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# THE ELECTRICAL RESPONSE OF AN INSULATING CIRCULAR DISK TO UNIFORM FIELDS

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**Abstract**—This paper presents a closed form solution for the electrical potential perturbation of a perfectly insulating flat circular disc embedded in a homogeneous half-space in a uniform primary electric field. This has applications to the detection of insulating underground targets such as hydrocarbon reservoirs or plastic mines, as well as to the electrical properties of composite materials. The solution method is an adaptation of Weber's method for the potential around a charged conducting disk. It yields closed form analytic solutions for the electric and magnetic fields and by straightforward numerical integration, an easily evaluated numerical solution for the electric potential and an explicit solution for the electrical resistivity of a composite material consisting of a dilute concentration of such embedded disks in an otherwise uniform conductor.

## 1. INTRODUCTION

An insulating object in a conducting medium perturbs the flow of an imposed current density. Such perturbations are important in a number of fields: the subsurface imaging by electrical methods of insulating underground structures such as hydrocarbon reservoirs (on the large scale) or plastic mines (on the small scale), or the computation of the electrical resistivity of composite materials. There are solutions for related problems for a disk more conductive than its surroundings in the literature, such as closed form solutions [20] for a perfectly conducting disk in a uniform normal electric field in free space, series expansions [16, 13] for a conductive disk subject to electromagnetic wave excitation, or solutions based on short-wavelength approximations [9]. Insulating disks in a conductive medium do not seem to have received similar analytic attention, although iterative techniques have no trouble dealing with them analytically, even in quite complex situations (*e.g.* [6])

In general, the availability of modern computing power means that numerical solutions for the electrical responses of realistically complex objects of this type are most efficiently done using (surface or volume) finite element methods. However, to demonstrate the essential properties of such solutions, as well as to calibrate and test such numerical solvers, it is useful to have simple closed form analytic or semi-analytic solutions as test cases. I present such a solution here: the electrical potential perturbation of a perfectly insulating flat circular disc embedded in a homogeneous half-space in a uniform primary electric field.

## 2. PROBLEM DEFINITION

### 2.1. The Geometry

Let the disc radius be $a$. Without loss of generality, let it be centred at the coordinate origin $\mathbf{r} = 0$ with surface normal in the $z$ direction. Assume that in the absence of the disc, the source would, in the vicinity of the coordinate origin, generate a uniform current density $\mathbf{j}_0$ (the "primary" current current density), which exists in the region of interest (that is, out to several disc radii from the origin). For a sufficiently small disc, or distant source, this will be a good approximation to reality. The current density parallel to the disc surface will be unperturbed by the disc, and thus, without loss of generality, we can assume it to be zero, and the background current to be of uniform amplitude $j_0$ and directed in the $-z$ direction (corresponding to a distant current source directly above the origin). Defining the electrical conductivity of the medium as $\sigma$, the corresponding primary electric potential $\Phi^{(P)}(x, y, z)$ is

$$\Phi^{(P)} = \frac{j_0 z}{\sigma} \tag{1}$$

### 2.2. The Boundary Conditions

The total electric potential $\Phi$ must obey Laplace's equation in the medium, and satisfy three boundary conditions. These are 1) that the potential tend to $\Phi^{(P)}$ as $\mathbf{r}$ goes to infinity, and 2) that the normal current density at the surface of the disk is zero. That is,

$$\lim_{|\mathbf{r}|\to\infty} \Phi(\mathbf{r}) = \Phi^{(P)} \tag{2}$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = 0 \qquad \text{for} \qquad \rho \equiv \sqrt{x^2+y^2} \leq a \tag{3}$$

## 3. FORMULATION OF THE PROBLEM FOR THE SECONDARY POTENTIAL

It is convenient to split $\Phi$ into that which the current source would produce in the homogeneous half-space in the absence of the disk (the primary potential $\Phi^{(P)} = z$), and the change produced by introducing the disk (the "secondary" potential $\Phi^{(S)}$). Each of these individually must satisfy Laplace's equation.

