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AN IMPROVED ALGORITHM FOR MATRIX BANDWIDTH AND PROFILE REDUCTION IN FINITE ELEMENT ANALYSIS

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Abstract—In finite element analysis, methods for the solution of sparse linear systems of equations usually start out with reordering the coefficient matrix to reduce its bandwidth or profile. The location of pseudo-peripheral nodes is an important factor in the bandwidth and profile reduction algorithm. This paper presents a heuristic parameter, called the "width-depth ratio" and denoted by κ. With such a parameter, suitable pseudo-peripheral nodes would be found; the distance between which could be much close to or even to be the diameter of a graph compared with Gibbs-Poole-Stockmeyer (GPS) algorithm. As the new parameter was implemented in GPS algorithm, an improved bandwidth and profile reduction algorithm is proposed. Simulation results show that with the proposed algorithm, sometimes bandwidth and profile could be reduced by as great as 33.33% and 11.65%, respectively, compared with the outcomes in GPS algorithm, while the execution time of both algorithms is close. Empirical results show that the proposed algorithm is superior to GPS algorithm in reducing bandwidth or profile.

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1. INTRODUCTION

Analysis of many problems in electromagnetics involves the solution of partial differential equations arising from the finite element method (FEM), with the form [1–5]:

$$Ax = B \tag{1}$$

where the $n \times n$ matrix A is the coefficient matrix, called the stiffness matrix. The matrix A is a large sparse symmetric matrix, so there is a direct correspondence between the structures of A and the structures of the FEM mesh. Both direct and iterative methods can be used for solving such a system.

In direct method, for the efficient solution and to compress the memory space, it is desirable to have a nodal renumbering before the construction of the stiffness matrix A to ensure that A has a narrow bandwidth or a small profile.

A lot of algorithms have been proposed for the problem of matrix bandwidth or profile reduction. The first extensive study of such a problem was done by Cuthill and McKee in 1969 as the Cuthill-McKee (CM) algorithm [6]. Methods proposed in the late 1970s and early 1980s include the Reverse Cuthill-McKee algorithm [7], Gibbs-King algorithm [8], and Gibbs-Poole-Stockmeyer (GPS) algorithm [9]. Methods proposed in the 1990s include the spectral method [10] and the algorithm using simulated annealing (SA) procedure [11].

One of the most popular algorithms among all the bandwidth or profile reduction methods is GPS algorithm. The success of GPS algorithm is dependent on the selection of starting nodes, finding a pair of nodes which are located at nearly maximal distance apart, called the pseudo-peripheral nodes. It has been demonstrated by extensive tests available in the literature that this strategy provides good starting nodes [12, 13].

However, GPS algorithm cannot always find the pseudo-peripheral nodes with the distance which is the "diameter" in a graph. There may be several candidates for good pseudo-peripheral nodes in a graph. The selection of the pair of nodes may make a difference. Consequently, if additional criteria were taken into account to pick the eligible nodes, the real "diameter" would be found.

In this paper, a new parameter "width-depth ratio" is introduced, denoted by κ, aiming at improving the selection of pseudo-peripheral nodes. We modify the part of finding pseudo-peripheral nodes in GPS algorithm to get the "diameter" nodes.

The paper is organized as follows: Section 2 introduces basic concepts and terminologies. Section 3 gives the definition of the

new parameter κ, the basic procedures of the new algorithm, and also a simple example to illustrate each step of the new algorithm. Section 4 compares simulation results and performances of the proposed algorithm against GPS algorithm with 16 classical electromagnetic models. Section 5 draws up our conclusions.

2. BASIC CONCEPTS AND TERMINOLOGIES

2.1. Matrix Bandwidth and Profile

Let the coefficient matrix A in Equation (1) be an n by n symmetric matrix, with entries a_{ij}. The ith bandwidth of A is defined by

$$\beta_i(A) = i - \min\{\, j \,|\, a_{ij} \neq 0\} \tag{2}$$

The bandwidth of A is defined by

$$\beta = \beta(A) = \max\{\beta_i(A) \,|\, 1 \leq i \leq n\} = \max\{|i-j| \,|\, a_{ij} \neq 0\} \tag{3}$$

For Cholesky decomposition, the relation between the number of operations N_{op} and the bandwidth β is [14]

$$N_{op} = \beta(\beta+2)n - \frac{2}{3}\beta^3 - \frac{3}{2}\beta^2 - \frac{5}{6}\beta \tag{4}$$

The relation between memory space M_{space} and β is [15]

$$M_{space} = (\beta+1)n - \frac{1}{2}\beta^2 - \frac{1}{2}\beta \tag{5}$$

If $\beta \ll n$, N_{op} is simplified as a function

$$N_{op} = O\left(n\beta^2\right) \tag{6}$$

The memory space required is also simplified as a function

$$M_{space} = O(n\beta) \tag{7}$$

Hence, reduction in β leads to a square ratio reduction in N_{op} and a homogeneous reduction in M_{space}.

