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ELECTROMAGNETIC RADIATION FROM SOURCES EMBEDDED IN A CYLINDRICALLY STRATIFIED UNBOUNDED GYROTROPIC MEDIUM

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Abstract—A study is made of the excitation of electromagnetic waves by spatially bounded, arbitrary sources in the presence of a cylindrical guiding structure immersed in an infinitely extended, homogeneous gyrotropic medium whose permittivity and permeability are both describable by tensors with nonzero off-diagonal elements. The axis of symmetry of the considered cylindrical structure is assumed to coincide with the gyrotropic axis. The total field is sought in terms of vector modal solutions of the source-free Maxwell equations. We determine the content of the modal spectrum and obtain an eigenfunction expansion of the source-excited field in terms of discrete- and continuous-spectrum modes. The expansion coefficients of the modes are derived in explicit form. An expression for the total power radiated from sources is deduced and analyzed. It is shown that the

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developed approach makes it possible to readily represent the source-excited field without preliminary calculation of the dyadic Green's functions, which significantly facilitates the field evaluation.

1. INTRODUCTION

Gyrotropic structures capable of guiding electromagnetic waves have been an important research topic for a long time because of many applications including, in particular, those related to the characteristics of electromagnetic waves in plasma waveguides [1–3], helicon waves in magnetized metals and semiconductors [4–6], waves in ferrites [7, 8], modes of gyrotropic fibers [9, 10], etc. The theory of excitation of closed waveguides filled with gyrotropic media has received much careful study, and there are many accounts of it (see, e.g., [11–13] and references therein). Open gyrotropic waveguides surrounded by an isotropic outer medium have been discussed in [10, 14–18]. The case where the outer medium is anisotropic is considered in [19]. Recently, open gyrotropic guiding structures located in a gyrotropic background medium have attracted considerable interest [3, 20]. Several workers have discussed representations of the dyadic Green's functions for such structures (see, e.g., [20–22]). Another approach is to obtain eigenfunction expansions of the source-excited fields without preliminary calculations of the dyadic Green's functions [3]. In particular, this approach was applied for representing the electromagnetic fields of given sources in the presence of cylindrical guiding structures immersed in an unbounded gyroelectric medium [3, 23–26] such as a magnetoplasma, for example.

Note that for gyrotropic media, use of eigenfunction expansions for the source-excited fields can have some advantages over the Green's function technique. This is related to the fact that even in the simplest case of a homogeneous gyrotropic medium without spatial dispersion, the dyadic Green's functions cannot be expressed in closed forms and are represented by improper integrals in the wave number space [11, 27–29]. Moreover, two sets of the Green's functions, which correspond to two characteristic waves of such a medium, should be introduced. Another, more substantial difficulty occurs when dealing with resonant gyrotropic media in which the refractive index surface for one of the characteristic waves extends to infinity when an angle between the wavenormal direction and the gyrotropic axis approaches a certain value determined by the medium parameters [3]. Under such conditions, the Green's functions turn out to be singular not only at the source point, but also on some conical surfaces commonly known as resonance cones [3, 12]. It is important that the structure of resonance surfaces of the spatially distributed sources may not coincide with that of the resonance cones for the dyadic Green's functions, which significantly complicates evaluation of convolution integrals comprising the Green's functions.

The above-mentioned features make the development of methods employing the eigenfunction expansions of the source-excited fields very topical. It is the purpose of the present paper to extend such methods, which were initially developed for the fields of sources in cylindrically stratified gyroelectric media [3], to the case where given sources are embedded in a gyrotropic medium whose permittivity and permeability are both describable by tensors with nonzero off-diagonal elements.

Our paper is organized as follows. In Sec. 2, we formulate the studied problem and present the basic equations. In Sec. 3, rigorous representations of cylindrical vector wave functions in a homogeneous gyrotropic medium are discussed. In Sec. 4, the total source-excited field in a cylindrically stratified unbounded gyrotropic medium is found in the form of an eigenfunction expansion over the discrete- and continuous-spectrum modes. Then, in Sec. 5, a reduction to the special case of a uniaxial medium is made. In Sec. 6, we apply the obtained field representation for determining the power radiated from electromagnetic sources. Sec. 7 states our conclusions along with suggestions for future work. Appendices A and B present coefficients entering the field expressions and the derivation of the mode orthogonality relations, respectively.

2. BASIC FORMULATION

Consider an unbounded gyrotropic medium containing a cylindrical nonuniformity whose axis is aligned with the z axis of a cylindrical coordinate system (ρ, ϕ, z). The medium can be frequency-dispersive and is described by the permittivity and permeability tensors which, for a monochromatic signal with a time dependence of $\exp(i\omega t)$, are written as

$$\boldsymbol{\varepsilon} = \epsilon_0 \begin{pmatrix} \varepsilon_1 & i\varepsilon_2 & 0 \\ -i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \qquad \boldsymbol{\mu} = \mu_0 \begin{pmatrix} \mu_1 & i\mu_2 & 0 \\ -i\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \tag{1}$$

where ϵ_0 and μ_0 are the electric and magnetic constants, respectively. Note that such tensors are typical of a general bigyrotropic anisotropic medium whose distinguished axis is parallel to the z axis [30]. The tensor elements in (1) are prescribed piecewise-continuous functions of distance ρ from the z axis. To avoid phenomena related to the resonant absorption of electromagnetic energy in an inhomogeneous medium in the case where the diagonal elements of tensors (1) reverse signs in the absence of losses, we assume that these elements pass zero in a jump-like manner.

Let an electromagnetic field be excited by time-harmonic given electric and magnetic currents whose densities, with time dependence dropped, are represented by the functions $\boldsymbol{J}^{\mathrm{e}}(\boldsymbol{r})$ and $\boldsymbol{J}^{\mathrm{m}}(\boldsymbol{r})$, respectively. The currents are located in a limited spatial region defined by $\rho < b$ and $z_1 < z < z_2$.

Although our main task in this paper is to study the excitation of electromagnetic waves by given currents in a gyrotropic medium in the presence of the above-mentioned nonuniformity, we start from the source-free Maxwell equations

$$\nabla \times \boldsymbol{E} = -i\omega\,\boldsymbol{\mu} \cdot \boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = i\omega\,\boldsymbol{\varepsilon} \cdot \boldsymbol{E}. \tag{2}$$

The symmetry of the problem dictates that the solution of these equations can be determined in terms of the vector wave functions

$$\begin{bmatrix} \boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}(\rho) \\ \boldsymbol{H}(\rho) \end{bmatrix} \exp(-im\phi - ik_0 p z), \tag{3}$$

where m is the azimuthal index ($m = 0, \pm 1, \pm 2, \ldots$), k_0 is the wave number in free space, and p is the axial wave number normalized to k_0. From the Maxwell equations, it can be shown that in the source-free regions, the radial and azimuthal components of vector functions (3) are related to their axial components by the expressions

$$\begin{aligned} E_\rho &= \frac{1}{k_0 \Delta}\left\{ ip\tau_{\mathrm{g}}\kappa^2 \frac{m}{\rho} E_z + ip\left(p^2 - \kappa^2 - \lambda^2\right)\frac{\partial E_z}{\partial \rho} \right. \\ &\quad \left. + \mu_1\left(p^2 - \tau_{\mathrm{e}}\kappa_{\mathrm{e}}^2\right)\frac{m}{\rho} Z_0 H_z + \left(\mu_2 p^2 + \varepsilon_2\mu_+\mu_-\right) Z_0 \frac{\partial H_z}{\partial \rho} \right\}, \end{aligned} \tag{4}$$

$$\begin{aligned} E_\phi &= \frac{1}{k_0 \Delta}\left\{ p\left(p^2 - \kappa^2 - \lambda^2\right)\frac{m}{\rho} E_z + p\tau_{\mathrm{g}}\kappa^2 \frac{\partial E_z}{\partial \rho} \right. \\ &\quad \left. - i\left(\mu_2 p^2 + \varepsilon_2\mu_+\mu_-\right)\frac{m}{\rho} Z_0 H_z - i\mu_1\left(p^2 - \tau_{\mathrm{e}}\kappa_{\mathrm{e}}^2\right) Z_0 \frac{\partial H_z}{\partial \rho} \right\}, \end{aligned} \tag{5}$$

$$\begin{aligned} H_\rho &= \frac{1}{k_0 \Delta}\left\{ -\varepsilon_1\left(p^2 - \tau_{\mathrm{m}}\kappa_{\mathrm{m}}^2\right)\frac{m}{\rho}\frac{E_z}{Z_0} - \left(\varepsilon_2 p^2 + \mu_2\varepsilon_+\varepsilon_-\right)\frac{1}{Z_0}\frac{\partial E_z}{\partial \rho} \right. \\ &\quad \left. + ip\tau_{\mathrm{g}}\kappa^2 \frac{m}{\rho} H_z + ip\left(p^2 - \kappa^2 - \lambda^2\right)\frac{\partial H_z}{\partial \rho} \right\}, \end{aligned} \tag{6}$$

$$H_\phi = \frac{1}{k_0\Delta}\left\{i\left(\varepsilon_2 p^2+\mu_2\varepsilon_+\varepsilon_-\right)\frac{m}{\rho}\frac{E_z}{Z_0}+i\varepsilon_1\left(p^2-\tau_{\rm m}\kappa_{\rm m}^2\right)\frac{1}{Z_0}\frac{\partial E_z}{\partial\rho}\right.$$
$$\left.+p\left(p^2-\kappa^2-\lambda^2\right)\frac{m}{\rho}H_z+p\tau_{\rm g}\kappa^2\frac{\partial H_z}{\partial\rho}\right\}, \qquad (7)$$

where

$$\begin{aligned}\Delta &= \left(p^2-\varepsilon_+\mu_+\right)\left(p^2-\varepsilon_-\mu_-\right),\ \varepsilon_\pm=\varepsilon_1\pm\varepsilon_2,\ \mu_\pm=\mu_1\pm\mu_2,\\ \tau_{\rm e} &= \frac{\varepsilon_1}{\varepsilon_3},\quad \tau_{\rm m}=\frac{\mu_1}{\mu_3},\quad \tau_{\rm g}=\frac{\varepsilon_2}{\varepsilon_1}+\frac{\mu_2}{\mu_1},\\ \kappa_{\rm e} &= \left(\varepsilon_3\frac{\mu_1^2-\mu_2^2}{\mu_1}\right)^{1/2},\quad \kappa_{\rm m}=\left(\mu_3\frac{\varepsilon_1^2-\varepsilon_2^2}{\varepsilon_1}\right)^{1/2},\\ \kappa &= (\varepsilon_1\mu_1)^{1/2},\quad \lambda=(\varepsilon_2\mu_2)^{1/2},\end{aligned} \qquad (8)$$

and $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space. The axial components E_z and H_z, in turn, satisfy the following system of equations:

$$\hat{L}_m E_z + k_0^2\left(\kappa_{\rm e}^2-\tau_{\rm e}^{-1}p^2\right)E_z = ik_0^2\tau_{\rm g}\mu_3 pZ_0H_z+\Psi_E, \qquad (9)$$
$$\hat{L}_m H_z + k_0^2\left(\kappa_{\rm m}^2-\tau_{\rm m}^{-1}p^2\right)H_z = -ik_0^2\tau_{\rm g}\varepsilon_3 pZ_0^{-1}E_z+\Psi_H. \qquad (10)$$

Here,

$$\hat{L}_m = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2},$$

$$\begin{aligned}\Psi_E &= ik_0\varepsilon_1^{-1}p\left(E_\rho\frac{d\varepsilon_1}{d\rho}+iE_\phi\frac{d\varepsilon_2}{d\rho}\right)\\ &\quad+\frac{\mu_+\mu_-}{\mu_1}\left[\left(\frac{m}{\rho}E_z-k_0pE_\phi\right)\frac{d}{d\rho}\left(\frac{\mu_2}{\mu_+\mu_-}\right)\right.\\ &\quad\left.-\left(\frac{\partial E_z}{\partial\rho}+ik_0pE_\rho\right)\frac{d}{d\rho}\left(\frac{\mu_1}{\mu_+\mu_-}\right)\right],\end{aligned} \qquad (11)$$

$$\begin{aligned}\Psi_H &= ik_0\mu_1^{-1}p\left(H_\rho\frac{d\mu_1}{d\rho}+iH_\phi\frac{d\mu_2}{d\rho}\right)\\ &\quad+\frac{\varepsilon_+\varepsilon_-}{\varepsilon_1}\left[\left(\frac{m}{\rho}H_z-k_0pH_\phi\right)\frac{d}{d\rho}\left(\frac{\varepsilon_2}{\varepsilon_+\varepsilon_-}\right)\right.\\ &\quad\left.-\left(\frac{\partial H_z}{\partial\rho}+ik_0pH_\rho\right)\frac{d}{d\rho}\left(\frac{\varepsilon_1}{\varepsilon_+\varepsilon_-}\right)\right].\end{aligned} \qquad (12)$$

Note that the components E_ρ, E_ϕ, H_ρ, and H_ϕ, which enter the expressions for the functions Ψ_E and Ψ_H, should be replaced by expressions (4)–(7) to represent these functions only in terms of E_z and H_z. However, the resulting formulas for Ψ_E and Ψ_H turn out to be very cumbersome and will not be written here in the interests of brevity.

When writing Equations (4)–(12), we did not specify the dependence of the tensor elements on ρ. We now assume that they are varying in the inner region of a column with radius a and are constant in its outer region (see Fig. 1). For $\rho > a$, we thus have

$$\Psi_E = 0, \qquad \Psi_H = 0. \tag{13}$$

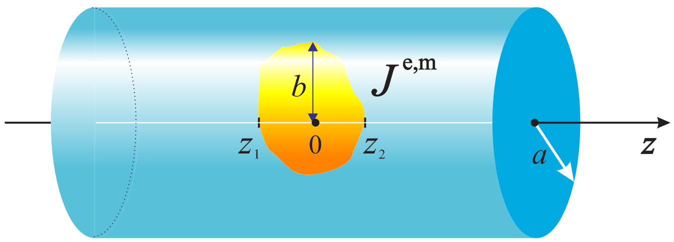

Figure 1. Geometry of the problem.

