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CHARGE TRANSPORT BY A PULSE E-WAVE IN A WAVEGUIDE WITH CONDUCTIVE MEDIUM

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Abstract—An E-wave excites longitudinal conductivity currents in a waveguide filled with lossy medium. Total flow of this current through a cross-section is nonzero for odd E-modes; it means that some charge is transported by the pulse along the waveguide. This phenomenon is considered in the paper using analytical approach based on mode expansion of the fields in Time Domain (known as Mode Basis Method) and using Finite Difference in Time Domain method (FDTD). Volume conductivity charges being moved by the pulse wave are determined. In order to set the problem as much physically as possible we consider diffraction of a pulse E-wave on the boundary between a hollow waveguide and a waveguide with conductive medium. Behavior of the transient fields and surface charges at the interface are also examined. It is shown that an incident pulse wave excites a surface wave at the interface that results in transversal resonance with relaxation time much greater than that of the conductive medium. Characteristics of the travelling charge pulse wave are studied based on the obtained closed-form solution.

1. INTRODUCTION

Many phenomena look very simple when considered in the Frequency Domain (FD). This is due to steadiness of the processes, all of them last from minus to plus infinity in time. It masks some effects that can be revealed only when considering transient processes. Among such effects we examine charge transportation by an E-wave in a waveguide with conductive medium. E-wave has longitudinal electric field component that results in longitudinal conductivity currents. For odd wave indexes the total current flow through the waveguide cross-section is nonzero, it means that charges are moved along the

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waveguide [1]. This situation when considered in FD is very simple: Half the period the charges are moved forth and another half the period they are moved back, in average the medium is neutral, no charge is moved farther than half wavelength [2]. Differently the situation looks for a transient process, e.g., for a pulse wave. At the wave front the medium is originally neutral and the front moves charges ahead, the pulse tail returns them back. Due to dispersion these processes may not compensate each other exactly and there is always some charge moving at the wave front. We are going to study this phenomenon in details using analytical solution based on mode expansion in TD and transport operators [3, 4]. In order to be more realistic let us consider the case when a pulse wave is excited in a hollow waveguide and then is incident on an interface of semi-infinite conductive medium in the waveguide. This allows avoiding problems concerning source current interaction with conductivity currents.

The problem of diffraction of a transient wave on a dielectric half-space has been studied by many authors before. Some approaches to its solution can be found in [5–7]. It should be noted that the problem of plane wave diffraction on a half-space under oblique incidence is equivalent to diffraction at a dielectric interface in a waveguide in accordance to the Brillouin concept [2]. As far as we know none of the previous authors have studied a pulse charge wave excited in the dielectric half-space with non-zero conductivity by a longitudinal component of a transient E-wave.

The problem of transient signal propagation in a waveguide has been considered in many papers (a detailed review can be found e.g., in [8]). The most suited analytical method for TD analysis of the problem is the Mode Expansion in Time Domain method also known as Mode Basis Method [8–11]. Within METD the sought fields are expanded in terms of waveguide modes with mode amplitudes being some functions of longitudinal coordinate and time. The latter are governed by Klein-Gordon equation describing waveform evolution with propagation. The solution to the evolutionary equation can be presented in the form of convolution of initial-boundary conditions with some transport operator. Such method has been used under different names by many other researches, see for example papers by Geyi [8,9], Kurokawa [10], Kristensson [11], and a series of papers by Tretyakov and his followers [3, 4, 12, 13–15].

2. PROBLEM STATEMENT

We consider a PEC bounded waveguide of arbitrary singly-connected cross-section (without TEM-modes). Part of the waveguide $z > 0$

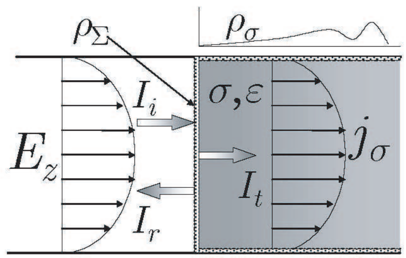


Figure 1. Problem geometry.

is filled with conductive dielectric medium with permittivity ε and conductivity σ (Fig. 1). From the hollow waveguide region ($z < 0$) a pulse E-wave arrives to the boundary $z = 0$ and causes reflected and transmitted waves. The transmitted wave in the conductive medium creates longitudinal conductivity currents that results in volume conductivity charges appearing. Also at the boundary some surface conductivity charge arises.