The vector that contains all the bandwidth lines is called the envelope of A and defined by

$$Env(A) = \{(i,j) \,|\, 0 < i-j \leq \beta_i(A), i = 1, \ldots, n\} \tag{8}$$

The quantity $|Env(A)|$ is called the profile of A and is defined by

$$P(A) = |Env(A)| = \sum_{i=1}^{n} \beta_i(A) \tag{9}$$

By minimizing the profile, we minimize the number of stored zero values of stiffness matrix.

2.2. Graph Theory

By renumbering nodes in corresponding graph G, we can change the structure of A to reduce its bandwidth and profile. Hence the graph theory can be used to solve bandwidth and profile reduction problem.

For $A = (a_{ij})_{n\times n}$ in Equation (1), we can define a graph $G = (V, E)$, where V has n nodes, $\{v_1, v_2, \ldots, v_n\}$ and $\{v_i, v_j\} \in E$ if $a_{ij} \neq 0$ and $i \neq j$. The elements of $V = V(G)$ and $E = E(G)$ are called nodes and edges, respectively.

Level structure is an important concept concerning bandwidth and profile reduction algorithms in graph theory. A level structure, $L(G)$, of a graph G is a partition of set $V(G)$ into levels L_1, L_2, $\ldots$, L_k, so the depth of $L(G)$, $d(L)$, is k, the number of levels. The essential properties of $L(G)$ are that all nodes adjacent to nodes in L_1 are in either L_1 or L_2; all nodes adjacent to nodes in L_k are in either L_k or L_{k-1}; for $1 < i < k$, all nodes adjacent to nodes in L_i are in either L_{i-1}, L_i or L_{i+1}. To each node $v \in V(G)$ there corresponds a particular level structure $L_v(G)$ called the level structure rooted at v. In any level structure $L(G)$, rooted or not, $w_i(L) = |L_i|$ (the number of nodes in L_i) is called the width of L_i, and $w(L) = \max\{w_i\}\,(i = 1, \ldots, k)$ is the width of the level structure $L(G)$.

3. DESCRIPTION OF THE NOVEL ALGORITHM

3.1. Basic Theory of Proposed Algorithm

It is easily observed that for any level structure, $L(G)$, a numbering f_L of G that assigns consecutive integers level by level, from L_1 to the last level, yields a bandwidth, β_{f_L}, satisfying [9]

$$\beta_{f_L} \leq 2w(L) - 1 \tag{10}$$

If $L(G)$ is rooted, then we also have

$$\beta_{f_L} \geq w(L) \tag{11}$$

The width of any level structure $L(G)$, $w(L)$, and the depth of $L(G)$, $d(L)$ satisfy

$$\frac{n}{d(L)} \leq w(L) \leq \frac{n^2}{d(L)} \tag{12}$$

Equation (12) shows that if level depth $d(L)$ becomes larger, level width $w(L)$ has the tendency to become smaller. So we put forward a new parameter "width-depth ratio", denoted by κ, defined by

$$\kappa = \frac{w(L)}{d(L)} \tag{13}$$

The pseudo-diameter of the graph, $PD(G)$, and the depth the level structure, $d(L)$, have the relation

$$PD(G) = d(L) - 1 \tag{14}$$

If we pick the pseudo-peripheral nodes with the minimal width-depth ratio, it means that the level structure or the graph tends to have the maximal $d(L)$, maximal $PD(G)$ and minimal $w(L)$, which is desired to get a smaller bandwidth and further a smaller profile. From Equations (10), (11), (12) and (13) we get a relation that smaller width-depth ratio leads to smaller bandwidth and profile.