In what follows, the notations $\varepsilon_{1,2,3}$ and $\mu_{1,2,3}$ as well as the notations introduced in (8) will be used only for the homogeneous outer region $\rho > a$. For the inner region $\rho < a$, the corresponding quantities will be denoted by the tilde ($\tilde{\varepsilon}_{1,2,3}, \tilde{\mu}_{1,2,3}$, etc.). It is to be emphasized that at first we will consider the case in which $\tau_{\rm g} \neq 0$ in (9) and (10). Then the special case $\tau_{\rm g} = 0$ where the outer medium has the property of uniaxial anisotropy [30] will be discussed.

3. CYLINDRICAL VECTOR WAVE FUNCTIONS IN A HOMOGENEOUS GYROTROPIC MEDIUM

Before proceeding further, it is necessary to discuss the solutions of the source-free Maxwell equations in cylindrical coordinates for a homogeneous medium whose parameters coincide with those of the outer region $\rho > a$. In this case, Equations (9) and (10) with allowance for (13) can be solved analytically in terms of cylindrical functions. Let $\mathcal{Z}_m$ be a cylindrical function, say a Bessel or Hankel function, of order m. Putting $E_z(\rho) = \mathcal{A}\mathcal{Z}_m(k_0 q\rho)$ and $H_z(\rho) = Z_0^{-1}\mathcal{B}\mathcal{Z}_m(k_0 q\rho)$ in (9) and (10), where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary constants, we get

$$\begin{pmatrix} q^2+\tau_{\rm e}^{-1}p^2-\kappa_{\rm e}^2 & i\tau_{\rm g}\mu_3 p \\ -i\tau_{\rm g}\varepsilon_3 p & q^2+\tau_{\rm m}^{-1}p^2-\kappa_{\rm m}^2 \end{pmatrix}\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix}=0. \tag{14}$$

The determinant of (14) must vanish, whence we obtain the following quartic equation relating q and p [30]:

$$\begin{aligned} &q^4+\left[\left(\tau_{\rm e}^{-1}+\tau_{\rm m}^{-1}\right)p^2-\kappa_{\rm e}^2-\kappa_{\rm m}^2\right]q^2+\left(\tau_{\rm e}\tau_{\rm m}\right)^{-1}p^4\\ &-\left(\tau_{\rm m}^{-1}\kappa_{\rm e}^2+\tau_{\rm e}^{-1}\kappa_{\rm m}^2+\tau_{\rm g}^2\varepsilon_3\mu_3\right)p^2+\kappa_{\rm e}^2\kappa_{\rm m}^2=0. \end{aligned} \tag{15}$$

It is worth noting that in the special case of a gyroelectric medium where $\mu_1=\mu_3=1$ and $\mu_2=0$, Equation (15) becomes the well-known Booker quartic [31]. If (15) is regarded as a quartic equation in q, then it determines four solutions $\pm q_1(p)$ and $\pm q_2(p)$ such that

$$q_k(p)=2^{-1/2}\left[\kappa_{\rm e}^2+\kappa_{\rm m}^2-\left(\tau_{\rm e}^{-1}+\tau_{\rm m}^{-1}\right)p^2+(-1)^k R_q(p)\right]^{1/2},\ k=1,2, \tag{16}$$

where

$$\begin{aligned} R_q(p)\ =\ &\Big\{\left(\tau_{\rm e}^{-1}-\tau_{\rm m}^{-1}\right)^2p^4-2\left[\left(\tau_{\rm e}^{-1}+\tau_{\rm m}^{-1}\right)\left(\kappa_{\rm e}^2+\kappa_{\rm m}^2\right)\right.\\ &\left.-\ 2\left(\tau_{\rm m}^{-1}\kappa_{\rm e}^2+\tau_{\rm e}^{-1}\kappa_{\rm m}^2+\tau_{\rm g}^2\varepsilon_3\mu_3\right)\right]p^2+\left(\kappa_{\rm e}^2-\kappa_{\rm m}^2\right)^2\Big\}^{1/2}. \end{aligned} \tag{17}$$

The ratio of the constants $\mathcal{A}$ and $\mathcal{B}$ for $q=q_k$ is

$$\left.\left(\frac{\mathcal{A}}{\mathcal{B}}\right)\right|_{q=q_k}=-i\frac{\mu_3}{\varepsilon_3}n_k, \tag{18}$$

where

$$n_k=\frac{q_k^2+\tau_{\rm m}^{-1}p^2-\kappa_{\rm m}^2}{\tau_{\rm g}\mu_3 p}=\frac{\tau_{\rm g}\varepsilon_3 p}{q_k^2+\tau_{\rm e}^{-1}p^2-\kappa_{\rm e}^2}. \tag{19}$$

Note that

$$n_1n_2=-\frac{\varepsilon_3}{\mu_3}. \tag{20}$$

If $q=q_k$ and $\mathcal{A}=in_kq_k/\varepsilon_3$, then, for a fixed k, the particular solution of Equations (9) and (10) in the case of a homogeneous medium takes the form

$$E_z(\rho)\ =\ i\varepsilon_3^{-1}n_kq_k\mathcal{Z}_m(k_0q_k\rho), \tag{21}$$

$$H_z(\rho)\ =\ -Z_0^{-1}\mu_3^{-1}q_k\mathcal{Z}_m(k_0q_k\rho). \tag{22}$$

Upon substituting (21) and (22) into (4)–(7), we get

$$E_\rho(\rho) = \frac{1}{2}\left[\beta_{k,+}\mathcal{Z}_{m+1}(k_0q_k\rho) - \beta_{k,-}\mathcal{Z}_{m-1}(k_0q_k\rho)\right], \tag{23}$$

$$E_\phi(\rho) = \frac{i}{2}\left[\beta_{k,+}\mathcal{Z}_{m+1}(k_0q_k\rho) + \beta_{k,-}\mathcal{Z}_{m-1}(k_0q_k\rho)\right], \tag{24}$$

$$H_\rho(\rho) = \frac{i}{2Z_0}n_k\left[\gamma_{k,+}\mathcal{Z}_{m+1}(k_0q_k\rho) + \gamma_{k,-}\mathcal{Z}_{m-1}(k_0q_k\rho)\right], \tag{25}$$

$$H_\phi(\rho) = -\frac{1}{2Z_0}n_k\left[\gamma_{k,+}\mathcal{Z}_{m+1}(k_0q_k\rho) - \gamma_{k,-}\mathcal{Z}_{m-1}(k_0q_k\rho)\right], \tag{26}$$

where

$$\beta_{k,\pm} = \frac{q_k^2}{\mu_3}\,\frac{p\varepsilon_3^{-1}\mu_3 n_k \mp \mu_\mp}{p^2 - \varepsilon_\mp\mu_\mp} = \pm\frac{\varepsilon_\pm \mp n_k p}{\varepsilon_1}, \tag{27}$$

$$\gamma_{k,\pm} = \pm\frac{q_k^2}{\mu_3}\,\frac{pn_k^{-1} \mp \varepsilon_\mp\varepsilon_3^{-1}\mu_3}{p^2 - \varepsilon_\mp\mu_\mp} = \frac{\mu_\pm \mp n_k^{-1}p}{\mu_1}. \tag{28}$$

Note that expressions (27) and (28) can be derived by straightforward manipulation.

Using the well-known formula

$$\mathcal{Z}_{m+1}(\zeta) + \mathcal{Z}_{m-1}(\zeta) = \frac{2m}{\zeta}\mathcal{Z}_m(\zeta), \tag{29}$$

which is valid for cylindrical functions [32], and the identities

$$\begin{aligned}\beta_{k,+} + \beta_{k,-} &= 2(\varepsilon_2 - n_k p)\varepsilon_1^{-1}, \quad \beta_{k,+} - \beta_{k,-} = 2,\\ \gamma_{k,+} + \gamma_{k,-} &= 2, \quad \gamma_{k,+} - \gamma_{k,-} = 2(\mu_2 - n_k^{-1}p)\mu_1^{-1},\end{aligned} \tag{30}$$

we can rewrite formulas (23)–(26) for the transverse field components in alternative forms:

$$\begin{aligned}E_\rho(\rho) &= \frac{\varepsilon_2 - n_k p}{\varepsilon_1}\mathcal{Z}_{m+1}(k_0q_k\rho) - \beta_{k,-}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\\ &= -\left[\frac{\varepsilon_2 - n_k p}{\varepsilon_1}\mathcal{Z}_{m-1}(k_0q_k\rho) - \beta_{k,+}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\right],\end{aligned} \tag{31}$$

$$\begin{aligned}E_\phi(\rho) &= i\left[\mathcal{Z}_{m+1}(k_0q_k\rho) + \beta_{k,-}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\right]\\ &= -i\left[\mathcal{Z}_{m-1}(k_0q_k\rho) - \beta_{k,+}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\right],\end{aligned} \tag{32}$$

$$H_\rho(\rho) = \frac{in_k}{Z_0}\left[\frac{\mu_2 - n_k^{-1}p}{\mu_1}\mathcal{Z}_{m+1}(k_0q_k\rho) + \gamma_{k,-}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\right]$$
$$= -\frac{in_k}{Z_0}\left[\frac{\mu_2 - n_k^{-1}p}{\mu_1}\mathcal{Z}_{m-1}(k_0q_k\rho) - \gamma_{k,+}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\right], \quad (33)$$

$$H_\phi(\rho) = -\frac{n_k}{Z_0}\left[\mathcal{Z}_{m+1}(k_0q_k\rho) - \gamma_{k,-}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\right]$$
$$= \frac{n_k}{Z_0}\left[\mathcal{Z}_{m-1}(k_0q_k\rho) - \gamma_{k,+}m\frac{\mathcal{Z}_m(k_0q_k\rho)}{k_0q_k\rho}\right]. \quad (34)$$

The above expressions yield the components of cylindrical vector wave functions in a homogeneous gyrotropic medium for a given axial wave number p. For later work, it will also be necessary to employ another representation of these functions, for which (15) is regarded as a quartic equation in p. Then it determines four functions $\pm p_1(q)$ and $\pm p_2(q)$ such that

$$p_\alpha(q) = 2^{-1/2}\left[\sigma - (\tau_\mathrm{e} + \tau_\mathrm{m})q^2 + \chi_\alpha R_p(q)\right]^{1/2}, \quad (35)$$

where

$$\sigma = \tau_\mathrm{e}\kappa_\mathrm{e}^2 + \tau_\mathrm{m}\kappa_\mathrm{m}^2 + \tau_\mathrm{g}^2\kappa^2,$$
$$R_p(q) = \left\{(\tau_\mathrm{e} - \tau_\mathrm{m})^2q^4 - 2\left[(\tau_\mathrm{e} + \tau_\mathrm{m})\sigma - 2\tau_\mathrm{e}\tau_\mathrm{m}\left(\kappa_\mathrm{e}^2 + \kappa_\mathrm{m}^2\right)\right]q^2 + \sigma^2 - 4\tau_\mathrm{e}\tau_\mathrm{m}\kappa_\mathrm{e}^2\kappa_\mathrm{m}^2\right\}^{1/2}, \quad (36)$$

$\alpha = 1, 2$, and $\chi_1 = -\chi_2 = -1$. It may be noted that substituting $p = p_\alpha(q)$ into (16) yields two quantities q_k one of which coincides with q, while the other, hereafter denoted by $q_\alpha(q)$, differs from q. Therefore, we have

$$q_\alpha^2(q) = q_{\hat{k}}^2(p_\alpha(q)) \neq q, \quad (37)$$

where the value of $\hat{k}$, equal to either 1 or 2, is chosen such as to ensure the inequality on the extreme right-hand side of (37). Another way to calculate q_α as a function of q is to put one of two quantities q_1 or q_2 in (19) equal to q, and the other, to $q_\alpha(q)$, provided that $p = p_\alpha(q)$. Then formula (20) yields

$$\left(q^2 + \tau_\mathrm{m}^{-1}p_\alpha^2(q) - \kappa_\mathrm{m}^2\right)\left(q_\alpha^2(q) + \tau_\mathrm{m}^{-1}p_\alpha^2(q) - \kappa_\mathrm{m}^2\right) = -\tau_\mathrm{g}^2\varepsilon_3\mu_3p_\alpha^2(q), \quad (38)$$

whence we get

$$q_\alpha(q)=\left\{\kappa_{\rm m}^2-\tau_{\rm m}^{-1}p_\alpha^2(q)-\tau_{\rm g}^2\varepsilon_3\mu_3p_\alpha^2(q)\left[q^2+\tau_{\rm m}^{-1}p_\alpha^2(q)-\kappa_{\rm m}^2\right]^{-1}\right\}^{1/2}. \quad (39)$$

The quantity $q_\alpha(q)$ may be called the auxiliary transverse wave number corresponding to the transverse wave number q. Similarly, two quantities n_k, regarded as functions of p and given by (19), can be replaced by the corresponding functions of q if we substitute q and $q_\alpha(q)$ for $q_{1,2}$, and $p_\alpha(q)$ for p in (19).

In what follows, to make the functions $p_\alpha(q)$ and $q_\alpha(q)$ single-valued, we require that

$$\operatorname{Im} p_\alpha(q) < 0 \quad (40)$$

and

$$\operatorname{Im} q_\alpha(q) < 0. \quad (41)$$

In a lossless medium, the functions $p_\alpha(q)$ and $q_\alpha(q)$ can be purely real for some values of q. In such a case, one should introduce small losses in the medium, choose the branches of $p_\alpha(q)$ and $q_\alpha(q)$ in accordance with (40) and (41), respectively, and then put the losses equal to zero in the resulting formulas for the discussed functions.

