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DIFFRACTION OF ELECTROMAGNETIC PLANE WAVE FROM A SLIT IN PEMC PLANE

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Abstract—In the present investigation, diffraction from a slit in perfectly electromagnetic conducting (PEMC) plane has been studied. Both the E- and H-polarization are considered and the method of analysis is Kobayashi Potential (KP). The mathematical formulation involves dual integral equations (DIEs). These DIEs are solved by using the discontinuous properties of Weber-Schafheitlin's integral. The resulting expressions, finally, reduce to matrix equations. These are then used to compute the values of unknown expansion coefficients. Numerical results are presented for different parameters of interest especially the dependance of co-polarized and cross-polarized components on the admittance parameter.

1. INTRODUCTION

In recent years, the concept of perfectly electromagnetic conductor (PEMC) got much popularity among the investigators working in the field of electromagnetics. This concept was introduced by Lindell and Sihvola [1]. Using differential-form formalism, they described that PEC and PMC media may be generalized to a medium called perfectly electromagnetic conducing medium. This medium is characterized by a single parameter M, PEMC admittance, which can vary from zero to infinity. A null admittance corresponds to a PMC medium and an admittance of infinity to a PEC medium when the field magnitudes are finite [2]. This medium is isotropic and the most notable property of

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this medium is its nonreciprocity when M has a finite nonzero value [3]. Because after scattering from a PEMC boundary, the electromagnetic wave also have cross-polarized component along with the co-polarized component [4–6]. The possible realization of such type of materials has been discussed in [3]. The boundary conditions to be satisfied on a PEMC surface can be written by using the PEC and PMC boundary conditions and the fact that PEMC is the generalization of PEC and PMC as follow

$$\hat{\mathbf{n}} \times (\mathbf{H} + M\mathbf{E}) = 0, \qquad \hat{\mathbf{n}} \cdot (\mathbf{D} - M\mathbf{B}) = 0$$

where M is defined as the PEMC admittance and $\hat{\mathbf{n}}$ is the unit normal to the boundary.

Scattering of electromagnetic wave from different PEMC geometries has been demonstrated theoretically by many authors [7–15]. In the present investigation, we studied the diffracting properties of a slit in PEMC plane of arbitrary admittance and of negligible thickness. The method of analysis adopted here is the Kobayashi Potential (KP) method. The method has been successfully applied to potential [16, 17] as well as scattering problems for different geometries [18–22]. Imposition of the boundary conditions result in dual integral equations (DIEs). These DIEs can be solved by using the discontinuous properties of Weber-Schafheitlin integral and the projection method like the method of moment (MoM), in which Jacobi's polynomials are used as the basis functions. Finally, the problem reduces to matrix equations whose matrix elements are the infinite integrals. These integrals are hard to solve analytically. Therefore numerical methods are adopted to compute these integrals and to solve the matrix equations for the determination of unknown expansion coefficients. Illustrative computations have been presented for the parameters of interest.

2. FORMULATION AND SOLUTION OF THE PROBLEM

2.1. E-polarization

The geometry of the problem is shown in Fig. 1. The plane is infinite in extent along z-axis which makes the problem two-dimensional. Electromagnetic plane wave is incident upon the slit in PEMC plane of negligible thickness. The width of the slit is $2a$. If ϕ_0 is the angle of incidence, then the incident field E_z^i, co-polarized component E_z^s and cross-polarized component H_z^s of the scattered field can be written as

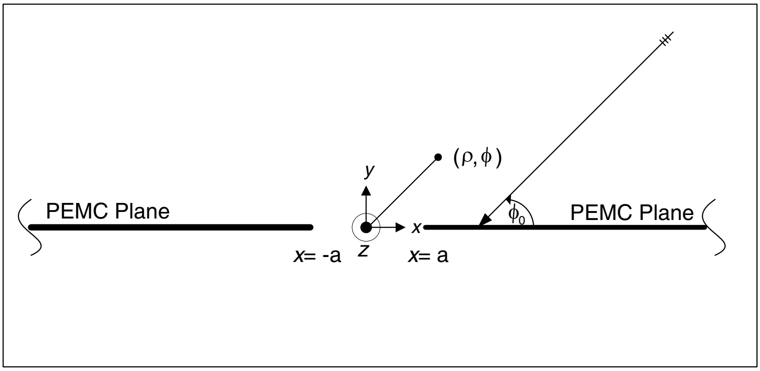

Figure 1. Geometry of the problem.

