

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