4. FIELD EXPANSION IN THE PRESENCE OF A CYLINDRICAL NONUNIFORMITY

4.1. The Boundary-value Problem for an Open Cylindrical Waveguide in a Gyrotropic Medium

We now proceed to seeking and discussing solutions of the source-free Maxwell equations in a gyrotropic medium with a cylindrical nonuniformity. Such solutions can be sought in terms of the modal fields

$$\begin{bmatrix}\boldsymbol{E}_{m,s,\alpha}(\boldsymbol{r},q)\\ \boldsymbol{H}_{m,s,\alpha}(\boldsymbol{r},q)\end{bmatrix}=\begin{bmatrix}\boldsymbol{E}_{m,s,\alpha}(\rho,q)\\ \boldsymbol{H}_{m,s,\alpha}(\rho,q)\end{bmatrix}\exp[-im\phi-ik_0p_{s,\alpha}(q)z], \quad (42)$$

where q is the normalized (to k_0) transverse wave number in the outer medium ($\rho > a$), which was introduced in Sec. 3; the functions $p_{s,\alpha}(q)$ describe the dependence of the normalized axial wave number p on q for two characteristic waves of the outer medium, denoted by the subscript values $\alpha = 1$ and $\alpha = 2$; the subscript s denotes the wave propagation direction ($s = -$ and $s = +$ designate waves propagating in the negative and positive directions of the z axis, respectively); and $\boldsymbol{E}_{m,s,\alpha}(\rho, q)$ and $\boldsymbol{H}_{m,s,\alpha}(\rho, q)$ are the vector wave functions describing the radial distribution of the field of a mode corresponding to the transverse wave number q and the indices m, s, and α. The functions $p_{s,\alpha}(q)$ obey the relation $p_{+,\alpha}(q) \equiv p_\alpha(q) = -p_{-,\alpha}(q)$. Recall that $p_\alpha(q)$ is given by (35) and satisfies condition (40). We note that any solution of the source-free Maxwell equations that satisfies the boundary conditions at $\rho = a$ and the required conditions at $\rho \to \infty$ is a mode if its z-dependence is represented by (42).

It is evident that the radial and azimuthal components of the vector functions $\boldsymbol{E}_{m,s,\alpha}(\rho, q)$ and $\boldsymbol{H}_{m,s,\alpha}(\rho, q)$ are related to their axial components by expressions coinciding in form with (4)–(7), and the axial components of these vector functions are determined by Equations (9) and (10) if we put $p = p_{s,\alpha}(q)$ and substitute $\tilde{\varepsilon}_{1,2,3}, \tilde{\mu}_{1,2,3}$ and $\varepsilon_{1,2,3}, \mu_{1,2,3}$ for the tensor elements in the regions $\rho < a$ and $\rho > a$, respectively.

To find the values of q over which the corresponding modal fields (42) should be summed for representing the total field, it is required that the quantities $\boldsymbol{E}_{m,s,\alpha}(\rho, q)$ and $\boldsymbol{H}_{m,s,\alpha}(\rho, q)$ be regular on the z axis and satisfy both the boundary conditions, which consist in continuity of the components $E_{\phi;\,m,s,\alpha}(\rho, q)$, $E_{z;\,m,s,\alpha}(\rho, q)$, $H_{\phi;\,m,s,\alpha}(\rho, q)$, and $H_{z;\,m,s,\alpha}(\rho, q)$ at the discontinuity points of the tensor elements, and the following boundedness conditions at $\rho \to \infty$ [3, 33]:

$$\rho^{1/2} \left|\boldsymbol{E}_{m,s,\alpha}(\rho, q)\right| < R^{(1)}_{m,\alpha}, \quad \rho^{1/2} \left|\boldsymbol{H}_{m,s,\alpha}(\rho, q)\right| < R^{(2)}_{m,\alpha}, \tag{43}$$

where $R^{(1)}_{m,\alpha}$ and $R^{(2)}_{m,\alpha}$ are finite constants. It can be shown that the total field yielded by summing (or integrating) modes over the found values of q will satisfy the radiation condition at infinity ($r = (\rho^2 + z^2)^{1/2} \to \infty$) [3, 19, 33]. Note that in the considered case, conditions (43) turn out to be sufficient for finding the eigenvalues q and the corresponding modes in contrast to the case of an isotropic outer medium where, along with the boundedness conditions, certain additional conditions should be imposed on the desired modal fields [10, 16, 19, 33, 34].

4.2. Discrete- and Continuous-spectrum Modes

In the homogeneous outer medium ($\rho > a$), the solution for the axial field components $E_{z;\,m,s,\alpha}(\rho, q)$ and $H_{z;\,m,s,\alpha}(\rho, q)$ can be written using the results of Sec. 3:

$$E_{z;\,m,s,\alpha}(\rho,q) = \frac{i}{\varepsilon_3}\left[\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q) n^{(1)}_{s,\alpha} q H^{(k)}_m(k_0 q\rho) + C_{m,s,\alpha}(q) n^{(2)}_{s,\alpha} q_\alpha H^{(2)}_m(k_0 q_\alpha \rho)\right], \tag{44}$$

$$H_{z;\,m,s,\alpha}(\rho,q) = -\frac{1}{Z_0\mu_3}\left[\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q) q H^{(k)}_m(k_0 q\rho) + C_{m,s,\alpha}(q) q_\alpha H^{(2)}_m(k_0 q_\alpha \rho)\right]. \tag{45}$$

Here, $C^{(1)}_{m,s,\alpha}$, $C^{(2)}_{m,s,\alpha}$, and $C_{m,s,\alpha}$ are coefficients to be determined, and $H^{(1)}_m$ and $H^{(2)}_m$ are Hankel functions of the first and second kinds, respectively, of order m. The functions $q_\alpha(q)$ are determined by formula (39), and

$$n^{(1,2)}_{s,\alpha}(q) = \frac{\left(q^{(1,2)}_\alpha\right)^2 + \tau_{\rm m}^{-1} p^2_\alpha(q) - \kappa^2_{\rm m}}{\tau_{\rm g}\mu_3 p_{s,\alpha}(q)}, \tag{46}$$

where $q^{(1)}_\alpha = q$ and $q^{(2)}_\alpha = q_\alpha(q)$. Formula (46) follows from the expression (19) derived in Sec. 3. When writing (44) and (45), we took into account condition (41) determining the choice of the branches of $q_\alpha(q)$. With this point of mind, the particular solution comprising $H^{(1)}_m(k_0 q_\alpha \rho)$ has been rejected in (44) and (45) in order that the functions $\boldsymbol{E}_{m,s,\alpha}(\rho,q)$ and $\boldsymbol{H}_{m,s,\alpha}(\rho,q)$ do not contradict the boundedness conditions (43).

Expressions for the transverse components of the field in the region $\rho > a$ with allowance for formulas (23)–(28) can be written as

$$E_{\rho;\,m,s,\alpha}(\rho,q) = \frac{1}{2}\left[\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q)\left(\mathfrak{b}^{(1)}_{\alpha,+} H^{(k)}_{m+1}(k_0 q\rho) - \mathfrak{b}^{(1)}_{\alpha,-} H^{(k)}_{m-1}(k_0 q\rho)\right) + C_{m,s,\alpha}(q)\left(\mathfrak{b}^{(2)}_{\alpha,+} H^{(2)}_{m+1}(k_0 q_\alpha\rho) - \mathfrak{b}^{(2)}_{\alpha,-} H^{(2)}_{m-1}(k_0 q_\alpha\rho)\right)\right], \tag{47}$$

$$E_{\phi;\,m,s,\alpha}(\rho,q) = \frac{i}{2}\left[\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q)\left(\mathfrak{b}^{(1)}_{\alpha,+} H^{(k)}_{m+1}(k_0 q\rho) + \mathfrak{b}^{(1)}_{\alpha,-} H^{(k)}_{m-1}(k_0 q\rho)\right)\right.$$

$$+ C_{m,s,\alpha}(q) \left(\mathfrak{b}^{(2)}_{\alpha,+} H^{(2)}_{m+1}(k_0 q_\alpha \rho) + \mathfrak{b}^{(2)}_{\alpha,-} H^{(2)}_{m-1}(k_0 q_\alpha \rho) \right) \Bigg], \qquad (48)$$

$$H_{\rho;\, m,s,\alpha}(\rho, q) = \frac{i}{2Z_0} \left[\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q) n^{(1)}_{s,\alpha} \left(\mathfrak{g}^{(1)}_{\alpha,+} H^{(k)}_{m+1}(k_0 q \rho) + \mathfrak{g}^{(1)}_{\alpha,-} H^{(k)}_{m-1}(k_0 q \rho) \right) + C_{m,s,\alpha}(q) n^{(2)}_{s,\alpha} \left(\mathfrak{g}^{(2)}_{\alpha,+} H^{(2)}_{m+1}(k_0 q_\alpha \rho) + \mathfrak{g}^{(2)}_{\alpha,-} H^{(2)}_{m-1}(k_0 q_\alpha \rho) \right) \right], \qquad (49)$$

$$H_{\phi;\, m,s,\alpha}(\rho, q) = -\frac{1}{2Z_0} \left[\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q) n^{(1)}_{s,\alpha} \left(\mathfrak{g}^{(1)}_{\alpha,+} H^{(k)}_{m+1}(k_0 q \rho) - \mathfrak{g}^{(1)}_{\alpha,-} H^{(k)}_{m-1}(k_0 q \rho) \right) + C_{m,s,\alpha}(q) n^{(2)}_{s,\alpha} \left(\mathfrak{g}^{(2)}_{\alpha,+} H^{(2)}_{m+1}(k_0 q_\alpha \rho) - \mathfrak{g}^{(2)}_{\alpha,-} H^{(2)}_{m-1}(k_0 q_\alpha \rho) \right) \right], \qquad (50)$$

where

$$\mathfrak{b}^{(1,2)}_{\alpha,\pm} = \pm \frac{\varepsilon_\pm \mp n^{(1,2)}_{s,\alpha} p_{s,\alpha}}{\varepsilon_1}, \quad \mathfrak{g}^{(1,2)}_{\alpha,\pm} = \frac{\mu_\pm \mp \left(n^{(1,2)}_{s,\alpha} \right)^{-1} p_{s,\alpha}}{\mu_1}. \qquad (51)$$

We now consider the special case where the medium in the inner region ($\rho < a$) is homogeneous. According to (21) and (22), the axial field components that are regular at $\rho = 0$ are represented for $\rho < a$ in the form

$$E_{z;\, m,s,\alpha}(\rho, q) = \frac{i}{\tilde{\varepsilon}_3} \sum_{k=1}^{2} B^{(k)}_{m,s,\alpha}(q)\, \tilde{n}^{(k)}_{s,\alpha}\, \tilde{q}^{(k)}_\alpha J_m \left(k_0 \tilde{q}^{(k)}_\alpha \rho \right), \qquad (52)$$

$$H_{z;\, m,s,\alpha}(\rho, q) = -\frac{1}{Z_0 \tilde{\mu}_3} \sum_{k=1}^{2} B^{(k)}_{m,s,\alpha}(q)\, \tilde{q}^{(k)}_\alpha J_m \left(k_0 \tilde{q}^{(k)}_\alpha \rho \right), \qquad (53)$$

where J_m are Bessel functions of the first kind of order m, $B^{(1)}_{m,s,\alpha}$ and $B^{(2)}_{m,s,\alpha}$ are undetermined coefficients,

$$\tilde{n}^{(1,2)}_{s,\alpha}(q) = \frac{\left(\tilde{q}^{(1,2)}_\alpha \right)^2 + \tilde{\tau}^{-1}_{\mathrm{m}} p^2_\alpha(q) - \tilde{\kappa}^2_{\mathrm{m}}}{\tilde{\tau}_{\mathrm{g}} \tilde{\mu}_3 p_{s,\alpha}(q)}, \qquad (54)$$

and the functions $\tilde{q}_\alpha^{(1)}(q)$ and $\tilde{q}_\alpha^{(2)}(q)$ are yielded by formula (16) for $q_1(p)$ and $q_2(p)$, respectively, if we substitute $p_\alpha(q)$ for p in (16) and make the replacements $\varepsilon_{1,2,3} \to \tilde{\varepsilon}_{1,2,3}$ and $\mu_{1,2,3} \to \tilde{\mu}_{1,2,3}$ in all quantities entering that formula and comprising the tensor elements. Such replacements are also used for obtaining the corresponding tilde quantities in (54).