The primary potential satisfies boundary condition (2) but not (3). Its current density normal to the disk is not zero but $-j_0$. Therefore, in order that the total potential $\Phi$ satisfy the required boundary conditions (2) and (3), it is necessary that the secondary potential satisfies the boundary conditions

$$\lim_{|\mathbf{r}|\to\infty} \Phi^{(S)}(\mathbf{r}) = 0 \tag{4}$$

$$\left.\frac{\partial\Phi^{(S)}}{\partial z}\right|_{z=0} = -\frac{j_0 z}{\sigma} \qquad \text{for} \qquad \rho \leq a \tag{5}$$

Because the secondary potential tends to that of a point current dipole at large distances, we expect $\Phi$ and $|\nabla\Phi|$ to fall off as $r^{-2}$ and $r^{-3}$ respectively as $r$ tends to infinity.

By standard methods [14], a solution for $\Phi^{(S)}$ that satisfies (4) with the required azimuthal symmetry is

$$\Phi^{(S)} = \int_0^\infty A(\lambda) e^{-\lambda|z|} J_0(\lambda\rho) d\lambda \tag{6}$$

The vertical current density (normal to the disk), on the surface of the disk, which is generated by this potential, is

$$j_z = \sigma E_z = -\sigma \left.\frac{\partial\Phi^{(S)}}{\partial z}\right|_{z=0} = \sigma \int_0^\infty \lambda A(\lambda) J_0(\lambda\rho) d\lambda \tag{7}$$

To solve the problem, our objective now is to find the function $A(\lambda)$ such that boundary condition (5) is also satisfied, namely that for

$$\frac{j_0}{\sigma} = const. = \int_0^\infty \lambda A(\lambda) J_0(\lambda\rho) d\lambda \qquad 0 < \rho < a \tag{8}$$

## 4. THE SOLUTION FOR THE SECONDARY POTENTIAL

It is tempting to closely follow Weber's [20] original procedure for the conducting disk (see [4, 14, 19]), but this fails because it does not generate a solution which falls off sufficiently rapidly at infinity. Therefore instead of utilizing just the integral $\mathcal{I}_1(\rho)$ (11.4.37 in [1] with $\mu$ set to zero) defined by

$$\int_0^\infty \frac{\sin\lambda a}{\lambda} J_0(\lambda\rho)d\lambda = \frac{\pi}{2}, \qquad 0<\rho<a \tag{9}$$

$$= \arcsin\frac{a}{\rho}, \quad 0<a<\rho \tag{10}$$

as in the conducting disk problem, as Weber did, we utilize a linear combination of it and the integral $\mathcal{I}_2(\rho)$ (11.4.35 in [1] again with $\mu$ set to zero)

$$\int_0^\infty \cos\lambda a J_0(\lambda\rho)d\lambda = 0, \qquad 0<\rho<a \tag{11}$$

$$= \frac{1}{\sqrt{\rho^2-a^2}}, \; 0<a<\rho \tag{12}$$

It is easily seen from (11) and (9) that $\mathcal{I}_1(\rho)-a\mathcal{I}_2(\rho)$ equals a constant $\pi/2$ for $\rho<a$. It is also easily shown that this falls off as $\rho^{-3}$ as $\rho$ tends to infinity (see the section below on the equivalent dipole moment for details), and can be used to represent $E_z$ for $z=0$. It therefore follows that the function $A(\lambda)$ that we require in (8) is

$$A(\lambda) = \frac{2j_0}{\pi\sigma}\left(\frac{\sin\lambda a}{\lambda^2} - \frac{a\cos\lambda a}{\lambda}\right) \tag{13}$$

The uniqueness of solutions of Laplace's equation for completely specified boundary conditions assures us that this is the correct solution.