3.2. Procedure of the Proposed Algorithm

The description of proposed algorithm is divided to three parts, composed of the following steps:

(i) Finding pseudo-peripheral nodes of G.
   1) Pick all the nodes with the minimal degree and call them the candidate nodes group.
   2) Compute the value of κ of each node in candidate nodes group.
   3) Pick all the nodes with the minimal value of κ in candidate nodes group; call them κ nodes group.
   4) Set k to be the depth of level structures rooted from κ nodes group.
   5) If there is only one node in κ nodes group, call it "v", and go to step 7); if there are no less than two nodes in κ nodes group, go to step 6).
   6) If we can find two nodes in κ nodes group to be the pseudo-peripheral nodes, set them to be node "v" and node "u", generate level structure rooted from each node, which are L_v and L_u, and the first part of the proposed algorithm stops here; if not, find the node with the smallest code and call it node "v", and go to step 7).
   7) Generate a level structure L_v rooted at node v and let S be the set of nodes which are located in the last level of L_v; find node in S with the smallest value of κ and call it node "u", and generate rooted level structure L_u.

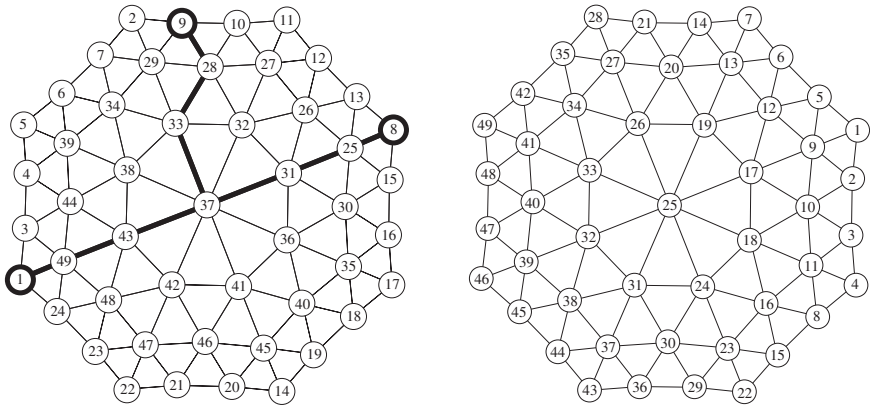
(ii) Minimize level width by combining the level structures rooted at "v" and "u" found in step (i).

(iii) Renumber nodes in G level by level.

Step (ii) and step (iii) are both similar to the second and third part in GPS algorithm.

3.3. An Example

To illustrate the proposed algorithm, consider the graph in Figure 1(a), which is composed of 49 nodes. The initial bandwidth and profile are 48 and 532. The bandwidth and profile after GPS algorithm are 12 and 329. The pseudo-peripheral nodes in GPS algorithm are node 1 and 9, while the pseudo-diameter is 6.



(a) Initial numbering and pseudo-diameters (b) Renumbering in proposed algorithm

Figure 1. Graph with 49 nodes.

First, pick all the nodes with the minimal degree. In this case the minimal degree is 3. The candidate nodes group is (1, 2, 5, 8, 11, 14, 17, 22). Then compute κ of each node in candidate nodes group, the results is (13/7, 13/7, 13/7, 13/7, 13/7, 13/7). All have the same value of κ, so κ nodes group is also (1, 2, 5, 8, 11, 14, 17, 22). Pick node 1 and generate its rooted level structure, node 8 is just in the last level of this level structure, thus set node 1 and node 8 as node "v" and "u", respectively. The pseudo-diameter is 6, which is the length of path between node 1 and 8.

After the second and third parts of proposed algorithm we have a new numbering as shown in Figure 1(b). The bandwidth and profile are 8 and 318, and the reduction ratios in bandwidth and profile are 33.3% and 3.34%, respectively, compared with the results of GPS algorithm.

It is easily observed that the path with the longest length in the graph is 6; hence the proposed algorithm just found the pseudo-diameter with the same length as GPS algorithm. However, the pseudo-peripheral nodes in two algorithms are different as well as the paths between pseudo-peripheral nodes. The pseudo-diameter in the proposed algorithm is just in the middle of the graph, cutting the graph

into two equal parts, while the pseudo-diameter in GPS algorithm is irregular, as shown in Figure 1.

4. DESCRIPTION OF TEST RESULTS

In this section, we report the results on a range of test problems. The authors chose 16 classical models, containing both solid and plane models. The solid models were meshed in tetrahedron elements, and the plane models were meshed in triangle elements. The obtained results are then compared to the results in GPS algorithm. The proposed algorithm and GPS algorithm were both coded with FORTRAN 95 programming language.

Table 1 shows the test results containing bandwidths, profiles, and pseudo-diameters of 16 models.

Table 1. Bandwidths, profiles and diameters of two algorithms, PD — pseudo-diameters, β_p — bandwidth in proposed algorithm, β_G — bandwidth in GPS algorithm.