It is now a simple matter to verify that the transverse components of the field in the homogeneous inner region, which correspond to the axial components (52) and (53), are written as follows:

$$\begin{aligned} E_{\rho;\,m,s,\alpha}(\rho,q) &= \frac{1}{2}\sum_{k=1}^{2} B_{m,s,\alpha}^{(k)}(q)\left(\tilde{\mathfrak{b}}_{\alpha,+}^{(k)} J_{m+1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right. \\ &\quad \left. - \tilde{\mathfrak{b}}_{\alpha,-}^{(k)} J_{m-1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right), \end{aligned} \tag{55}$$

$$\begin{aligned} E_{\phi;\,m,s,\alpha}(\rho,q) &= \frac{i}{2}\sum_{k=1}^{2} B_{m,s,\alpha}^{(k)}(q)\left(\tilde{\mathfrak{b}}_{\alpha,+}^{(k)} J_{m+1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right. \\ &\quad \left. + \tilde{\mathfrak{b}}_{\alpha,-}^{(k)} J_{m-1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right), \end{aligned} \tag{56}$$

$$\begin{aligned} H_{\rho;\,m,s,\alpha}(\rho,q) &= \frac{i}{2Z_0}\sum_{k=1}^{2} B_{m,s,\alpha}^{(k)}(q)\tilde{n}_{s,\alpha}^{(k)}\left(\tilde{\mathfrak{g}}_{\alpha,+}^{(k)} J_{m+1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right. \\ &\quad \left. + \tilde{\mathfrak{g}}_{\alpha,-}^{(k)} J_{m-1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right), \end{aligned} \tag{57}$$

$$\begin{aligned} H_{\phi;\,m,s,\alpha}(\rho,q) &= -\frac{1}{2Z_0}\sum_{k=1}^{2} B_{m,s,\alpha}^{(k)}(q)\tilde{n}_{s,\alpha}^{(k)}\left(\tilde{\mathfrak{g}}_{\alpha,+}^{(k)} J_{m+1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right. \\ &\quad \left. - \tilde{\mathfrak{g}}_{\alpha,-}^{(k)} J_{m-1}\left(k_0\tilde{q}_\alpha^{(k)}\rho\right)\right), \end{aligned} \tag{58}$$

where

$$\tilde{\mathfrak{b}}_{\alpha,\pm}^{(1,2)} = \pm\frac{\tilde{\varepsilon}_\pm \mp \tilde{n}_{s,\alpha}^{(1,2)} p_{s,\alpha}}{\tilde{\varepsilon}_1}, \quad \tilde{\mathfrak{g}}_{\alpha,\pm}^{(1,2)} = \frac{\tilde{\mu}_\pm \mp \left(\tilde{n}_{s,\alpha}^{(1,2)}\right)^{-1} p_{s,\alpha}}{\tilde{\mu}_1} \tag{59}$$

with $\tilde{\varepsilon}_\pm = \tilde{\varepsilon}_1 \pm \tilde{\varepsilon}_2$ and $\tilde{\mu}_\pm = \tilde{\mu}_1 \pm \tilde{\mu}_2$.

It may be noted that using formulas (29) and (30), expressions (47)–(50) and (55)–(58) for the transverse field components can be rewritten in another form that is similar to (31)–(34). Such a representation is more convenient when analyzing the field behavior at $\rho \to \infty$.

Satisfying the continuity conditions for the tangential field components at the boundary $\rho = a$, we arrive at the system of linear equations for unknown coefficients $B^{(1,2)}_{m,s,\alpha}$, $C^{(1,2)}_{m,s,\alpha}$, and $C_{m,s,\alpha}$. This system can thus be represented in matrix form:

$$\mathbf{S}\cdot\boldsymbol{G}=C^{(1)}_{m,s,\alpha}\boldsymbol{F}. \tag{60}$$

The elements of the matrix $\mathbf{S}$ and the column vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ are written in Appendix A.

The coefficients $B^{(1,2)}_{m,s,\alpha}$, $C^{(1,2)}_{m,s,\alpha}$, and $C_{m,s,\alpha}$ related by (60) ensure the fulfillment of the boundary conditions. However, matrix Equation (60) gives only four linear relationships for five coefficients $B^{(1,2)}_{m,s,\alpha}$, $C^{(1,2)}_{m,s,\alpha}$, and $C_{m,s,\alpha}$, so that one of these coefficients can be taken arbitrary. This circumstance reflects the fact that the mode fields are defined up to an arbitrary factor independent of the spatial coordinates. For convenience of the further calculations, we put $C^{(1)}_{m,s,\alpha}=i^m\det||\mathbf{S}||$. In this case, all the coefficients turn out to be finite, as is evident from their expressions given in Appendix A.

Some general remarks can be made concerning the coefficients $B^{(1,2)}_{m,s,\alpha}$, $C^{(1,2)}_{m,s,\alpha}$, and $C_{m,s,\alpha}$. It is seen from the differential equations for the axial fields that the solutions for the field components in the inner region $\rho<a$ are even functions of q since they depend on q^2. This is also evident from the expressions for $B^{(1,2)}_{m,s,\alpha}$. The field expressions for the outer region $\rho>a$ comprise the Hankel functions $H^{(1,2)}_m(k_0q\rho)$ having a branch point $q=0$, from which a branch cut goes along the negative real q axis. Therefore, it is necessary to distinguish the field quantities corresponding to the q values belonging to different sides of the branch cut. Using the relations

$$J_m\left(e^{\mp i\pi}\zeta\right)=(-1)^mJ_m(\zeta),\quad H^{(1)}_m\left(e^{i\pi}\zeta\right)=-e^{-im\pi}H^{(2)}_m(\zeta), \tag{61}$$

which are valid for cylindrical functions [32], and the fact that $p_\alpha(q)$, $q_\alpha(q)$, and $n^{(1,2)}_{s,\alpha}(q)$ are even functions of q, we get

$$\begin{aligned}C^{(1)}_{m,s,\alpha}\left(e^{-i\pi}q\right)&=(-1)^{m+1}C^{(2)}_{m,s,\alpha}(q),\\ C^{(2)}_{m,s,\alpha}\left(e^{i\pi}q\right)&=(-1)^{m+1}C^{(1)}_{m,s,\alpha}(q),\\ \Delta C_{m,s,\alpha}\left(e^{\mp i\pi}q\right)&=(-1)^{m+1}\Delta C_{m,s,\alpha}(q),\end{aligned} \tag{62}$$

where

$$\Delta C_{m,s,\alpha}(q)=C^{(2)}_{m,s,\alpha}(q)-C^{(1)}_{m,s,\alpha}(q).$$

Bearing this in mind, we make the rearrangements

$$\begin{aligned}&C^{(1)}_{m,s,\alpha}\left(e^{-i\pi}q\right)H^{(1)}_m\left(k_0e^{-i\pi}q\rho\right)+C^{(2)}_{m,s,\alpha}\left(e^{-i\pi}q\right)H^{(2)}_m\left(k_0e^{-i\pi}q\rho\right)\\ &=2C^{(1)}_{m,s,\alpha}\left(e^{-i\pi}q\right)J_m\left(k_0e^{-i\pi}q\rho\right)+\Delta C_{m,s,\alpha}\left(e^{-i\pi}q\right)H^{(2)}_m\left(k_0e^{-i\pi}q\rho\right)\end{aligned} \tag{63}$$

and

$$C^{(1)}_{m,s,\alpha}\left(e^{i\pi}q\right)H^{(1)}_m\left(k_0e^{i\pi}q\rho\right)+C^{(2)}_{m,s,\alpha}\left(e^{i\pi}q\right)H^{(2)}_m\left(k_0e^{i\pi}q\rho\right)$$
$$=-\Delta C_{m,s,\alpha}\left(e^{i\pi}q\right)H^{(1)}_m\left(k_0e^{i\pi}q\rho\right)+2C^{(2)}_{m,s,\alpha}\left(e^{i\pi}q\right)J_m\left(k_0e^{i\pi}q\rho\right)\quad(64)$$

in the expressions for $E_{z;\,m,s,\alpha}(\rho,q)$ and $H_{z;\,m,s,\alpha}(\rho,q)$ in the region $\rho > a$. Using the relationship $C_{m,s,\alpha}(e^{\mp i\pi}q)=C_{m,s,\alpha}(q)$ and formulas (61)–(64), we can deduce that in the outer region,

$$\begin{aligned}E_{z;\,m,s,\alpha}\left(\rho,e^{\mp i\pi}q\right)&=E_{z;\,m,s,\alpha}(\rho,q),\\ H_{z;\,m,s,\alpha}\left(\rho,e^{\mp i\pi}q\right)&=H_{z;\,m,s,\alpha}(\rho,q).\end{aligned}\quad(65)$$

Taking into account the expressions for the transverse components via the axial components, we find that the transverse components obey relationships that are similar to (65). We thus have

$$\begin{aligned}\boldsymbol{E}_{m,s,\alpha}\left(\rho,e^{\mp i\pi}q\right)&=\boldsymbol{E}_{m,s,\alpha}(\rho,q),\\ \boldsymbol{H}_{m,s,\alpha}\left(\rho,e^{\mp i\pi}q\right)&=\boldsymbol{H}_{m,s,\alpha}(\rho,q)\end{aligned}\quad(66)$$

for all values of ρ.

The foregoing analysis has been performed for a uniform cylindrical column immersed in a homogeneous background medium. In the case where the medium parameters inside the column depend on ρ, the field solutions in the region $\rho < a$ cannot be expressed in terms of known functions, and Equations (9) and (10) should be solved numerically for assumed radial profiles of the medium parameters. Two independent solutions, $\tilde{E}^{(1)}_{z;\,m,s,\alpha}(\rho,q)$, $\tilde{H}^{(1)}_{z;\,m,s,\alpha}(\rho,q)$ and $\tilde{E}^{(2)}_{z;\,m,s,\alpha}(\rho,q)$, $\tilde{H}^{(2)}_{z;\,m,s,\alpha}(\rho,q)$, that are regular at $\rho=0$ can be found, and they should be used instead of the Bessel functions for the field inside the radially nonuniform column. This yields

$$\begin{aligned}E_{z;\,m,s,\alpha}(\rho,q)&=\sum_{k=1}^{2}B^{(k)}_{m,s,\alpha}(q)\tilde{E}^{(k)}_{z;\,m,s,\alpha}(\rho,q),\\ H_{z;\,m,s,\alpha}(\rho,q)&=\sum_{k=1}^{2}B^{(k)}_{m,s,\alpha}(q)\tilde{H}^{(k)}_{z;\,m,s,\alpha}(\rho,q).\end{aligned}\quad(67)$$

Note that the particular solutions denoted by the superscripts $k=1$ and $k=2$ in (67) are distinguished by their behavior near the axis $\rho=0$, in the vicinity of which they are described by Bessel functions with the arguments $k_0\tilde{q}^{(1)}_\alpha(0)\rho$ and $k_0\tilde{q}^{(2)}_\alpha(0)\rho$, respectively, where the local transverse wave numbers $\tilde{q}^{(1)}_\alpha(0)$ and $\tilde{q}^{(2)}_\alpha(0)$ correspond to the parameters of the medium at $\rho = 0$. The solutions for the field outside the column remain intact. Next, satisfying the boundary conditions at $\rho = a$, we again arrive at a matrix equation in the form (60). Taking into account the above-discussed properties of Equations (9) and (10) and repeating the foregoing analysis for solutions in the inner and outer regions, we can show that the resulting field will satisfy the relationships (66) in the case of a nonuniform column, as well. Therefore, allowance for radial nonuniformity of a cylindrical column surrounded by a homogeneous background medium does not require any significant modifications of the present procedure, and only a consistent use of new field quantities for the region $\rho < a$ is needed.

The obtained field representation allows us to find the spectrum of eigenvalues q and the corresponding eigenfunctions of the guiding structure. First, it is easy to verify that the field represented by (44), (45), and (47)–(50) for $\rho > a$ satisfies the boundedness conditions (43) for all real transverse wave numbers q. Next, it follows from (66) that the negative values of q do not yield new solutions and can be excluded from the analysis. Thus, all positive values of q constitute the continuous eigenvalue spectrum.

Along with the continuous spectrum of q, conditions (43) can also be ensured for certain discrete complex values $q = q_{m,n}$ $(n = 1, 2, \dots)$ which satisfy the equations

$$C^{(1)}_{m,s,\alpha}(q_{m,n}) = 0 \quad \text{for} \quad \operatorname{Im} q_{m,n} < 0 \tag{68}$$

and

$$C^{(2)}_{m,s,\alpha}(q_{m,n}) = 0 \quad \text{for} \quad \operatorname{Im} q_{m,n} > 0. \tag{69}$$

With allowance for (62) and (66), it can easily be verified that the roots of Equation (69) do not give new solutions for the field compared with those yielded by the solutions of Equation (68) and can therefore be rejected. Since the quantities $\hat{q}_{m,n} = q_\alpha(q_{m,n})$ also satisfy Equation (68), we should unambiguously distinguish between $q_{m,n}$ and $\hat{q}_{m,n}$, to choose only one set of the discrete values satisfying Equation (68). To this end, we assume that the discrete part of the spectrum of eigenvalues q is constituted only by the roots of Equation (68) for which the inequality $|\operatorname{Im} q_{m,n}| < |\operatorname{Im} q_\alpha(q_{m,n})|$ takes place. The subscript α for which this inequality holds will further be denoted by $\hat{\alpha}$. It is evident that the waves corresponding to the discrete values $q_{m,n}$ are localized eigenmodes (discrete-spectrum modes) of the considered guiding structure. The fields of these modes decay to zero for $\rho \to \infty$ and will further be written in the form

$$\begin{bmatrix} \boldsymbol{E}_{m,n_s}(\boldsymbol{r}) \\ \boldsymbol{H}_{m,n_s}(\boldsymbol{r}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}_{m,n_s}(\rho) \\ \boldsymbol{H}_{m,n_s}(\rho) \end{bmatrix} \exp(-im\phi - ik_0 p_{m,n_s} z), \tag{70}$$

where the vector functions $\boldsymbol{E}_{m,n_s}(\rho) = \boldsymbol{E}_{m,s,\hat{\alpha}}(\rho, q_{m,n})$ and $\boldsymbol{H}_{m,n_s}(\rho) = \boldsymbol{H}_{m,s,\hat{\alpha}}(\rho, q_{m,n})$ describe the field distribution of an eigenmode with indices m and n_s, and $p_{m,n_s} = p_{s,\hat{\alpha}}(q_{m,n})$ is the axial wave number of this mode. It is assumed here that the indices $n_+ = n > 0$ and $n_- = -n < 0$ correspond to discrete-spectrum modes propagating in the positive and negative directions of the z axis, respectively.

Thus, we have established that a complete set of normal modes of the considered open structure comprises the discrete spectrum of guided eigenmodes and the continuous spectrum of unguided modes. It is important that the developed formulation enables us to use only one transverse wave number q in the outer medium as an eigenvalue determining a particular mode for fixed m, s, and α. This is ensured by using the concept of the auxiliary transverse wave number $q_\alpha(q)$ which represents the other transverse wave number of the mode in the outer region as a function of q.