First of all let's recall how the charges can be calculated. One way is to use the continuity equation:

$$\partial_t \rho_\sigma = -\nabla \cdot \vec{j}_\sigma, \tag{1}$$

Another way consists in exploiting the divergence equation:

$$\mathrm{div}\vec{D} = \rho \tag{2}$$

At this point we should remind that when considering in FD this equation assumes that only the impressed charges but not the conductivity ones are taken at the right hand side. This is due to including conductivity into the definition of complex permittivity ($i = \sqrt{-1}$):

$$\varepsilon^* = \varepsilon + \sigma / i\omega\varepsilon_0$$

In contrast in TD conductivity charges are considered separately from the polarization ones and thus can be obtained from (2). The surface charge can be found from the boundary condition for the electric displacement normal component that can be derived from (2) as:

$$\rho_\Sigma = \varepsilon_0 \left(\varepsilon E_z^2 - E_z^1\right) \tag{3}$$

indexes 1 and 2 denote fields in the hollow waveguide and in the lossy dielectric correspondingly.

This formula can also be found from its FD counterpart in the following way

$$\begin{aligned}
&\dot{D}_z^1 - \dot{D}_z^2 = 0 \Rightarrow \varepsilon_0 \dot{E}_z^1 - \varepsilon_0 \left(\varepsilon + \tfrac{\sigma}{i\omega\varepsilon_0}\right) \dot{E}_z^2 = 0 \\
&\tfrac{1}{i\omega} \dot{E}(\omega) \xrightarrow{\mathcal{F}^{-1}} \int_{-\infty}^{t} E(\tau)\, d\tau \\
&\varepsilon_0 \left(E_z^1 - \varepsilon E_z^2\right) = \int_{-\infty}^{t} \sigma E_z^2 \, dt = \int_{-\infty}^{t} j_z \sigma \, dt = -\rho_\Sigma
\end{aligned} \tag{4}$$

From this derivation it is clear that the surface charge appears due to the longitudinal conductivity current component that removes charges from the surface. The derivation similar to (4) can also be applied to the surface between the conductive medium and the waveguide walls. Then we'll see that some surface conductivity charges also occur at the side walls due to existence of transversal components of the conductivity currents there. There is one thing to point out here: Transversal current components don't move charges from one cross-section to other, they just borrow some charge from the wall into the cross-section and later return them back. Since we investigate current transport **along** the waveguide we will not take into account the volume conductivity charges caused by the transversal currents. Mathematically it can be expressed by leaving in (1) only z component of the conductivity current:

$$\partial_t \rho'_\sigma = -\partial_z j_{\sigma z} \tag{5}$$

The prime indicates that we consider only the part of the volume charge that is moved along the waveguide. This equation can also be justified by considering the integral form of the charge conservation law and accounting for the fact that there is no conductivity charge leakage into the waveguide walls.

3. SOLUTION BY METHOD OF MODE EXPANSION IN TIME DOMAIN

Now that we are defined with what we are looking for let us determine a suitable calculation method. One possible way is to use FD solution, which is very simple for the case under consideration, and then convert it into TD using inverse Fourier transform. Such an approach has several disadvantages. First of all when considering a pulse wave we are dealing with an ultrawideband signal that requires one to calculate the data at large number of frequency points in order to obtain sufficient time resolution. This objection is not critical for the case under consideration since FD solution is easy to calculate. Another much more important issue is that the FD data contain a singular point at the cut-off frequency. This obstacle is much heavier since commonly used FFT algorithm assumes even sampling and consequently the convergence with decreasing frequency step is very slow [16, 17]. It can be clarified by comparing with calculating a singular integral with a quadrature that uses evenly sampled points.

Reasoning from the above said we will derive the solution directly in TD. It does use Laplace transform at intermediate steps but arrives to a close-form representation as convolution operators at the end, i.e., the singular point is integrated analytically. The convolutions involved in the resulting expressions are easy to calculate directly via quadrature due to short duration of the input signal.