$$E_z^i = \exp\left[jk(x\cos\phi_0 + y\sin\phi_0)\right] \tag{1a}$$

$$E_z^s = \int_0^\infty \left\{ f_e(\xi)\cos(x_a\xi) + g_e(\xi)\sin(x_a\xi) \right\} \exp\left[-\sqrt{\xi^2-\kappa^2}y_a\right] d\xi \qquad y>0 \tag{1b}$$

$$H_z^s = \int_0^\infty \left\{ f_h(\xi)\cos(x_a\xi) + g_h(\xi)\sin(x_a\xi) \right\} \exp\left[-\sqrt{\xi^2-\kappa^2}y_a\right] d\xi \qquad y>0 \tag{1c}$$

where $\kappa = ka, x_a = \frac{x}{a}, y_a = \frac{y}{a}$ and k is the propagation constant of the free space. The $f_{e,h}(\xi)$ and $g_{e,h}(\xi)$ are the weighting functions to be determined from the boundary conditions.

The required boundary conditions are given by

(i) The fields are continuous at $|x_a| \leq 1$ and $y = 0$,

(ii) $H_x^s + ME_x^s = 0$ and $H_z^s + ME_z^s = 0$ for $|x_a| \geq 1$ and $y = 0$.

Applying boundary condition given in (ii), we have

$$\int_0^\infty \left[f_h(\xi)+Mf_e(\xi)\right]\cos(x_a\xi)+\left[g_h(\xi)+Mg_e(\xi)\right]\sin(x_a\xi)d\xi = 0 \tag{2a}$$

$$\int_0^\infty \left[\sqrt{\xi^2-\kappa^2}\right]\left[f_e(\xi) - MZ^2 f_h(\xi)\right]\cos(x_a\xi) + \left[g_e(\xi) - MZ^2 g_h(\xi)\right]\sin(x_a\xi)d\xi = 0 \tag{2b}$$

where time dependance is taken as $\exp(j\omega t)$ in the calculations. The above expressions can be used to decide the nature of weighting functions $f_{e,h}(\xi)$ and $g_{e,h}(\xi)$ by making using of the discontinuous properties of Weber-Schafheitlin's integrals, as follow

$$f_h(\xi) + Mf_e(\xi) = \sum_{m=0}^{\infty} A_m J_{2m+1}(\xi)\xi^{-1} \tag{3a}$$

$$g_h(\xi) + Mg_e(\xi) = \sum_{m=0}^{\infty} B_m J_{2m+2}(\xi)\xi^{-1} \tag{3b}$$

$$f_e(\xi) - MZ^2 f_h(\xi) = \sum_{m=0}^{\infty} C_m \frac{J_{2m}(\xi)}{\sqrt{\xi^2-\kappa^2}} \tag{3c}$$

$$g_e(\xi) - MZ^2 g_h(\xi) = \sum_{m=0}^{\infty} D_m \frac{J_{2m+1}(\xi)}{\sqrt{\xi^2-\kappa^2}} \tag{3d}$$

where $J_m(.)$ be the Bessel's function of order m and A_m, B_m, C_m and D_m are the expansion coefficients. Manipulating the above expressions, we get

$$f_e(\xi) = \frac{1}{1+M^2Z^2} \sum_{m=0}^{\infty} C_m \frac{J_{2m}(\xi)}{\sqrt{\xi^2-\kappa^2}} + MZ^2 A_m \frac{J_{2m+1}(\xi)}{\xi} \tag{4a}$$

$$f_h(\xi) = \frac{1}{1+M^2Z^2} \sum_{m=0}^{\infty} -MC_m \frac{J_{2m}(\xi)}{\sqrt{\xi^2-\kappa^2}} + A_m \frac{J_{2m+1}(\xi)}{\xi} \tag{4b}$$

$$g_e(\xi) = \frac{1}{1+M^2Z^2} \sum_{m=0}^{\infty} D_m \frac{J_{2m+1}(\xi)}{\sqrt{\xi^2-\kappa^2}} + MZ^2 B_m \frac{J_{2m+2}(\xi)}{\xi} \tag{4c}$$