4.3. Mode Orthogonality and Expansion Coefficients

The discrete- and continuous-spectrum modes that were determined in the foregoing section satisfy the following orthogonality relations:

$$J_{\tilde{m},\tilde{n}}^{m,n} = \int_0^{2\pi}\int_0^{\infty} \left[\boldsymbol{E}_{m,n}(\boldsymbol{r}) \times \boldsymbol{H}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r}) - \boldsymbol{E}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r}) \times \boldsymbol{H}_{m,n}(\boldsymbol{r})\right] \cdot \hat{\boldsymbol{z}}_0\, \rho\, d\rho\, d\phi = N_{m,n}\delta_{m,-\tilde{m}}\, \delta_{n,-\tilde{n}}, \tag{71}$$

$$J_{\tilde{m},\tilde{n}}^{m,s,\alpha} = \int_0^{2\pi}\int_0^{\infty} \left[\boldsymbol{E}_{m,s,\alpha}(\boldsymbol{r}, q) \times \boldsymbol{H}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r}) - \boldsymbol{E}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r}) \times \boldsymbol{H}_{m,s,\alpha}(\boldsymbol{r}, q)\right] \cdot \hat{\boldsymbol{z}}_0\, \rho\, d\rho\, d\phi = 0, \tag{72}$$

$$J_{\tilde{m},\tilde{s},\tilde{\alpha}}^{m,s,\alpha} = \int_0^{2\pi}\int_0^{\infty}\left[\boldsymbol{E}_{m,s,\alpha}(\boldsymbol{r}, q)\times\boldsymbol{H}_{\tilde{m},\tilde{s},\tilde{\alpha}}^{(\mathrm{T})}(\boldsymbol{r}, \tilde{q}) - \boldsymbol{E}_{\tilde{m},\tilde{s},\tilde{\alpha}}^{(\mathrm{T})}(\boldsymbol{r}, \tilde{q})\times\boldsymbol{H}_{m,s,\alpha}(\boldsymbol{r}, q)\right] \cdot \hat{\boldsymbol{z}}_0\, \rho\, d\rho\, d\phi = N_{m,s,\alpha}(q)\, \delta(q - \tilde{q})\delta_{m,-\tilde{m}}\, \delta_{s,-\tilde{s}}\delta_{\alpha,\tilde{\alpha}}, \tag{73}$$

where $\hat{\boldsymbol{z}}_0$ is the unit vector aligned with the z axis, $\delta_{\alpha,\beta}$ is the Kronecker delta, $\delta(q)$ is the Dirac function, and the superscript (T) denotes fields taken in an auxiliary ("transposed") medium described by the transposed tensors $\boldsymbol{\varepsilon}^{\mathrm{T}}$ and $\boldsymbol{\mu}^{\mathrm{T}}$. The normalization quantities $N_{m,n}$ and $N_{m,\alpha} = N_{m,+,\alpha}$ for waves propagating in the positive direction of the z axis are given by the formulas

$$N_{m,\alpha}(q) = -\frac{16\pi}{Z_0 k_0^2}\left(\frac{dp_\alpha(q)}{dq}\right)^{-1} \times \left[\mu_3^{-1} + \varepsilon_3^{-1}\left(n_{s,\alpha}^{(1)}\right)^2\right] C_{m,s,\alpha}^{(1)}(q) C_{m,s,\alpha}^{(2)}(q), \tag{74}$$

$$N_{m,n} = \frac{1}{2\pi i} \frac{dN_{m,\hat{\alpha}}}{dq}\bigg|_{q=q_{m,n}}. \tag{75}$$

It is evident that $N_{m,-n} = -N_{m,n}$ and $N_{m,-,\alpha}(q) = -N_{m,+,\alpha}(q)$. Orthogonality relations (71)–(73) and formulas (74) and (75) are derived in Appendix B.

Using the above orthogonality relations, we can calculate the amplitude coefficients of the discrete- and continuous-spectrum modes due to given sources. Outside the source region, i.e., for $z < z_1$ and $z > z_2$, the total field can be expanded in the form

$$\begin{bmatrix} \boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) \end{bmatrix} = \sum_{m=-\infty}^{\infty} \left(\sum_{n_s} a_{m,n_s} \begin{bmatrix} \boldsymbol{E}_{m,n_s}(\boldsymbol{r}) \\ \boldsymbol{H}_{m,n_s}(\boldsymbol{r}) \end{bmatrix} + \sum_{\alpha} \int_0^{\infty} a_{m,s,\alpha}(q) \begin{bmatrix} \boldsymbol{E}_{m,s,\alpha}(\boldsymbol{r}, q) \\ \boldsymbol{H}_{m,s,\alpha}(\boldsymbol{r}, q) \end{bmatrix} dq \right), \tag{76}$$

where a_{m,n_s} and $a_{m,s,\alpha}$ are the expansion coefficients of the discrete- and continuous-spectrum modes, respectively. In (76), one should take $n_s = n > 0$ and $s = +$ for $z > z_2$, and $n_s = -n$ and $s = -$ for $z < z_1$ (see Fig. 1). Then, following the well-known method for determining the expansion coefficients of waves of closed and open waveguides with gyrotropic filling (see, e.g., [3, 12, 16, 33]) on the basis of Lorentz's theorem in the transposed form [35, 36], we find that the expansion coefficients in (76) are given by the following expressions:

$$a_{m,\pm n} = \frac{1}{N_{m,n}} \int \left[\boldsymbol{J}^{\mathrm{e}}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(\mathrm{T})}_{-m,\mp n}(\boldsymbol{r}) - \boldsymbol{J}^{\mathrm{m}}(\boldsymbol{r}) \cdot \boldsymbol{H}^{(\mathrm{T})}_{-m,\mp n}(\boldsymbol{r}) \right] d\boldsymbol{r}, \tag{77}$$

$$a_{m,\pm,\alpha}(q) = \frac{1}{N_{m,\alpha}(q)} \int \left[\boldsymbol{J}^{\mathrm{e}}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(\mathrm{T})}_{-m,\mp,\alpha}(\boldsymbol{r}, q) - \boldsymbol{J}^{\mathrm{m}}(\boldsymbol{r}) \cdot \boldsymbol{H}^{(\mathrm{T})}_{-m,\mp,\alpha}(\boldsymbol{r}, q) \right] d\boldsymbol{r}. \tag{78}$$

Here, integration is performed over the region occupied by currents [3]. We do not dwell on the procedure of deriving expressions (77) and (78) in detail since it fully coincides with that discussed for open gyroelectric waveguides in [3, 16]. Following the line of approach of [3], it is also possible to derive an expression for the total field in the source region

$z_1 < z < z_2$:

$$\begin{bmatrix} \boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) \end{bmatrix} = \sum_{m=-\infty}^{\infty} \sum_{s=-}^{+} \left(\sum_{n_s} a_{m,n_s}(z) \begin{bmatrix} \boldsymbol{E}_{m,n_s}(\boldsymbol{r}) \\ \boldsymbol{H}_{m,n_s}(\boldsymbol{r}) \end{bmatrix} + \sum_{\alpha} \int_0^{\infty} a_{m,s,\alpha}(z,q) \begin{bmatrix} \boldsymbol{E}_{m,s,\alpha}(\boldsymbol{r},q) \\ \boldsymbol{H}_{m,s,\alpha}(\boldsymbol{r},q) \end{bmatrix} dq \right) + \frac{i}{k_0} \begin{bmatrix} \varepsilon_3^{-1}(\rho) Z_0 J_z^e \\ \mu_3^{-1}(\rho) Z_0^{-1} J_z^m \end{bmatrix} \hat{\boldsymbol{z}}_0, \tag{79}$$

where $\varepsilon_3(\rho)$ and $\mu_3(\rho)$ denote the values of the corresponding elements of tensors (1) at the considered point of space. In contrast to the preceding expansion (76), the expansion coefficients in (79) are z-dependent. They are found to be

$$a_{m,n_s}(z) = \frac{1}{N_{m,n}} \int_{\left(z_s^{(1)},\, z_s^{(2)}\right)} \left[\boldsymbol{J}^{\mathrm{e}}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(\mathrm{T})}_{-m,-n_s}(\boldsymbol{r}) - \boldsymbol{J}^{\mathrm{m}}(\boldsymbol{r}) \cdot \boldsymbol{H}^{(\mathrm{T})}_{-m,-n_s}(\boldsymbol{r}) \right] d\boldsymbol{r}, \tag{80}$$

$$a_{m,s,\alpha}(z,q) = \frac{1}{N_{m,\alpha}(q)} \int_{\left(z_s^{(1)},\, z_s^{(2)}\right)} \left[\boldsymbol{J}^{\mathrm{e}}(\boldsymbol{r}) \cdot \boldsymbol{E}^{(\mathrm{T})}_{-m,-s,\alpha}(\boldsymbol{r},q) - \boldsymbol{J}^{\mathrm{m}}(\boldsymbol{r}) \cdot \boldsymbol{H}^{(\mathrm{T})}_{-m,-s,\alpha}(\boldsymbol{r},q) \right] d\boldsymbol{r}, \tag{81}$$

where the notation $\left(z_s^{(1)}, z_s^{(2)}\right)$ stands to designate the interval of integration with respect to z:

$$z_s^{(1)} = \begin{cases} z_1 \text{ for } s = +, \\ z \ \text{ for } s = -, \end{cases} \qquad z_s^{(2)} = \begin{cases} z \ \text{ for } s = +, \\ z_2 \text{ for } s = -. \end{cases} \tag{82}$$

Now it remains to check that the total field obtained in the above satisfies the radiation condition at infinity ($r = (\rho^2 + z^2)^{1/2} \to \infty$). To this end, we should note that the contributions of the discrete-spectrum modes and the terms comprising Hankel functions with the argument $k_0 q_\alpha \rho$ vanish in the outer medium in the limit $r \to \infty$. Assuming for simplicity that the medium is lossless, we can thus conclude that the nonzero contributions to the far field come only from the remaining terms in integrals over the values of q for which the functions $p_\alpha(q)$ are real-valued. Taking the large-argument approximations of Hankel functions with the argument $k_0 q \rho$ and applying the saddle-point method for evaluation of the integrals over the corresponding

q values, we can arrive at the sum of spherical waves in the far-field representation. The number of such waves at an observation point is determined by the number of rays coming to it. This implies that the obtained field indeed satisfies the radiation condition at infinity. We do not present the corresponding far-field expressions since the procedure of their derivation is similar to that discussed in [3] for a gyroelectric outer medium. Furthermore, we leave aside the discussion of the possible influence of leaky modes, which can be separated from the integral over the continuous spectrum in some cases, on the radiation field [3, 23, 25]. This needs separate consideration in itself, and falls beyond the scope of this paper.

It is important to emphasize that the derived formulas allow one to immediately obtain the source-excited field without preliminary calculation of the dyadic Green's functions. This is the main implication of the presented approach. Of course, if required, this approach can be used to derive the components of the dyadic Green's functions in the form of eigenfunction expansions. To do this, one should take expansion (76) and substitute the appropriate point sources for $\boldsymbol{J}^{\mathrm{e}}(\boldsymbol{r})$ and $\boldsymbol{J}^{\mathrm{m}}(\boldsymbol{r})$ into (77) and (78).

4.4. Limiting Transition to the Case of a Homogeneous Medium

It is instructive to consider the transition from the eigenfunction expansion obtained for a cylindrically nonuniform medium to the case of a homogeneous medium described by tensors (1). By solving the above-posed boundary-value problem with the boundedness conditions (43) at $\rho \to \infty$, it can be found that in the case of a homogeneous medium, the eigenvalue spectrum consists only of positive real values of q, and the axial components of the corresponding continuous-spectrum modes are written as

$$E_{z;\,m,s,\alpha}(\rho, q) = i\varepsilon_3^{-1}\mathcal{E}_{m,s,\alpha} n_{s,\alpha}^{(1)} q J_m(k_0 q\rho), \tag{83}$$

$$H_{z;\,m,s,\alpha}(\rho, q) = -Z_0^{-1}\mu_3^{-1}\mathcal{E}_{m,s,\alpha} q J_m(k_0 q\rho), \tag{84}$$

where $\mathcal{E}_{m,s,\alpha}$ is an arbitrary factor independent of ρ. It is evident that the modes given by (83) and (84) can formally be obtained from the expressions (44), (45), and (47)–(50) by putting

$$C_{m,s,\alpha}^{(1)} = \mathcal{E}_{m,s,\alpha}/2, \quad C_{m,s,\alpha}^{(2)} = \mathcal{E}_{m,s,\alpha}/2, \quad C_{m,s,\alpha} = 0. \tag{85}$$

Note that a particular form of the coefficient $\mathcal{E}_{m,s,\alpha}$ is unimportant since it cancels in the resulting expansion for the total source-excited field. Nevertheless, it is convenient to take this coefficient in the form $\mathcal{E}_{m,s,\alpha} = i^m$. With such a choice, the relationships in (B6) will immediately be ensured, which facilitates the procedure of establishing mode orthogonality. In the considered case, the continuous-spectrum modes obey the orthogonality relations (73) in which the normalization quantity is again given by (74), with allowance for (85). The fields inside and outside the source region are given by expansions (76) and (79), respectively, in which one should omit the discrete wave terms that are absent in the case of an unbounded homogeneous medium. The expansion coefficients of the remaining continuous-spectrum waves are given by the previous formulas (78) and (81).

5. THE CASE OF A UNIAXIALLY ANISOTROPIC OUTER MEDIUM

If $\tau_{\rm g} = 0$ but $\varepsilon_2 \neq 0$ and $\mu_2 \neq 0$, then the outer medium becomes a generalized uniaxial one. The case where $\tau_{\rm g} = 0$ also includes the special case of an uniaxially anisotropic medium with $\varepsilon_2 = \mu_2 = 0$.