No.	Nodes	GPS			proposed algorithm			$1-\beta_p/\beta_G$
		β_G	profile	PD	β_p	profile	PD	
1	49	12	P_1	7	8	$0.9666P_1$	7	33.33%
2	85	15	P_2	12	15	$0.9945P_2$	12	0.00%
3	194	40	P_3	12	38	$0.9724P_3$	12	5.00%
4	199	17	P_4	16	17	$0.9978P_4$	16	0.00 %
5	204	16	P_5	22	15	$0.9784P_5$	22	6.25%
6	366	22	P_6	30	21	$0.9884P_6$	30	4.55%
7	521	16	P_7	53	16	$0.9572P_7$	53	0.00%
8	677	36	P_8	30	34	$0.9800P_8$	30	5.56%
9	738	24	P_9	44	24	$0.9880P_9$	46	0.00%
10	745	60	P_{10}	24	47	$0.8835P_{10}$	24	21.67%
11	750	34	P_{11}	31	34	$0.9915P_{11}$	31	0.00%
12	3683	82	P_{12}	91	80	$0.9750P_{12}$	93	2.44%
13	12029	1128	P_{13}	31	1059	$0.9703P_{13}$	31	6.12%
14	22298	1123	P_{14}	44	1090	$0.9980P_{14}$	44	2.94%
15	72610	340	P_{15}	299	322	$0.9978P_{15}$	300	5.29%
16	149610	864	P_{16}	294	783	$0.9654P_{16}$	300	9.38%

From Table 1, we can see the renumberings generated by proposed algorithm; all have smaller bandwidths and profiles, compared with the outcomes in GPS algorithm. The bandwidth can be reduced by 21.67% in the No. 10 model, or even by 33.33% in the No. 1 model; the profile can be even reduced by 11.65% in the No. 10 model.

In some cases, the pseudo-diameters are longer than the ones in GPS algorithm. In the No. 9 and No. 12 models, pseudo-diameters are both increased by 2; in the No. 15 model, pseudo-diameter is increased by 1; in the last model, pseudo-diameter is even increased by 6. The models without increase in pseudo-diameters also get reductions in bandwidth and profile.

Figure 2 graphically displays the time ratio between two algorithms, where T_{Pro} is the time cost by proposed algorithm; T_{GPS} is the time cost by GPS algorithm. From Figure 2, we find that most time ratio between two algorithms is 1. The biggest time ratio is around 1.4 in the No. 13, No. 15, and No. 16 models, while the reduction ratios in bandwidths of these models are all more than 5%.

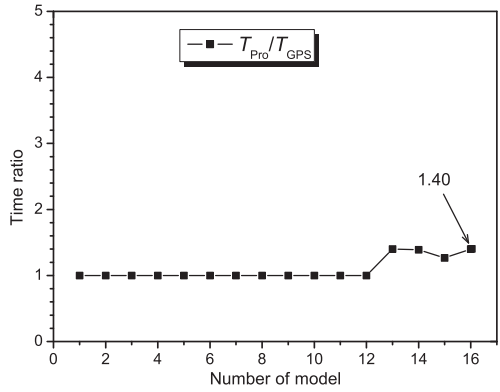


Figure 2. Computation time ratio between two algorithms.

5. CONCLUSIONS

Based on GPS algorithm, we put forward a new parameter κ (width-depth ratio). Upon κ, we develop a novel algorithm to supply proper pseudo-peripheral nodes, in order to obtain high quality results in minimizing bandwidth and profile of a stiffness matrix in the finite element analysis. The examples presented in this paper concerning a variety of real problems confirm the insufficient of GPS algorithm in the selection of pseudo-peripheral nodes.

We have tested both algorithms by the same 16 classical models and compared their performance. Both approaches give significant improvements. However, our implementation was shown to be more competitive. The pseudo-diameter in proposed algorithm is even longer than the one of GPS by 6. The bandwidth reduction are impressive, and some of them are as high as 33.33%. The profile reduction are also desirable, and the biggest reduction ratio is 11.65%. The computation time of proposed algorithm is very close to the time cost by GPS algorithm, meaning the cost in time is worthy in the bandwidth and profile reduction. We can reach a conclusion that the proposed algorithm is more effective in reducing bandwidth and profile.

For all models mentioned in this paper, the proposed algorithm produces better renumbering and is only slightly worse in computation time, compared with GPS algorithm. It is concluded that the proposed algorithm has a more suitable rule; hence, the renumbering in proposed algorithm can be always better than the renumbering in GPS algorithm. The proposed algorithm can be widely used in the renumbering of stiffness matrix in the finite element method.

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