The required solution for the secondary potential $\Phi^{(S)}(\rho,z)$ everywhere is therefore, as a Hankel transform,

$$\frac{2j_0}{\sigma\pi}\int_0^\infty \left(\frac{\sin\lambda a}{\lambda^2} - \frac{a\cos\lambda a}{\lambda}\right) e^{-\lambda|z|}J_0(\lambda\rho)d\lambda \tag{14}$$

This has a square root singularity at the edge of the disk, limiting to zero as $\rho$ approaches $a$ from below as $\sqrt{a-\rho}$. The electric fields derived from this will necessarily have divergences of the form $(a-\rho)^{-1/2}$ at the disk edge.

## 5. CLOSED FORM SOLUTIONS FOR THE FIELDS

There are three field components which occur in this problem: the vertical electric field $E_z$, the horizontal (radial) electric field $E_\rho$, and the horizontal (azimuthal) magnetic field $H_\phi$. In this section, we obtain closed form solutions for all three by identifying their Hankel transforms as known tabulated integrals (with some care because of the significant number of erroneous entries in many tables).

### 5.1. The Vertical Electric Field

The vertical secondary electric field is the negative of the $z$ derivative of (14) with respect to $z$. It is

$$E_z^{(S)} = \frac{2j_0}{\sigma\pi}\int_0^\infty \left(\frac{\sin\lambda a}{\lambda} - a\cos\lambda a\right) e^{-\lambda|z|} J_0(\lambda\rho)d\lambda \tag{15}$$

Using the tabulated integrals 2.12.25.1 and 2.12.25.4 in [17], with appropriately changed notation,

$$\int_0^\infty e^{-\lambda|z|}\cos\lambda a J_0(\lambda\rho)d\lambda = \frac{\sqrt{(R_+ + R_-)^2 - 4a^2}}{2R_+R_-} \tag{16}$$

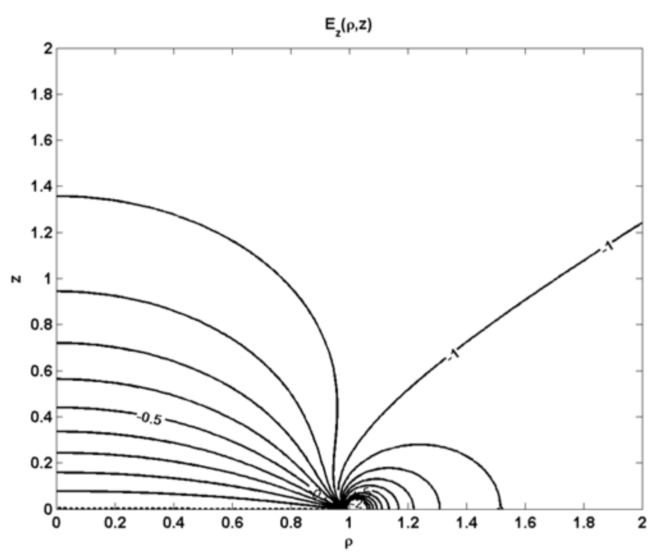


**Figure 1.** Contours of $E_z$ in the $\rho$-$z$ plane. The disk has unit radius and lies between the origin and $\rho = 1$.

and

$$\int_0^\infty e^{-\lambda|z|}\frac{\sin\lambda a}{\lambda}J_0(\lambda\rho)d\lambda = \arcsin\left(\frac{2a}{R_+ + R_-}\right) \tag{17}$$

where

$$R_\pm = \sqrt{z^2 + (\rho \pm a)^2} \tag{18}$$

the vertical secondary electric field can be written in closed form as

$$\frac{2j_0}{\sigma\pi}\left[\arcsin\left(\frac{2a}{R_+ + R_-}\right) - \frac{a\sqrt{(R_+ + R_-)^2 - 4a^2}}{2R_+R_-}\right] \tag{19}$$

## 5.2. The Horizontal Electric Field

The horizontal secondary electric field is the negative of the derivative of (14) with respect to $\rho$. It is

$$E_\rho^{(S)} = -\frac{2j_0}{\sigma\pi}\int_0^\infty\left(\frac{\sin\lambda a}{\lambda} - a\cos\lambda a\right)e^{-\lambda|z|}J_1(\lambda\rho)d\lambda \tag{20}$$