Consider a cylindrical guiding system located in a generalized uniaxial outer medium. In such a medium, i.e., for $\tau_{\rm g} = 0$, the set of coupled Equations (9) and (10) splits into two independent equations for $E_{z;\,m,s,\alpha}(\rho, q)$ and $H_{z;\,m,s,\alpha}(\rho, q)$:

$$\hat{L}E_{z;\,m,s,\alpha} + k_0^2\varepsilon_3\left(\frac{\mu_1^2-\mu_2^2}{\mu_1} - \frac{p_\alpha^2}{\varepsilon_1}\right)E_{z;\,m,s,\alpha} = 0, \tag{86}$$

$$\hat{L}H_{z;\,m,s,\alpha} + k_0^2\mu_3\left(\frac{\varepsilon_1^2-\varepsilon_2^2}{\varepsilon_1} - \frac{p_\alpha^2}{\mu_1}\right)H_{z;\,m,s,\alpha} = 0. \tag{87}$$

The functions $p_\alpha(q)$ for $\alpha = 1$ and $\alpha = 2$ can be written as

$$\begin{aligned} p_1(q) &= \left(\varepsilon_1\frac{\mu_1^2-\mu_2^2}{\mu_1} - \frac{\varepsilon_1}{\varepsilon_3}q^2\right)^{1/2}, \\ p_2(q) &= \left(\mu_1\frac{\varepsilon_1^2-\varepsilon_2^2}{\varepsilon_1} - \frac{\mu_1}{\mu_3}q^2\right)^{1/2}. \end{aligned} \tag{88}$$

The corresponding auxiliary transverse wave numbers $q_\alpha(q)$ have the form

$$q_1(q) = \left(\frac{\varepsilon_1\mu_3}{\varepsilon_3\mu_1}\right)^{1/2} q, \quad q_2(q) = \left(\frac{\varepsilon_3\mu_1}{\varepsilon_1\mu_3}\right)^{1/2} q. \tag{89}$$

Recall that conditions (40) and (41) are assumed to hold. For $\alpha \neq \beta$, we have

$$p_\alpha(q) = p_\beta(q_\alpha(q)).$$

It is important to emphasize that the function p_α is taken for the same value of q in each of Equations (86) and (87) in order that their solutions yield fields satisfying the boundary conditions at $\rho = a$. For $\alpha = 1$, Equations (86) and (87) reduce to

$$\hat{L}E_{z;\,m,s,\alpha} + (k_0 q)^2 E_{z;\,m,s,\alpha} = 0, \tag{90}$$

$$\hat{L}H_{z;\,m,s,\alpha} + (k_0 q_\alpha(q))^2 H_{z;\,m,s,\alpha} = 0. \tag{91}$$

The solution of these equations is written as

$$E_{z;\,m,s,\alpha}(\rho,q) = \frac{ip_{s,\alpha}}{\varepsilon_3}\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q)\,q H^{(k)}_m(k_0 q\rho), \tag{92}$$

$$H_{z;\,m,s,\alpha}(\rho,q) = -\frac{1}{Z_0\mu_3} C_{m,s,\alpha}(q)\,q_\alpha H^{(2)}_m(k_0 q_\alpha \rho). \tag{93}$$

For $\alpha = 2$, from (86) and (87) we have the equations

$$\hat{L}E_{z;\,m,s,\alpha} + (k_0 q_\alpha(q))^2 E_{z;\,m,s,\alpha} = 0, \tag{94}$$

$$\hat{L}H_{z;\,m,s,\alpha} + (k_0 q)^2 H_{z;\,m,s,\alpha} = 0, \tag{95}$$

whose solution is

$$E_{z;\,m,s,\alpha}(\rho,q) = \frac{ip_{s,\alpha}}{\varepsilon_3} C_{m,s,\alpha}(q)\,q_\alpha H^{(2)}_m(k_0 q_\alpha \rho), \tag{96}$$

$$H_{z;\,m,s,\alpha}(\rho,q) = -\frac{1}{Z_0\mu_3}\sum_{k=1}^{2} C^{(k)}_{m,s,\alpha}(q)\,q H^{(k)}_m(k_0 q\rho). \tag{97}$$

The meanings of the notations used in (90)–(97) are the same as in Sec. 4. Thus, it is seen that for fixed indices m, s, and α, the solutions of two field equations again comprise both the transverse wavenumber q and the auxiliary transverse wave number $q_\alpha(q)$.

If the medium of the inner region remains gyrotropic, then we have two solutions with coefficients $B^{(1)}_{m,s,\alpha}$ and $B^{(2)}_{m,s,\alpha}$ for $\rho < a$. Satisfying the boundary conditions at $\rho = a$, we arrive at four equations for five coefficients $B^{(1,2)}_{m,s,\alpha}$, $C^{(1,2)}_{m,s,\alpha}$, and $C_{m,s,\alpha}$ in matrix form (60). The problem has become one coinciding with that considered in Sec. 4. In the case of a uniaxial medium in the inner region, we again have a similar problem if modes are nonsymmetric, i.e., $m \neq 0$. This fact is stipulated by the hybrid nature of nonsymmetric modes in the considered case. In contrast to this, the axisymmetric ($m = 0$) modes of a uniaxially anisotropic open waveguide split into TE and

TM waves, and the boundary conditions are ensured for them without terms comprising the auxiliary transverse wave number.

Thus, the approach of Sec. 4 remains applicable when dealing with a cylindrically stratified uniaxially anisotropic medium. Note that in such a case, the superscript (T) may be omitted in formulas representing orthogonality relations and modal expansion coefficients. In addition, it is worth mentioning that the discrete- and continuous-spectrum modes in the case discussed in this section can be obtained by an alternative approach based on the so-called S-operator method (see for details [19] and references therein).

6. POWER RADIATED

The total power P_Σ radiated from given currents $\boldsymbol{J}^{\mathrm{e}}$ and $\boldsymbol{J}^{\mathrm{m}}$ can be calculated by the formula

$$P_\Sigma = -\frac{1}{2}\,\mathrm{Re}\int\left[(\boldsymbol{J}^{\mathrm{e}}(\boldsymbol{r}))^*\cdot\boldsymbol{E}(\boldsymbol{r}) + (\boldsymbol{J}^{\mathrm{m}}(\boldsymbol{r}))^*\cdot\boldsymbol{H}(\boldsymbol{r})\right]d\boldsymbol{r}, \tag{98}$$

where the integration is performed over the source region and the asterisk denotes complex conjugation. In the absence of losses in a medium, this power can also be calculated by integrating the Poynting vector over two infinite cross-sectional areas on both sides of the source region. For evaluations of integrals in this case, it is convenient to take into account the fact that in a loss-free medium, the discrete- and continuous-spectrum modes with real axial wave numbers satisfy the power orthogonality relations

$$\int_0^{2\pi}\int_0^\infty\left[\boldsymbol{E}_{m,n}(\boldsymbol{r})\times\boldsymbol{H}^*_{\tilde{m},\tilde{n}}(\boldsymbol{r}) + \boldsymbol{E}^*_{\tilde{m},\tilde{n}}(\boldsymbol{r})\times\boldsymbol{H}_{m,n}(\boldsymbol{r})\right] \cdot\hat{\boldsymbol{z}}_0\rho\,d\rho\,d\phi = 4\mathcal{P}_{m,n}\delta_{m,\tilde{m}}\delta_{n,\tilde{n}}, \tag{99}$$

$$\int_0^{2\pi}\int_0^\infty\left[\boldsymbol{E}_{m,s,\alpha}(\boldsymbol{r},q)\times\boldsymbol{H}^*_{\tilde{m},\tilde{n}}(\boldsymbol{r}) + \boldsymbol{E}^*_{\tilde{m},\tilde{n}}(\boldsymbol{r})\times\boldsymbol{H}_{m,s,\alpha}(\boldsymbol{r},q)\right] \cdot\hat{\boldsymbol{z}}_0\rho\,d\rho\,d\phi = 0, \tag{100}$$

$$\int_0^{2\pi}\int_0^\infty\left[\boldsymbol{E}_{m,s,\alpha}(\boldsymbol{r},q)\times\boldsymbol{H}^*_{\tilde{m},\tilde{s},\tilde{\alpha}}(\boldsymbol{r},\tilde{q}) + \boldsymbol{E}^*_{\tilde{m},\tilde{s},\tilde{\alpha}}(\boldsymbol{r},\tilde{q})\times\boldsymbol{H}_{m,s,\alpha}(\boldsymbol{r},q)\right] \cdot\hat{\boldsymbol{z}}_0\rho\,d\rho\,d\phi = 4\mathcal{P}_{m,s,\alpha}(q)\delta(q-\tilde{q})\delta_{m,\tilde{m}}\delta_{s,\tilde{s}}\delta_{\alpha,\tilde{\alpha}}, \tag{101}$$

where

$$\mathcal{P}_{m,+,\alpha}(q) = -\mathcal{P}_{m,-,\alpha}(q) = -\frac{2\pi}{Z_0k_0^2}\left(\frac{dp_\alpha(q)}{dq}\right)^{-1}\left[{\mu_3}^{-1}+\varepsilon_3^{-1}\left(n^{(1)}_{+,\alpha}\right)^2\right] \times\left[\left|C^{(1)}_{m,+,\alpha}(q)\right|^2+\left|C^{(2)}_{m,+,\alpha}(q)\right|^2\right]. \tag{102}$$

The derivation of relations (99)–(101) is basically analogous to that of orthogonality relations (71)–(73) and the mathematical details are not given here. However, there is an important difference between relations (71)–(73) and (99)–(101). It consists in that in contrast to the power orthogonality relations which cease to hold in lossy media, relations (71)–(73) remain valid regardless of whether the considered medium is lossy or lossless.

With allowance for (98)–(101), the expression for the total radiated power takes the following form:

$$P_\Sigma = \sum_{m=-\infty}^{+\infty} \Big\{ \sum_n \left[|a_{m,n}|^2 + |a_{m,-n}|^2\right] \mathcal{P}_{m,n} + \sum_\alpha \int \left[|a_{m,+,\alpha}(q)|^2 + |a_{m,-,\alpha}(q)|^2\right] \mathcal{P}_{m,+,\alpha}(q)\, dq \Big\}, \quad (103)$$

where the integration is performed over the positive real values of q for which the axial wave numbers $p_{s,\alpha}(q)$ are purely real. Note that the term $P_{m,\pm n} = |a_{m,\pm n}|^2 \mathcal{P}_{m,n}$ is the partial power going to a mode with indices m and $\pm n$. It is seen from (103) that in a lossless medium, the total radiated power is given by the sum of the partial powers going to individual discrete-spectrum modes and the power going to the continuous-spectrum modes. This fact obviously results from the power orthogonality which takes place in lossless media.

7. CONCLUSION

In this paper, we presented the complete eigenfunction expansion of the total electromagnetic field excited by spatially bounded given sources in a cylindrically stratified gyrotropic medium. The field has been expanded in terms of modes whose spectrum comprises both the discrete and continuous parts, and the expansion coefficients of the discrete- and continuous-spectrum modes have been calculated using a method based on Lorentz's theorem in the transposed formulation. Our analysis extends the theory of excitation of open waveguides in a gyroelectric medium to the case of open guiding structures located in a generalized gyrotropic medium whose permittivity and permeability are both described by off-diagonal tensors. Although the problem of excitation of guiding structures in such a medium, also called bigyrotropic, can be solved using the dyadic Green's functions, the approach developed herein makes it possible to immediately obtain the source-excited field without preliminary calculation of the dyadic Green's functions, which significantly facilitates evaluation of the field.

Finally, we note that this study is a necessary step towards considering special cases of wave excitation in natural or artificial gyrotropic media containing cylindrical nonuniformities, which can be of interest from academic or applied viewpoints.

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APPENDIX A. EXPRESSIONS FOR THE FIELD COEFFICIENTS

The elements of the matrix $\mathbf{S}$ and the column vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ in (60) are written as follows:

$$\begin{aligned}
&S_{1k} = \tilde{\mu}_3^{-1}\tilde{Q}_\alpha^{(k)} J_m\left(\tilde{Q}_\alpha^{(k)}\right),\\
&S_{13} = -\mu_3^{-1} Q H_m^{(2)}(Q),\ \ S_{14} = -\mu_3^{-1} Q_\alpha H_m^{(2)}(Q_\alpha),\\
&S_{2k} = \tilde{\varepsilon}_3^{-1}\tilde{n}_{s,\alpha}^{(k)}\tilde{Q}_\alpha^{(k)} J_m\left(\tilde{Q}_\alpha^{(k)}\right),\\
&S_{23} = -\varepsilon_3^{-1} n_{s,\alpha}^{(1)} Q H_m^{(2)}(Q),\ \ S_{24} = -\varepsilon_3^{-1} n_{s,\alpha}^{(2)} Q_\alpha H_m^{(2)}(Q_\alpha),\\
&S_{3k} = \tilde{Q}_\alpha^{(k)} J_m\left(\tilde{Q}_\alpha^{(k)}\right) J_m^{(k)},\\
&S_{33} = -Q H_m^{(2)}(Q)\mathcal{H}_m^{(2)},\ \ S_{34} = -Q_\alpha H_m^{(2)}(Q_\alpha)\mathcal{H}_m,\\
&S_{4k} = \tilde{n}_{s,\alpha}^{(k)}\tilde{Q}_\alpha^{(k)} J_m\left(\tilde{Q}_\alpha^{(k)}\right)\hat{J}_m^{(k)},\\
&S_{43} = -n_{s,\alpha}^{(1)} Q H_m^{(2)}(Q)\hat{\mathcal{H}}_m^{(2)},\ S_{44} = -n_{s,\alpha}^{(2)} Q_\alpha H_m^{(2)}(Q_\alpha)\hat{\mathcal{H}}_m; \quad \text{(A1)}\\
&F_1 = \mu_3^{-1} Q H_m^{(1)}(Q),\ \ F_2 = \varepsilon_3^{-1} n_{s,\alpha}^{(1)} Q H_m^{(1)}(Q),\\
&F_3 = Q H_m^{(1)}(Q)\mathcal{H}_m^{(1)},\ \ F_4 = n_{s,\alpha}^{(1)} Q H_m^{(1)}(Q)\hat{\mathcal{H}}_m^{(1)}; \quad \text{(A2)}\\
&G_k = B_{m,s,\alpha}^{(k)},\ \ G_3 = C_{m,s,\alpha}^{(2)},\ \ G_4 = C_{m,s,\alpha}, \quad \text{(A3)}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{Q}_{\alpha}^{(k)} &= k_0 \tilde{q}_{\alpha}^{(k)} a, \qquad Q = k_0 q a, \\
Q_{\alpha} &= k_0 q_{\alpha} a, \qquad k = 1, 2, \\
J_m^{(k)} &= \tilde{\mathfrak{b}}_{\alpha,+}^{(k)} \frac{J_{m+1}\left(\tilde{Q}_{\alpha}^{(k)}\right)}{\tilde{Q}_{\alpha}^{(k)} J_m\left(\tilde{Q}_{\alpha}^{(k)}\right)} + \tilde{\mathfrak{b}}_{\alpha,-}^{(k)} \frac{J_{m-1}\left(\tilde{Q}_{\alpha}^{(k)}\right)}{\tilde{Q}_{\alpha}^{(k)} J_m\left(\tilde{Q}_{\alpha}^{(k)}\right)}, \\
\hat{J}_m^{(k)} &= \tilde{\mathfrak{g}}_{\alpha,+}^{(k)} \frac{J_{m+1}\left(\tilde{Q}_{\alpha}^{(k)}\right)}{\tilde{Q}_{\alpha}^{(k)} J_m\left(\tilde{Q}_{\alpha}^{(k)}\right)} - \tilde{\mathfrak{g}}_{\alpha,-}^{(k)} \frac{J_{m-1}\left(\tilde{Q}_{\alpha}^{(k)}\right)}{\tilde{Q}_{\alpha}^{(k)} J_m\left(\tilde{Q}_{\alpha}^{(k)}\right)}, \\
\mathcal{H}_m^{(k)} &= \mathfrak{b}_{\alpha,+}^{(1)} \frac{H_{m+1}^{(k)}(Q)}{Q H_m^{(k)}(Q)} + \mathfrak{b}_{\alpha,-}^{(1)} \frac{H_{m-1}^{(k)}(Q)}{Q H_m^{(k)}(Q)}, \\
\hat{\mathcal{H}}_m^{(k)} &= \mathfrak{g}_{\alpha,+}^{(1)} \frac{H_{m+1}^{(k)}(Q)}{Q H_m^{(k)}(Q)} - \mathfrak{g}_{\alpha,-}^{(1)} \frac{H_{m-1}^{(k)}(Q)}{Q H_m^{(k)}(Q)}, \\
\mathcal{H}_m &= \mathfrak{b}_{\alpha,+}^{(2)} \frac{H_{m+1}^{(2)}(Q_{\alpha})}{Q_{\alpha} H_m^{(2)}(Q_{\alpha})} + \mathfrak{b}_{\alpha,-}^{(2)} \frac{H_{m-1}^{(2)}(Q_{\alpha})}{Q_{\alpha} H_m^{(2)}(Q_{\alpha})}, \\
\hat{\mathcal{H}}_m &= \mathfrak{g}_{\alpha,+}^{(2)} \frac{H_{m+1}^{(2)}(Q_{\alpha})}{Q_{\alpha} H_m^{(2)}(Q_{\alpha})} - \mathfrak{g}_{\alpha,-}^{(2)} \frac{H_{m-1}^{(2)}(Q_{\alpha})}{Q_{\alpha} H_m^{(2)}(Q_{\alpha})}.
\end{aligned} \tag{A4}$$

The coefficients $B_{m,s,\alpha}^{(1,2)}$, $C_{m,s,\alpha}^{(1,2)}$, and $C_{m,s,\alpha}$ are given by the expressions

$$\begin{aligned}
B_{m,s,\alpha}^{(1,2)} &= \Delta_{m,\alpha} \chi^{(1,2)} \frac{8}{i\pi} \left[\frac{n_{s,\alpha}^{(1)} - n_{s,\alpha}^{(2)}}{\varepsilon_3 \mu_3} \left(n_{s,\alpha}^{(1)} J_m^{(2,1)} - \tilde{n}_{s,\alpha}^{(2,1)} \hat{J}_m^{(2,1)} \right) \right. \\
&\quad + \left(\frac{\tilde{n}_{s,\alpha}^{(2,1)}}{\tilde{\varepsilon}_3 \mu_3} - \frac{n_{s,\alpha}^{(1)}}{\varepsilon_3 \tilde{\mu}_3} \right) \left(n_{s,\alpha}^{(1)} \mathcal{H}_m - n_{s,\alpha}^{(2)} \hat{\mathcal{H}}_m \right) \\
&\quad \left. -2mn_{s,\alpha}^{(1)} \frac{\mathfrak{b}_{\alpha,-}^{(1)} + \mathfrak{g}_{\alpha,-}^{(1)}}{Q^2} \left(\frac{\tilde{n}_{s,\alpha}^{(2,1)}}{\tilde{\varepsilon}_3 \mu_3} - \frac{n_{s,\alpha}^{(2)}}{\varepsilon_3 \tilde{\mu}_3} \right) \right] \left(\tilde{Q}_{\alpha}^{(1,2)} J_m\left(\tilde{Q}_{\alpha}^{(1,2)}\right) \right)^{-1},
\end{aligned} \tag{A5}$$

$$\begin{aligned}
C_{m,s,\alpha}^{(1,2)} &= \Delta_{m,\alpha} \chi^{(1,2)} Q H_m^{(2,1)}(Q) \\
&\quad \times \left[\frac{\tilde{n}_{s,\alpha}^{(2)} - \tilde{n}_{s,\alpha}^{(1)}}{\tilde{\varepsilon}_3 \tilde{\mu}_3} \left(n_{s,\alpha}^{(2)} \mathcal{H}_m^{(2,1)} \hat{\mathcal{H}}_m - n_{s,\alpha}^{(1)} \hat{\mathcal{H}}_m^{(2,1)} \mathcal{H}_m \right) \right. \\
&\quad + \frac{n_{s,\alpha}^{(2)} - n_{s,\alpha}^{(1)}}{\varepsilon_3 \mu_3} \left(\tilde{n}_{s,\alpha}^{(2)} \hat{J}_m^{(2)} J_m^{(1)} - \tilde{n}_{s,\alpha}^{(1)} \hat{J}_m^{(1)} J_m^{(2)} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\tilde{n}_{s,\alpha}^{(2)}}{\tilde{\varepsilon}_3\mu_3}-\frac{n_{s,\alpha}^{(2)}}{\varepsilon_3\tilde{\mu}_3}\right)\left(n_{s,\alpha}^{(1)}J_m^{(1)}\hat{\mathcal{H}}_m^{(2,1)}-\tilde{n}_{s,\alpha}^{(1)}\hat{J}_m^{(1)}\mathcal{H}_m^{(2,1)}\right)\\
&+\left(\frac{\tilde{n}_{s,\alpha}^{(1)}}{\tilde{\varepsilon}_3\mu_3}-\frac{n_{s,\alpha}^{(1)}}{\varepsilon_3\tilde{\mu}_3}\right)\left(n_{s,\alpha}^{(2)}J_m^{(2)}\hat{\mathcal{H}}_m-\tilde{n}_{s,\alpha}^{(2)}\hat{J}_m^{(2)}\mathcal{H}_m\right)\\
&+\left(\frac{n_{s,\alpha}^{(2)}}{\varepsilon_3\tilde{\mu}_3}-\frac{\tilde{n}_{s,\alpha}^{(1)}}{\tilde{\varepsilon}_3\mu_3}\right)\left(n_{s,\alpha}^{(1)}J_m^{(2)}\hat{\mathcal{H}}_m^{(2,1)}-\tilde{n}_{s,\alpha}^{(2)}\hat{J}_m^{(2)}\mathcal{H}_m^{(2,1)}\right)\\
&+\left(\frac{n_{s,\alpha}^{(1)}}{\varepsilon_3\tilde{\mu}_3}-\frac{\tilde{n}_{s,\alpha}^{(2)}}{\tilde{\varepsilon}_3\mu_3}\right)\left(n_{s,\alpha}^{(2)}J_m^{(1)}\hat{\mathcal{H}}_m-\tilde{n}_{s,\alpha}^{(1)}\hat{J}_m^{(1)}\mathcal{H}_m\right)\Bigg], \qquad \text{(A6)}
\end{aligned}
$$

$$
\begin{aligned}
C_{m,s,\alpha} \;=\; &\Delta_{m,\alpha}\frac{8}{i\pi}\left[\left(\frac{\tilde{n}_{s,\alpha}^{(2)}}{\tilde{\varepsilon}_3\mu_3}-\frac{n_{s,\alpha}^{(1)}}{\varepsilon_3\tilde{\mu}_3}\right)\left(n_{s,\alpha}^{(1)}J_m^{(1)}-\tilde{n}_{s,\alpha}^{(1)}\hat{J}_m^{(1)}\right)\right.\\
&+\left(\frac{n_{s,\alpha}^{(1)}}{\varepsilon_3\tilde{\mu}_3}-\frac{\tilde{n}_{s,\alpha}^{(1)}}{\tilde{\varepsilon}_3\mu_3}\right)\left(n_{s,\alpha}^{(1)}J_m^{(2)}-\tilde{n}_{s,\alpha}^{(2)}\hat{J}_m^{(2)}\right)-2mn_{s,\alpha}^{(1)}\\
&\left.\times\frac{\mathfrak{b}_{\alpha,-}^{(1)}+\mathfrak{g}_{\alpha,-}^{(1)}}{Q^2}\frac{\tilde{n}_{s,\alpha}^{(2)}-\tilde{n}_{s,\alpha}^{(1)}}{\tilde{\varepsilon}_3\tilde{\mu}_3}\right]\left(Q_\alpha H_m^{(2)}(Q_\alpha)\right)^{-1}, \qquad \text{(A7)}
\end{aligned}
$$

where

$$
\begin{aligned}
\chi^{(1)} &= -\chi^{(2)} = 1,\\
\Delta_{m,\alpha} &= i^m\tilde{Q}_\alpha^{(1)}J_m\left(\tilde{Q}_\alpha^{(1)}\right)\tilde{Q}_\alpha^{(2)}J_m\left(\tilde{Q}_\alpha^{(2)}\right)Q_\alpha H_m^{(2)}\left(Q_\alpha\right). \qquad \text{(A8)}
\end{aligned}
$$

The coefficients $B_{m,s,\alpha}^{(1,2)}$, $C_{m,s,\alpha}^{(1,2)}$, and $C_{m,s,\alpha}$ satisfy the relationships

$$
\begin{aligned}
\left(B_{-m,-s,\alpha}^{(1,2)}\right)^{(\mathrm{T})} &= (-1)^m B_{m,s,\alpha}^{(1,2)},\\
\left(C_{-m,-s,\alpha}^{(1,2)}\right)^{(\mathrm{T})} &= (-1)^m C_{m,s,\alpha}^{(1,2)}, \qquad \text{(A9)}\\
\left(C_{-m,-s,\alpha}\right)^{(\mathrm{T})} &= (-1)^m C_{m,s,\alpha},
\end{aligned}
$$

which are useful for deriving the orthogonality relations (71)–(73). Note that the common factor $\Delta_{m,\alpha}$, which is invariant with respect to the replacements $m \to -m$, $s \to -s$, and $\tau_{\mathrm{g}} \to -\tau_{\mathrm{g}}$, can be canceled out in formulas (A5)–(A7) without any violation of the relationships in (A9).

APPENDIX B. DERIVATION OF THE ORTHOGONALITY RELATIONS FOR MODES

To prove the orthogonality relations (71)–(73) and formulas (74) and (75), we consider two fields, $\boldsymbol{E}_{\mathrm{I}}, \boldsymbol{H}_{\mathrm{I}}$ and $\boldsymbol{E}_{\mathrm{II}}^{(\mathrm{T})}, \boldsymbol{H}_{\mathrm{II}}^{(\mathrm{T})}$, that have the same frequencies and are taken in different media described by the tensors $\boldsymbol{\varepsilon}, \boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}$, respectively. It is a straightforward matter to deduce from the Maxwell equations that

$$\nabla \cdot \left(\boldsymbol{E}_{\mathrm{I}} \times \boldsymbol{H}_{\mathrm{II}}^{(\mathrm{T})} - \boldsymbol{E}_{\mathrm{II}}^{(\mathrm{T})} \times \boldsymbol{H}_{\mathrm{I}}\right) = 0. \tag{B1}$$

Applying the divergence theorem in a two-dimensional form, we get

$$\begin{aligned}&\frac{\partial}{\partial z}\int_{\Sigma}\left(\boldsymbol{E}_{\mathrm{I}} \times \boldsymbol{H}_{\mathrm{II}}^{(\mathrm{T})} - \boldsymbol{E}_{\mathrm{II}}^{(\mathrm{T})} \times \boldsymbol{H}_{\mathrm{I}}\right) \cdot \hat{\boldsymbol{z}}_0\, \rho\, d\rho\, d\phi \\ &= -\oint_{L}\left(\boldsymbol{E}_{\mathrm{I}} \times \boldsymbol{H}_{\mathrm{II}}^{(\mathrm{T})} - \boldsymbol{E}_{\mathrm{II}}^{(\mathrm{T})} \times \boldsymbol{H}_{\mathrm{I}}\right) \cdot \hat{\boldsymbol{n}}\, d\ell,\end{aligned} \tag{B2}$$

where Σ is an arbitrary cross-sectional area of the open guiding structure, the line integral is along the boundary L of Σ, and $\hat{\boldsymbol{n}}$ is the unit outward normal on L in the plane of Σ.