$$g_h(\xi) = \frac{1}{1+M^2Z^2} \sum_{m=0}^{\infty} -MD_m \frac{J_{2m+1}(\xi)}{\sqrt{\xi^2-\kappa^2}} + B_m \frac{J_{2m+2}(\xi)}{\xi} \tag{4d}$$

where Z be the impedance of free space. Boundary condition given in (i) gives

$$\int_0^{\infty} \sqrt{\xi^2-\kappa^2}\Big[f_e(\xi)\cos(x_a\xi) + g_e(\xi)\sin(x_a\xi)\Big]d\xi$$
$$= j\kappa \sin\phi_0 \exp[j\kappa x_a \cos\phi_0] \tag{5a}$$

$$\int_0^{\infty} \Big[f_e(\xi)\cos(x_a\xi) + g_e(\xi)\sin(x_a\xi)\Big]d\xi = -\exp[j\kappa x_a \cos\phi_0] \tag{5b}$$

Separating even and odd functions of the Expressions (5a) and (5b) and then expanding the trigonometric functions in terms of Jacobi's polynomials $u_n^{\pm\frac{1}{2}}(x_a^2)$ and $v_n^{\pm\frac{1}{2}}(x_a^2)$ [23], we obtain finally, the matrix equations for the expansion coefficients

$$\sum_{m=0}^{\infty} H(2m, 2n+1; \kappa)C_m + MZ^2 K(2m+1, 2n+1; \kappa)A_m$$
$$= j\kappa \sin\phi_0 \Psi(MZ) \frac{J_{2n+1}(\kappa\cos\phi_0)}{(\kappa\cos\phi_0)} \tag{6a}$$
$$\sum_{m=0}^{\infty} H(2m+1, 2n+2; \kappa)D_m + MZ^2 K(2m+2, 2n+2; \kappa)B_m$$
$$= -\kappa \sin\phi_0 \Psi(MZ) \frac{J_{2n+2}(\kappa\cos\phi_0)}{(\kappa\cos\phi_0)} \tag{6b}$$
$$\sum_{m=0}^{\infty} G(2m, 2n; \kappa)C_m - MZ^2 H(2m+1, 2n; \kappa)A_m$$
$$= -\Psi(MZ)J_{2n}(\kappa\cos\phi_0) \tag{6c}$$
$$\sum_{m=0}^{\infty} G(2m+1, 2n+1; \kappa)D_m - MZ^2 H(2m+2, 2n+1; \kappa)B_m$$
$$= j\Psi(MZ)J_{2n+1}(\kappa\cos\phi_0) \qquad n = 0, 1, 2, \ldots \tag{6d}$$

where

$$\Psi(MZ) = 1 + M^2Z^2, G(\alpha, \beta; \kappa) = \int_0^\infty \frac{J_\alpha(\xi)J_\beta(\xi)}{\sqrt{\xi^2-\kappa^2}} d\xi \tag{7a}$$
$$H(\alpha,\beta;\kappa) = \int_0^\infty \frac{J_\alpha(\xi)J_\beta(\xi)}{\xi} d\xi, K(\alpha,\beta;\kappa) = \int_0^\infty \frac{\sqrt{\xi^2-\kappa^2}}{\xi^2} J_\alpha(\xi)J_\beta(\xi)d\xi \tag{7b}$$

In writing the Equation (6), we have used the following relations

$$\cos x = \sqrt{\frac{\pi x}{2}} J_{-\frac{1}{2}}(x), \qquad \sin x = \sqrt{\frac{\pi x}{2}} J_{\frac{1}{2}}(x) \tag{7c}$$
$$x^{-m/2} J_m(\xi\sqrt{x})$$
$$= \sum_{n=0}^{\infty} \frac{\sqrt{2}(2n+m+\frac{1}{2})\Gamma(n+m+\frac{1}{2})}{\Gamma(n+1)\Gamma(m+1)} \frac{J_{2n+m+\frac{1}{2}}(\xi)}{\xi^{\frac{1}{2}}} u_n^m(x) \tag{7d}$$
$$= \sum_{n=0}^{\infty} \frac{\sqrt{8}\left(2n+m+\frac{3}{2}\right)\Gamma\left(n+m+\frac{3}{2}\right)}{\Gamma(n+1)\Gamma(m+1)} \frac{J_{2n+m+\frac{3}{2}}(\xi)}{\xi^{\frac{3}{2}}} v_n^m(x) \tag{7e}$$