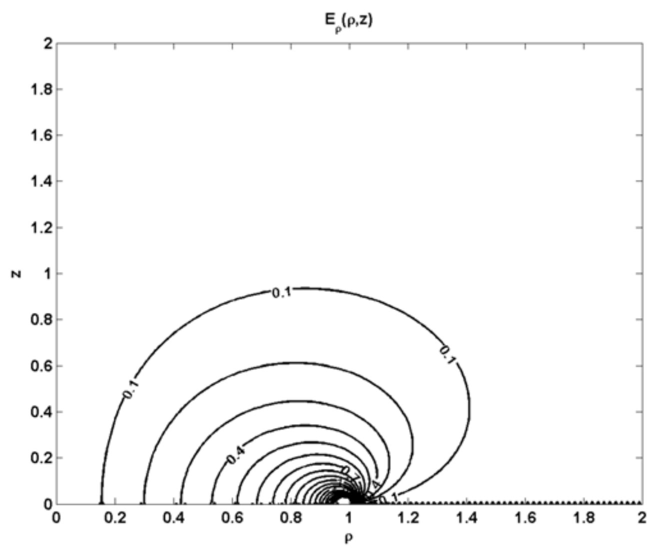


**Figure 2.** Contours of $E_\rho$ in the $\rho$-$z$ plane. The disk has unit radius and lies between the origin and $\rho = 1$.

Using the tabulated integrals 2.12.25.2 and 2.12.25.5 in [17], with appropriately changed notation,

$$\int_0^\infty e^{-\lambda|z|}\cos\lambda a J_1(\lambda\rho)d\lambda = \frac{1}{\rho} - \frac{\cos(\theta-\eta)\sqrt{a^2+z^2}}{\rho P} \tag{21}$$

and

$$\int_0^\infty e^{-\lambda|z|}\frac{\sin\lambda a}{\lambda}J_1(\lambda\rho)d\lambda = \frac{r(a+P\sin\eta)}{(z+P\cos\eta)^2+(a+P\sin\eta)^2} \tag{22}$$

where $P$, $\theta$ and $\eta$ are defined by

$$P^4 = \left(\rho^2+z^2-a^2\right)^2 + 4a^2z^2 \tag{23}$$

$$\tan\theta = a/z \tag{24}$$

$$2az\cot 2\eta = \rho^2+z^2-a^2 \tag{25}$$

Substituting (21) and (22) into (20) gives the closed form solution for the horizontal (radial) electric field as

$$\frac{2j_0}{\sigma\pi}\left[\frac{a}{\rho} - \frac{a\cos(\theta-\eta)\sqrt{a^2+z^2}}{\rho P} - \frac{\rho(a+P\sin\eta)}{(z+P\cos\eta)^2+(a+P\sin\eta)^2}\right] \tag{26}$$

Note that care must be taken to evaluate $\eta$ in (25) consistently on the branch $0<\eta<\pi/2$.

### 5.3. The Horizontal Magnetic Field

The magnetic field is the sum of whatever magnetic field is associated with the source (primary) field, and the magnetic field of the secondary currents. Here we solve for the secondary magnetic field $H_\phi$, which is purely azimuthal. From Ampere's law, the line integral of $H_\phi$ around a horizontal circle of radius $\rho$, centered on the $z$-axis at height $z$ above the origin, must equal the (secondary) electric current through the circle, the surface integral of $j_z$ over the area of the circle. Since, by symmetry, $H_\phi$ and $E_z$ are constants on contours of constants $\rho$ and $z$, this is equivalent to

$$2\pi\rho H_\phi(\rho,z) = \int_0^\rho 2\pi\rho'\sigma E_z^{(S)}(\rho',z)d\rho' \tag{27}$$