To obtain the relation (71), we put $\boldsymbol{E}_{\mathrm{I}} = \boldsymbol{E}_{m,n}(\boldsymbol{r})$, $\boldsymbol{H}_{\mathrm{I}} = \boldsymbol{H}_{m,n}(\boldsymbol{r})$ and $\boldsymbol{E}_{\mathrm{II}}^{(\mathrm{T})} = \boldsymbol{E}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r})$, $\boldsymbol{H}_{\mathrm{II}}^{(\mathrm{T})} = \boldsymbol{H}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r})$, where

$$\begin{bmatrix}\boldsymbol{E}_{m,n}(\boldsymbol{r}) \\ \boldsymbol{H}_{m,n}(\boldsymbol{r})\end{bmatrix} = \begin{bmatrix}\boldsymbol{E}_{m,n}(\rho) \\ \boldsymbol{H}_{m,n}(\rho)\end{bmatrix} \exp(-im\phi - ik_0\, p_{m,n}\, z)$$

and

$$\begin{bmatrix}\boldsymbol{E}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r}) \\ \boldsymbol{H}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\boldsymbol{r})\end{bmatrix} = \begin{bmatrix}\boldsymbol{E}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\rho) \\ \boldsymbol{H}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\rho)\end{bmatrix} \exp\left(-i\tilde{m}\phi - ik_0\, p_{\tilde{m},\tilde{n}}^{(\mathrm{T})} z\right),$$

and perform integration in (B2) over a cross-sectional circular area Σ of infinite radius. After some algebra, we get

$$\begin{aligned}J_{\tilde{m},\tilde{n}}^{m,n} = &\int_0^{2\pi} \exp\left[-i\left(m+\tilde{m}\right)\phi\right] d\phi \\ &\times \lim_{\rho\to\infty} \frac{\exp[-ik_0\left(p_{m,n} - p_{-\tilde{m},-\tilde{n}}\right) z]}{ik_0\left(p_{m,n} - p_{-\tilde{m},-\tilde{n}}\right)} \\ &\times\rho\Big[\boldsymbol{E}_{m,n}(\rho) \times \boldsymbol{H}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\rho) - \boldsymbol{E}_{\tilde{m},\tilde{n}}^{(\mathrm{T})}(\rho) \times \boldsymbol{H}_{m,n}(\rho)\Big] \cdot \hat{\boldsymbol{\rho}}_0.\end{aligned} \tag{B3}$$

Here, we made use of the relationship

$$p^{(\mathrm{T})}_{\tilde{m},\tilde{n}} = p_{-\tilde{m},\tilde{n}} = -p_{-\tilde{m},-\tilde{n}}$$

deduced from the dispersion relation (68) for the discrete-spectrum modes, and $\hat{\boldsymbol{\rho}}_0$ is the radial unit vector lying in the cross-sectional plane $z = \text{const}$. Noting that

$$\int_0^{2\pi} \exp[-i\,(m+\tilde{m})\,\phi]d\phi = 2\pi\,\delta_{m,-\tilde{m}} \tag{B4}$$

and observing that the fields of the discrete-spectrum modes vanish with ρ, one sees that the right-hand side of (B3) is equal to zero if $p_{m,n} \neq p_{-\tilde{m},-\tilde{n}}$. In the absence of mode degeneration this implies that $J^{m,n}_{\tilde{m},\tilde{n}} = 0$ for $m \neq -\tilde{m}$ and $n \neq -\tilde{n}$. In addition, it is easily verified that $N_{m,-n} = -N_{m,n}$. We have thus derived the orthogonality relation (71) for the discrete-spectrum modes. For degenerate modes, an additional orthogonalization procedure can always be performed to arrive at the desired orthogonality relation.

Relation (72) can be established analogously if we put

$$\begin{bmatrix}\boldsymbol{E}_{\mathrm{I}}(\boldsymbol{r})\\ \boldsymbol{H}_{\mathrm{I}}(\boldsymbol{r})\end{bmatrix} = \begin{bmatrix}\boldsymbol{E}_{m,s,\alpha}(\rho,q)\\ \boldsymbol{H}_{m,s,\alpha}(\rho,q)\end{bmatrix}\exp[-im\phi - ik_0\,p_{s,\alpha}(q)\,z]$$

and take the same field $\boldsymbol{E}^{(\mathrm{T})}_{\mathrm{II}}$, $\boldsymbol{H}^{(\mathrm{T})}_{\mathrm{II}}$ as in the preceding derivation. Following the above procedure and making use of the fact that $p_{s,\alpha}(q) \neq p_{-\tilde{m},-\tilde{n}}$ for all real values of q, one can readily obtain relation (72). Hence, each discrete-spectrum mode is orthogonal to each continuous-spectrum mode, and, therefore, to the total outward radiating field.

Next, to derive relation (73), we take the last form for $\boldsymbol{E}_{\mathrm{I}}$ and $\boldsymbol{H}_{\mathrm{I}}$, put

$$\begin{bmatrix}\boldsymbol{E}^{(\mathrm{T})}_{\mathrm{II}}(\boldsymbol{r})\\ \boldsymbol{H}^{(\mathrm{T})}_{\mathrm{II}}(\boldsymbol{r})\end{bmatrix} = \begin{bmatrix}\boldsymbol{E}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{\alpha}}(\rho,\tilde{q})\\ \boldsymbol{H}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{\alpha}}(\rho,\tilde{q})\end{bmatrix}\exp[-i\tilde{m}\phi - ik_0\,p_{\tilde{s},\tilde{\alpha}}(\tilde{q})\,z]$$

and then apply the same procedure. We thus find

$$J^{m,s,\alpha}_{\tilde{m},\tilde{s},\tilde{\alpha}} = \int_0^{2\pi}\exp[-i\,(m+\tilde{m})\,\phi]d\phi\,\lim_{\rho\to\infty}\frac{\exp\{-ik_0[p_{s,\alpha}(q)-p_{-\tilde{s},\tilde{\alpha}}(\tilde{q})]z\}}{ik_0[p_{s,\alpha}(q)-p_{-\tilde{s},\tilde{\alpha}}(\tilde{q})]}$$
$$\times\rho\left[\boldsymbol{E}_{m,s,\alpha}(\rho,q)\times\boldsymbol{H}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{\alpha}}(\rho,\tilde{q})-\boldsymbol{E}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{\alpha}}(\rho,\tilde{q})\times\boldsymbol{H}_{m,s,\alpha}(\rho,q)\right]\cdot\hat{\boldsymbol{\rho}}_0. \tag{B5}$$

At first, consider the case where the conditions $m = -\tilde{m}$, $s = -\tilde{s}$, and $\alpha = \tilde{\alpha}$ are simultaneously satisfied. By inspection of Equations (4)–(12), it is found that the following conventions can be adopted without any loss of generality:

$$\begin{aligned} E^{(\mathrm{T})}_{\phi;\,-m,-s,\alpha}(\rho,q) &= E_{\phi;\,m,s,\alpha}(\rho,q), \quad E^{(\mathrm{T})}_{z;\,-m,-s,\alpha}(\rho,q) = E_{z;\,m,s,\alpha}(\rho,q), \\ H^{(\mathrm{T})}_{\phi;\,-m,-s,\alpha}(\rho,q) &= H_{\phi;\,m,s,\alpha}(\rho,q), \quad H^{(\mathrm{T})}_{z;\,-m,-s,\alpha}(\rho,q) = H_{z;\,m,s,\alpha}(\rho,q). \end{aligned} \tag{B6}$$

Note that these relationships are ensured under conditions (A9). Using the relationships in (B6) along with the large-argument approximations of the Hankel functions $H^{(1,2)}_m(k_0 q\rho)$ and $H^{(2)}_m(k_0 q_\alpha \rho)$, we obtain the following representation:

$$\begin{aligned} &J^{m,s,\alpha}_{-m,-s,\alpha} \\ &= \lim_{\rho\to\infty} \frac{4(\tilde{q}-q)\exp\{-ik_0[p_{s,\alpha}(q)-p_{s,\alpha}(\tilde{q})]z\}}{Z_0 k_0^2 (q\,\tilde{q})^{1/2}\,[p_{s,\alpha}(q)-p_{s,\alpha}(\tilde{q})]} \left[\mu_3^{-1}+\varepsilon_3^{-1} n^{(1)}_{s,\alpha}(q) n^{(1)}_{s,\alpha}(\tilde{q})\right] \\ &\quad\times\Big\{(q+\tilde{q})\Big[C^{(1)}_{m,s,\alpha}(q)\,C^{(2)}_{m,s,\alpha}(\tilde{q})+C^{(1)}_{m,s,\alpha}(\tilde{q})C^{(2)}_{m,s,\alpha}(q)\Big]\,\frac{\sin[k_0\rho(q-\tilde{q})]}{q-\tilde{q}} \\ &\quad - i(q+\tilde{q})\Big[C^{(1)}_{m,s,\alpha}(q)\,C^{(2)}_{m,s,\alpha}(\tilde{q})-C^{(1)}_{m,s,\alpha}(\tilde{q})C^{(2)}_{m,s,\alpha}(q)\Big]\frac{\cos[k_0\rho\,(q-\tilde{q})]}{q-\tilde{q}} \\ &\quad + i(-1)^m\Big[C^{(1)}_{m,s,\alpha}(q)C^{(1)}_{m,s,\alpha}(\tilde{q})-C^{(2)}_{m,s,\alpha}(q)C^{(2)}_{m,s,\alpha}(\tilde{q})\Big]\sin[k_0\rho\,(q+\tilde{q})] \\ &\quad + (-1)^m\Big[C^{(1)}_{m,s,\alpha}(q)C^{(1)}_{m,s,\alpha}(\tilde{q})+C^{(2)}_{m,s,\alpha}(q)C^{(2)}_{m,s,\alpha}(\tilde{q})\Big]\cos[k_0\rho(q+\tilde{q})]\Big\}. \end{aligned} \tag{B7}$$

In the above, use was made of the fact that the functions $H^{(2)}_n(k_0 q_\alpha \rho)$ vanish with ρ because of condition (41). Passing to the limit $\rho \to \infty$ in (B7) in accordance with the theory of distributions and using the well-known relation

$$\delta(\xi) = \lim_{R\to\infty} \frac{\sin R\xi}{\pi\xi}\,,$$

we obtain from (B7) the orthogonality relation (73) for the continuous-spectrum modes, with $N_{m,s,\alpha}(q)$ given by (74). In other possible cases where either $s \neq -\tilde{s}$ or $\alpha \neq \tilde{\alpha}$ (here, $p_{s,\alpha}(q) \neq p_{-\tilde{s},\tilde{\alpha}}(\tilde{q})$ for all real values of q and $\tilde{q}$), or $m \neq -\tilde{m}$, we have the trivial result for $J^{m,s,\alpha}_{\tilde{m},\tilde{s},\tilde{\alpha}}$, as expressed by (73).

We note that the continuous-spectrum modes possessing the same eigenvalues q and indices s and α, but different azimuthal indices are degenerate since they have the same axial wave number $p = p_{s,\alpha}(q)$. Nevertheless, any two such modes are orthogonal due to the "angular" orthogonality expressed by (B4).

It now remains to establish relation (75) for $N_{m,n}$. The normalization quantity $N_{m,n}$ can be represented as

$$N_{m,n} = 2\pi \lim_{R\to\infty} \lim_{q\to q_{m,n}} \int_0^R \Big[\boldsymbol{E}_{m,n}(\boldsymbol{r}) \times \boldsymbol{H}^{(\mathrm{T})}_{-m,-,\hat{\alpha}}(\boldsymbol{r}, q) - \boldsymbol{E}^{(\mathrm{T})}_{-m,-,\hat{\alpha}}(\boldsymbol{r}, q) \times \boldsymbol{H}_{m,n}(\boldsymbol{r}) \Big] \cdot \hat{\boldsymbol{z}}_0 \, \rho \, d\rho. \tag{B8}$$

Following steps similar to those that led to (B3), and making use of the fact that $p_{m,n} = p_{\hat{\alpha}}(q_{m,n})$, we rewrite (B8) in the form

$$N_{m,n} = 2\pi \lim_{R\to\infty} \lim_{q\to q_{m,n}} \frac{\exp\{-ik_0 \left[p_{\hat{\alpha}}(q_{m,n}) - p_{\hat{\alpha}}(q)\right] z\}}{ik_0 \left[p_{\hat{\alpha}}(q_{m,n}) - p_{\hat{\alpha}}(q)\right]} \times \rho \Big[\boldsymbol{E}_{m,n}(\rho) \times \boldsymbol{H}^{(\mathrm{T})}_{-m,-,\hat{\alpha}}(\rho, q) - \boldsymbol{E}^{(\mathrm{T})}_{-m,-,\hat{\alpha}}(\rho, q) \times \boldsymbol{H}_{m,n}(\rho) \Big] \cdot \hat{\boldsymbol{\rho}}_0 \bigg|_{\rho=R}. \tag{B9}$$

Performing the operations indicated here and taking into account the relationships in (B6) and the chain of inequalities $\operatorname{Im} q_{\hat{\alpha}}(q_{m,n}) < \operatorname{Im} q_{m,n} < 0$, we find

$$N_{m,n} = \frac{8i}{Z_0 k_0^2} \left(\frac{dp_{\hat{\alpha}}}{dq} \right)^{-1} \left[\mu_3^{-1} + \varepsilon_3^{-1} \left(n^{(1)}_{s,\hat{\alpha}} \right)^2 \right] \frac{dC^{(1)}_{m,s,\hat{\alpha}}}{dq} C^{(2)}_{m,s,\hat{\alpha}} \bigg|_{q=q_{m,n}}. \tag{B10}$$

Using (68) and (74), we come from (B10) to the resulting expression (75) for $N_{m,n}$.

Finally, we note that the above-derived orthogonality relations can be generalized to be valid for transversely unbounded modes, as well. To do this, one should use a method similar to that employed in [19, 37–39] for open waveguides located in a nongyrotropic medium. The generalized orthogonality relations turn out to be useful when analyzing leaky modes.

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