3.1. Fields and Charges Expansion with METD

Let's start our consideration from Maxwell equations:

$$\begin{cases} \nabla \times \vec{\mathcal{H}} = \varepsilon_0 \varepsilon \partial_t \vec{\mathcal{E}} + \sigma \vec{\mathcal{E}}, \\ \nabla \times \vec{\mathcal{E}} = -\mu_0 \partial_t \vec{\mathcal{H}}, \\ \nabla \cdot \varepsilon_0 \varepsilon \vec{\mathcal{E}} = \rho_\sigma, \quad \nabla \cdot \vec{\mathcal{H}} = 0, \end{cases} \tag{6}$$

Introducing normal-tangential notation:

$$\vec{\mathcal{E}} = \vec{E} + \vec{z}_0 E_z;\ \vec{\mathcal{H}} = \vec{H} + \vec{z}_0 H_z;\ \nabla = \nabla_\perp + \vec{z}_0 \partial_z;\ \vec{r} = \vec{r}_\perp + \vec{z}_0$$

the equations can be separated into subsystems for E- and H-waves. E-waves are governed by the following equations subset [18]:

$$\begin{cases} \left[\vec{z}_0 \times \nabla_\perp\right] E_z = \mu_0 \partial_t \vec{H} + \partial_z \left[\vec{z}_0 \times \vec{E}\right] \\ \varepsilon_0 \varepsilon \partial_t E_z = \nabla_\perp \cdot \left[\vec{H} \times \vec{z}_0\right] - \sigma E_z \\ \partial_z E_z + \nabla_\perp \cdot \vec{E} = \rho_\sigma / \varepsilon_0 \varepsilon \end{cases} \tag{7}$$

The constitutive parameters ε and σ are assumed to be piecewise constant. The system can be reduced by applying partial separation of variables [3, 4]. Separation of transversal coordinates results in the following presentation for the solution (mode expansion):

$$\begin{aligned} \vec{E}\left(\vec{r}_\perp, z, t\right) =& (\varepsilon_0 \varepsilon)^{-1/2} \\ & \sum_n \left[V_n(z,t)\, \nabla_\perp \varphi_n\left(\vec{r}_\perp\right) + \vec{z}_0 e_n(z,t)\, \kappa_n^2 \varphi_n\left(\vec{r}_\perp\right)\right], \\ \vec{H}\left(\vec{r}_\perp, z, t\right) =& (\mu_0)^{-1/2} \sum_n I_n(z,t) \left[\vec{z}_0 \times \nabla_\perp \varphi_n\left(\vec{r}_\perp\right)\right], \end{aligned} \tag{8}$$

where V_n is the transversal electric field mode amplitude, e_n is the longitudinal electric field mode amplitude, and I_n is the transversal magnetic field mode amplitude.

The transversal mode configuration $\varphi_n(\vec{r}_\perp)$ and the separation constant κ_n can be found from the following boundary problem:

$$\left(\nabla_\perp^2 + \kappa_n^2\right)\varphi_n(\vec{r}_\perp) = 0; \quad \varphi_n|_L = 0,$$
$$\frac{\kappa_n^2}{S}\int_S \varphi_n\varphi_{n'}\,dS = \frac{1}{S}\int_S \nabla_\perp\varphi_n \cdot \nabla_\perp\varphi_{n'}\,dS = \delta_{nn'}$$

Index n indicates the mode number. Since the considered problem doesn't result in mode coupling let's further work with a single mode and omit the mode index n.

The volume charges ρ_σ and surface charges Σ_σ on the medium interface can be expanded in a similar way:

$$\begin{aligned}\rho_\sigma(\vec{r}_\perp, z, t) &= (\varepsilon_0\varepsilon)^{-1/2}P(z,t)\,\kappa^2\varphi(\vec{r}_\perp)\\ \Sigma_\sigma(\vec{r}_\perp, t) &= (\varepsilon_0)^{-1/2}S(t)\,\kappa^2\varphi(\vec{r}_\perp)\end{aligned} \tag{9}$$

where $P(z,t)$ is the mode amplitude of the volume charges and $S(t)$ is the mode amplitude of the surface charge.

3.2. System of Evolutionary Equations

Substituting the expansion (8) into system (7) one can find the following governing equations for the mode amplitudes V, I, e, and P:

$$\begin{cases}\kappa^2 e = \partial_{ct}I + \partial_z V\\ \partial_{ct}e = -I - \tilde{\sigma}e;\\ \partial_z e = V + P;\\ \partial_z I + \partial_{ct}V + \tilde{\sigma}V = 0\end{cases} \tag{10}$$

where $\tilde{\sigma} = \sigma\sqrt{\mu_0/\varepsilon_0\varepsilon}$, $\partial_{ct} = c^{-1}\partial/\partial t = \sqrt{\varepsilon_0\varepsilon\mu_0}\,\partial_t$, $c = 1/\sqrt{\varepsilon_0\varepsilon\mu_0}$ is the speed of light in the medium.