$$u_n^m(x) = F\left(n+m+\frac{1}{2}, -n, m+1; x\right)$$
$$v_n^m(x) = F\left(n+m+\frac{3}{2}, -n, m+1; x\right) \tag{7f}$$

where $F(m, n, l; x)$ is the hypergeometric series [23].

The Equations (6a)–(6d) may solved to evaluate the expansion coefficients A_m, B_m, C_m, D_m. The co-polarized component E_z^s and cross-polarized component H_z^s may be computed from the Equations (2a), (2b) using the saddle point method. The final results are

$$\begin{aligned} E_z^s &= C(k\rho)\frac{1}{\Psi(MZ)} \\ &\quad \Big[\sum_{m=0}^{\infty} MZ^2\left[A_m J_{2m+1}(\kappa\cos\phi) + B_m J_{2m+2}(\kappa\cos\phi)\right]\tan\phi \\ &\quad -j\left[C_m J_{2m}(\kappa\cos\phi) + D_m J_{2m+1}(\kappa\cos\phi)\right]\Big] \end{aligned} \tag{8a}$$

$$\begin{aligned} H_z^s &= C(k\rho)\frac{1}{\Psi(MZ)} \\ &\quad \Big[\sum_{m=0}^{\infty} \left[A_m J_{2m+1}(\kappa\cos\phi) + B_m J_{2m+2}(\kappa\cos\phi)\right]\tan\phi \\ &\quad +jM\left[C_m J_{2m}(\kappa\cos\phi) + D_m J_{2m+1}(\kappa\cos\phi)\right]\Big] \end{aligned} \tag{8b}$$

where $C(k\rho) = \sqrt{\frac{\pi}{2k\rho}}\exp\left[-jk\rho - j\frac{\pi}{4}\right]$ and (ρ, ϕ) are the cylindrical coordinates of the observation point. A far field in the lower region can also be derived similarly.

2.2. *H*-polarization

The field expressions corresponding to Expressions (1) for H-polarization may be written as

$$H_z^i = \exp\Big[jk(x\cos\phi_0 + y\sin\phi_0)\Big] \tag{9a}$$

$$E_z^s = \int_0^\infty \Big\{f_e(\xi)\cos(x_a\xi) + g_e(\xi)\sin(x_a\xi)\Big\}\exp\Big[-\sqrt{\xi^2-\kappa^2}y_a\Big]d\xi \tag{9b}$$

$$H_z^s = \int_0^\infty \Big\{f_h(\xi)\cos(x_a\xi) + g_h(\xi)\sin(x_a\xi)\Big\}\exp\Big[-\sqrt{\xi^2-\kappa^2}y_a\Big]d\xi \tag{9c}$$

where E_z^s is the cross component and H_z^s the co-component of the scattered field for H-polarized incident field. All the notations used in the above expressions have the same meaning as described in last section. Applying the boundary conditions (ii), we get the same expressions for the weighting functions $f_{e,h}(\xi)$ and $g_{e,h}(\xi)$ as given by (4a)–(4d). Imposition of boundary conditions (i) and following the same procedure as in the above case we finally get