Using (15), $\rho H_\phi$ can be expressed as

$$\frac{2j_0}{\pi}\int_0^\infty\left(\frac{\sin\lambda a}{\lambda} - a\cos\lambda a\right)e^{-\lambda|z|}\int_0^\rho \rho' J_0(\lambda\rho')d\rho' d\lambda \tag{28}$$

where we have interchanged the order of integration. Since

$$\int_0^x x' J_0(x')dx' = xJ_1(x) \tag{29}$$

(28) becomes

$$\frac{2j_0}{\pi}\int_0^\infty \left(\frac{\sin\lambda a}{\lambda} - a\cos\lambda a\right) e^{-\lambda|z|}\left[\frac{\rho}{\lambda}J_1(\lambda\rho)\right] d\lambda \tag{30}$$

or

$$H_\phi = \frac{2j_0}{\pi}\int_0^\infty \left(\frac{\sin\lambda a}{\lambda^2} - a\frac{\cos\lambda a}{\lambda}\right) e^{-\lambda|z|} J_1(\lambda\rho)d\lambda \tag{31}$$

This can be integrated explicitly using 2.12.25.6 in [17],

$$\begin{aligned}&\int_0^\infty e^{-\lambda|z|}\frac{\sin\lambda a}{\lambda^2}J_1(\lambda\rho)d\lambda\\ =&\ \frac{1}{2a^2\rho^2}\left[P\sqrt{a^2+z^2}\sin(\theta+\eta) - \left(a^2+z^2\right)\sin 2\theta\right]\\ &+\frac{\rho}{2a^2}\arctan\left(\frac{P\sin\eta + \sqrt{a^2+z^2}\sin\theta}{P\cos\eta + \sqrt{a^2+z^2}\cos\theta}\right)\end{aligned} \tag{32}$$

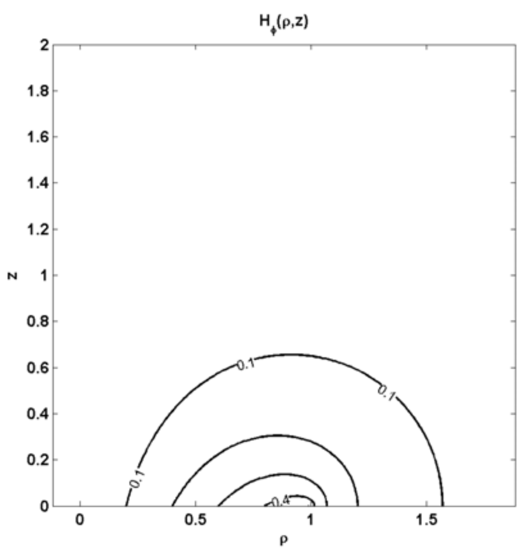


**Figure 3.** Contours of $H_\phi$ in the $\rho$-$z$ plane. The disk has unit radius and lies between the origin and $\rho = 1$.

and 3.12.10.15 (cosine form) in [18]

$$\int_0^\infty e^{-\lambda|z|}\frac{\cos\lambda a}{\lambda}J_1(\lambda\rho)d\lambda = \frac{P\cos\eta - z}{\rho} \tag{33}$$

where $P$, $\theta$ and $\eta$ are defined in (23)–(25) in the previous section. The horizontal magnetic field can thus be expressed in closed form as

$$\begin{aligned} H_\phi = &\frac{j_0}{\pi a^2\rho^2}\left[P\sqrt{a^2+z^2}\sin(\theta+\eta) - \left(a^2+z^2\right)\sin 2\theta\right] \\ &+\frac{j_0\rho}{\pi a^2}\arctan\left(\frac{P\sin\eta+\sqrt{a^2+z^2}\sin\theta}{P\cos\eta+\sqrt{a^2+z^2}\cos\theta}\right) - \frac{2j_0 a}{\pi\rho}(P\cos\eta-z) \end{aligned} \tag{34}$$