$$\sum_{m=0}^{\infty}\int_0^{\infty}\left[A_m\frac{J_{2m+1}(\xi)J_{2n}(\xi)}{\xi}-MC_m\frac{J_{2m}(\xi)J_{2n}(\xi)}{\sqrt{\xi^2-\kappa^2}}\right]d\xi$$
$$=-\Psi(MZ)J_{2n}(\kappa\cos\phi_0) \tag{10a}$$
$$\sum_{m=0}^{\infty}\int_0^{\infty}\left[B_m\frac{J_{2m+2}(\xi)J_{2n+1}(\xi)}{\xi}-MD_m\frac{J_{2m+1}(\xi)J_{2n+1}(\xi)}{\sqrt{\xi^2-\kappa^2}}\right]d\xi$$
$$=-j\Psi(MZ)J_{2n+1}(\kappa\cos\phi_0) \tag{10b}$$
$$\sum_{m=0}^{\infty}\int_0^{\infty}\left[A_mJ_{2m+1}(\xi)J_{2n+1}(\xi)\frac{\sqrt{\xi^2-\kappa^2}}{\xi^2}+MC_m\frac{J_{2m}(\xi)J_{2n+1}(\xi)}{\xi}\right]d\xi$$
$$=j\kappa\sin\phi_0\Psi(MZ)\frac{J_{2n+1}(\kappa\cos\phi_0)}{(\kappa\cos\phi_0)} \tag{10c}$$
$$\sum_{m=0}^{\infty}\int_0^{\infty}\left[B_mJ_{2m+2}(\xi)J_{2n+2}(\xi)\frac{\sqrt{\xi^2-\kappa^2}}{\xi^2}+MD_m\frac{J_{2m+1}(\xi)J_{2n+2}(\xi)}{\xi}\right]d\xi$$
$$=-\kappa\sin\phi_0\Psi(MZ)\frac{J_{2n+2}(\kappa\cos\phi_0)}{(\kappa\cos\phi_0)} \tag{10d}$$

The above expressions may be expressed in more precise form as follow

$$\sum_{m=0}^{\infty}H(2m+1,2n;\kappa)A_m-MG(2m,2n;\kappa)C_m$$
$$=-\Psi(MZ)J_{2n}(\kappa\cos\phi_0) \tag{11a}$$
$$\sum_{m=0}^{\infty}H(2m+2,2n+1;\kappa)B_m-MG(2m+1,2n+1;\kappa)D_m$$
$$=-j\Psi(MZ)J_{2n+1}(\kappa\cos\phi_0) \tag{11b}$$
$$\sum_{m=0}^{\infty}K(2m+1,2n+1;\kappa)A_m+MH(2m,2n+1;\kappa)C_m$$
$$=j\kappa\sin\phi_0\Psi(MZ)\frac{J_{2n+1}(\kappa\cos\phi_0)}{(\kappa\cos\phi_0)} \tag{11c}$$

$$\sum_{m=0}^{\infty} K(2m+2, 2n+2; \kappa) B_m + MH(2m+1, 2n+2; \kappa) D_m$$
$$= -\kappa \sin\phi_0 \Psi(MZ) \frac{J_{2n+2}(\kappa\cos\phi_0)}{(\kappa\cos\phi_0)} \qquad n = 0, 1, 2, \ldots \quad \text{(11d)}$$

The above expressions are the matrix equations and can be solved for the expansion coefficients A_m, B_m, C_m, D_m by any standard method.

Far diffracted fields for the co- and cross-polarized components are the same as that of (8a) and (8b) but the expansion coefficients are given by (11) instead of (6).

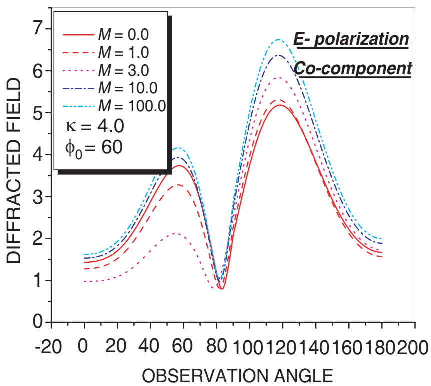

Figure 2. Variations of diffracted field as a function of M.

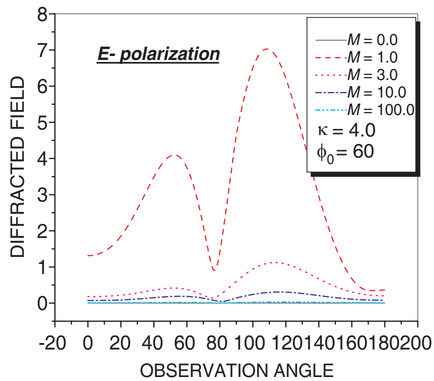

Figure 3. Variation of cross-component as a function of M.

3. RESULTS AND DISCUSSIONS

The Equations (6) and (11) are the matrix equations and are derived to compute the unknown coefficients A_m, B_m, C_m and D_m for E- and H-polarization respectively. These equations contain the integrals $G(\alpha, \beta; \kappa)$, $K(\alpha, \beta; \kappa)$ and $H(\alpha, \beta; \kappa)$ and how to compute these integrals are discussed in detail in [18]. We have taken the matrix size $(2\kappa + 1) \times (2\kappa + 1)$ in our simulations. Once the expansion coefficients are calculated, we can use them to compute the far field patterns for co-polarized and cross-polarized components from (8a) and (8b). Since M, the admittance parameter is most important in our work, therefore we have tried to explore the dependence of field patterns on this parameter. Fig. 2 and Fig. 3 show the variations in the field patterns as a function of M for $\phi_0 = 60°$, $\kappa = 4$ for E-polarization. It turns out

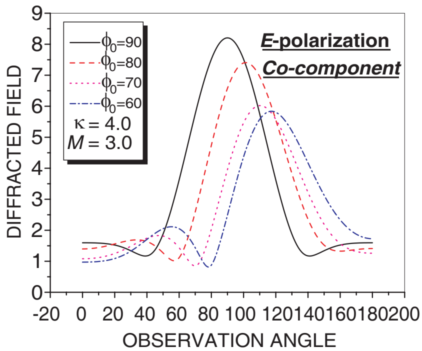

Figure 4. Effect of angle of incidence on field patterns.

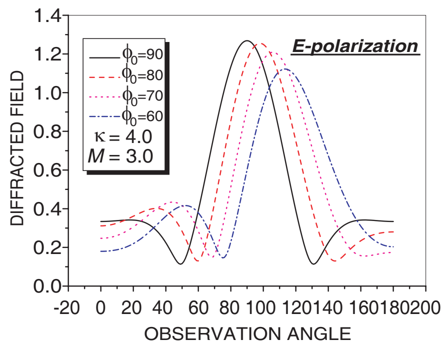

Figure 5. Effect of angle of incidence on cross-component.

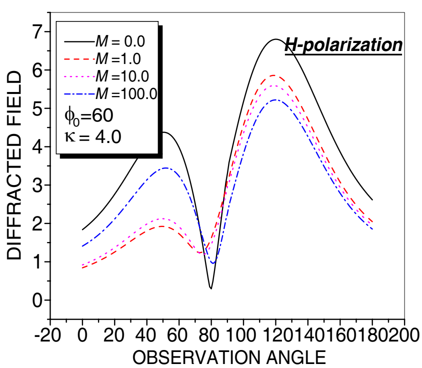

Figure 6. Dependence of co-component on M (H-polarization).

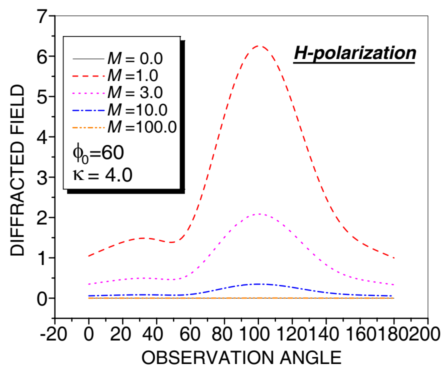

Figure 7. Dependence of cross-component on M (H-polarization).

that there exist no cross-polarized component H_z for PMC and PEC case and it dominates for $M = 1.0$ and as we increase the value of M, the amplitude of the cross-component gradually decrease. While, the co-polarized component E_z increases gradually, at $\phi = \pi - \phi_0$, as we increase the value of M. The same trends may also be seen for other values of angle of incidence. Fig. 4 and Fig. 5 give the dependence of the field patterns on angle of incidence for E-polarization case. It is obvious that, for a particular value of ϕ_0, the main lobe for the diffracted field for co-component occur approximately at $\phi = \pi - \phi_0$. And as we increase angle of incidence, the main lobe shifts towards the lower values of ϕ. For H-polarization, the dependence of field patterns on M is slightly different as is shown in Fig. 6 and Fig. 7. As we increase the value of M the amplitude of the co-component, H_z decreases (at $\phi = \pi - \phi_0$) and the behavior of cross-component E_z is similar to that of E-polarization case.

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