## 6. THE EQUIVALENT DIPOLE MOMENT

The behaviour of the potential and fields at large distances from the disk is of interest. To determine it, substitute $\lambda = u/\rho$ and $z = \rho\tan\theta$ in (14)

$$\frac{2j_0}{\sigma\pi}\int_0^\infty \left(\frac{\sin\lambda a}{\lambda^2} - \frac{a\cos\lambda a}{\lambda}\right)e^{-\lambda|z|}J_0(\lambda\rho)d\lambda \tag{35}$$

yielding

$$\frac{2j_0}{\sigma\pi}\int_0^\infty \left(\frac{\rho^2\sin(ua/\rho)}{u^2} - \frac{a\rho\cos(ua/\rho)}{u}\right)e^{-u\tan\theta}J_0(u)\frac{du}{\rho} \tag{36}$$

Taking the limit as $\rho$ tends to infinity and eliminating $\tan\theta$ in favor of $\rho$ and $z$ give

$$\frac{2a^3j_0}{3\pi\sigma}\frac{z}{(\rho^2+z^2)^{3/2}} \tag{37}$$

which is identical with the expression for the electric potential of a current dipole in a conducting medium if we take the dipole moment $m$ (current-separation product) to be

$$m = \frac{8}{3}a^3j_0 \tag{38}$$

## 7. NUMERICAL EVALUATION OF THE POTENTIAL

Unlike the integrals required for the field components, the integral for the electric potential in Equation (14) is not obviously evaluated analytically. However, it can be evaluated numerically with one

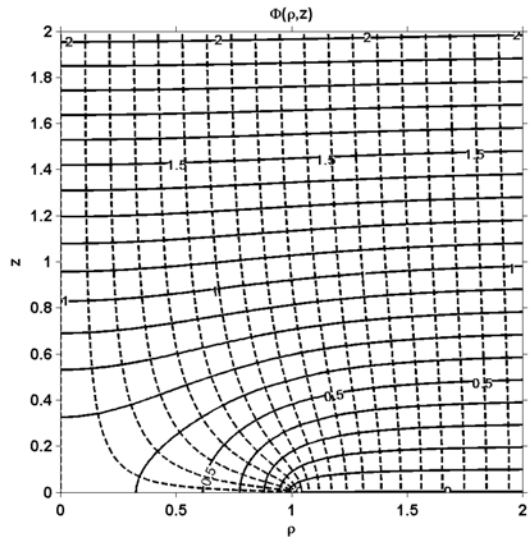


**Figure 4.** Equipotentials (solid lines) and current streamlines (dashed lines) shown in the $\rho$-$z$ plane. The disk has unit radius and lies between the origin and $\rho = 1$.

of the many algorithms (e.g., [2, 3, 5, 7–10, 12]) for numerical Hankel transforms. Figure 4 shows the equipotentials and current streamlines for a disk of unit radius, computed using a modified version of the method in [12].

The peak potential difference across the disk (at its center) is, however, easily evaluated analytically as

$$\Delta\Phi(0,0) = \frac{4aj_0}{\pi\sigma} \tag{39}$$

## 8. RESISTIVITY OF A COMPOSITE MATERIAL

The above results can be used to calculate the resistivity of a uniform medium with randomly embedded insulating disks, if those disks are, on average, far enough apart not to interact. The calculation is of interest because most analyses of the effect of inclusions on electrical resistivity (see [7, 21]) are done in terms of the volume fraction of inclusions, which is here zero (since the disks have zero thickness). The analytic solution provided below therefore explores an extreme case of shape anisotropy of the disks and furthermore, does so for the case of an infinite property contrast.

The calculation is most easily done by computing the extra power dissipation contributed by each disk and deducing the equivalent medium resistivity which has the same power dissipation. The power dissipated in any volume $\mathcal{V}$ of the medium can be calculated as the net input power applied on its boundary $\mathcal{S}$. A power equal to the product of current and potential is required to drive a current $\mathbf{j} \cdot \mathbf{n}\, \mathbf{d}\mathcal{S}$ across an infinitesmal boundary patch of area $d\mathcal{S}$ with normal $\mathbf{n}$. The net power $P$ delivered to a volume $\mathcal{V}$ is therefore

$$P = \int_{\mathcal{S}} \Phi \mathbf{j} \cdot \mathbf{n} d\mathcal{S} = \int_{\mathcal{S}} \Phi \sigma \mathbf{n} \cdot \nabla \Phi d\mathcal{S} \tag{40}$$

For ease of evaluation, consider as the volume $\mathcal{V}$ a vertical right circular cylinder (pillbox) of height $2h$ and radius $b$ centered on the disk, Choose $b$ much larger than $h$ so that the integral over the cylinder sides can be neglected and the integral in (40) need only be evaluated over the (circular) top and bottom of the cylinder. The integral is then (by symmetry)

$$P = 2\int_0^b \sigma \Phi(\rho, h) \left[\frac{\partial \Phi(\rho, z)}{\partial z}\right]_{z=h} 2\pi\rho d\rho \tag{41}$$

Choose $b$ large enough that the far-field approximation (37) for $\Phi^{(S)}$ is accurate, namely the total potential and its vertical derivative are (using $r = \sqrt{\rho^2 + z^2}$)

$$\Phi = \frac{j_0 z}{\sigma} + \frac{2 j_0 a^3}{3\pi\sigma} \frac{z}{r^3} \tag{42}$$

$$\frac{\partial \Phi}{\partial z} = \frac{j_0}{\sigma} + \frac{2 j_0 a^3}{3\pi\sigma}\left[\frac{1}{r^3} - \frac{3z^2}{r^5}\right] \tag{43}$$

so that (abbreviating $2a^3/(3\pi)$ as $V_e$)

$$\begin{aligned} P &= \frac{4\pi z j_0^2}{\sigma} \int_0^b \left(1 + \frac{V_e}{r^3}\right)\left(1 + \frac{V_e}{r^3} - \frac{3z^2 V_e}{r^5}\right) \rho d\rho \\ &= \frac{4\pi z j_0^2}{\sigma} \int_0^b \left(1 + \frac{2V_e}{r^3} - \frac{3z^2 V_e}{r^5} + \frac{V_e^2}{r^6} - \frac{3z^2 V_e}{r^8}\right) \rho d\rho \end{aligned} \tag{44}$$

Using the result

$$\int_0^b r^{-n} \rho d\rho = \frac{1}{2-n}\left[\frac{1}{(z^2+b^2)^{n-2}} - \frac{1}{z^{n-2}}\right] \tag{45}$$

at $z = h$, and letting $b$ become much larger than $h$ and $h$ much larger than $a$, we find the total power dissipated in the cylinder to be. We get

$$P = V_c\mathcal{P}_0 + \frac{16}{3}a^3\mathcal{P}_0 \tag{46}$$

$$= V_c\mathcal{P}_0\left[1 + \frac{16}{3}\frac{a^3}{V_c}\right] \tag{47}$$

where $\mathcal{P}_0 = j_0^2/\sigma$ is the power dissipated per unit volume in the unperturbed medium and the volume of the cylinder is $V_c = 2h\pi b^2$. The first term is simply the power that would be dissipated in the cylinder in the absence of an insulating disk. If $V_c$ is taken as the volume of medium associated with each disk, so that $N = V_c$ is the number of disks per unit volume, the relative power dissipation per unit volume, and thus the relative effective resistivity of the medium associated with the disks is

$$1 + \frac{16}{3}Na^3 \tag{48}$$

This is of course valid only for dilute disk distributions in which $Na^3 \ll 1$ (A solution for high disk concentrations cannot be obtained from the analytic closed form theory presented here). It also assumes that the disk planes are all oriented perpendicular to the direction of mean current flow. For disk planes oriented parallel to the direction of mean current flow, the effect of the disks will vanish. For a material in which the disks are randomly oriented, the mean value of the square of the component of the current density $\mathbf{j}$ perpendicular to disks will average to $j_0^2/3$, making the resistivity enhancement reduce to

$$1 + \frac{16}{9}Na^3 \tag{49}$$

Finally, if the disks are of different radii, $a^3$ in (48) and (49) represents the mean-cubed-radius.

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