This is the System of Evolutionary Equations (SEE) [3]. It can be rearranged as follows:

$$\begin{cases}\left(\partial_t^2 + 2\gamma\partial_t - c^2\partial_z^2 + \lambda^2\right)I = 0;\\ (\partial_t + 2\gamma)\,e = -c\,I;\\ (\partial_t + 2\gamma)\,V = -c\,\partial_z I;\end{cases} \tag{11}$$

where $\gamma = c\tilde{\sigma}/2 = \sigma/2\varepsilon_0\varepsilon$, $\lambda = c\,\kappa$. Here the first equation is the Klein-Gordon equation to be solved against I, while the second and the third ones allow finding mode amplitudes V and e when the mode amplitude I is already known.

3.3. Solution of the Klein-Gordon Equation

The solution of the Klein-Gordon equation can be found using Laplace transform ($t \to s$) as described in [19]. In the image domain Klein-Gordon equation looks as follows

$$\left(c^2\partial_z^2 - \left(s^2 + 2\gamma s + \lambda^2\right)\right)I = 0$$

The solution to this equation can be represented as

$$\begin{aligned} I(z,s) \;=\; & C_1 \exp\left(-c^{-1} z\sqrt{s^2+2\gamma s+\lambda^2}\right) \\ & + C_2 \exp\left(+c^{-1} z\sqrt{s^2+2\gamma s+\lambda^2}\right), \end{aligned}$$

In this expression the first term describes the wave moving from a source to $+\infty$ and the second term corresponds to the wave moving to $-\infty$. Accounting for the condition that all the possible sources should stay at $z<0$ we keep only the first term

$$I(z,s) = C_1 \exp\left(-c^{-1} z\sqrt{s^2+2\gamma s+\lambda^2}\right), \tag{12}$$

3.4. Propagation Operators

Substituting $z=0$ into formula (12) we can express C_1 via $I(0,s)$ and obtain

$$I(z,s) = I(0,s) \exp\left(-c^{-1} z\sqrt{s^2+2\gamma s+\lambda^2}\right), \tag{13}$$

Similarly applying z-derivative to (12) at $z=0$ we can express C_1 via $\partial_z I(0,s)$ that results in

$$I(z,s) = -c\partial_z I(0,s)\frac{\exp\left(-c^{-1} z\sqrt{s^2+2\gamma s+\lambda^2}\right)}{\sqrt{s^2+2\gamma s+\lambda^2}}, \tag{14}$$

Considering formulae (13) and (14) in TD and using the convolution theorem [19] one can arrive to the following transport operators Z and Z' [4]:

$$\begin{aligned} I(z,t) &= Z(z,t) * I(0,t) \\ I(z,t) &= Z'(z,t) * \partial_z I(0,t) \\ Z(z,t) &= \delta(t-z/c)e^{-\gamma z/c} \\ &\qquad - \eta(t-z/c)\, e^{-\gamma t}\tilde{\omega}^2 \tilde{J}_1\left(\tilde{\omega}\sqrt{t^2-(z/c)^2}\right) z/c \\ Z'(z,t) &= -c\, e^{-\gamma t} J_0\left(\tilde{\omega}\sqrt{t^2-(z/c)^2}\right)\eta(t-z/c) \end{aligned} \tag{15}$$

where $\tilde{\omega} = \sqrt{\lambda^2-\gamma^2}$, $\tilde{J}_1(x) = J_1(x)/x$, $J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions, $\delta(\cdot)$ is the Dirac delta-function, $\eta(\cdot)$ is the Heaviside step function. The time convolution is introduced in (15) as follows [19]:

$$f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

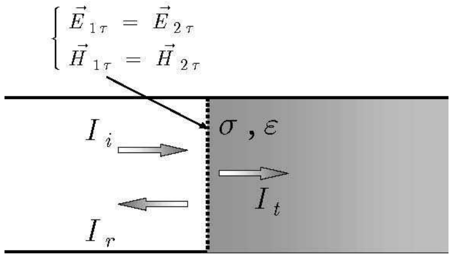


Figure 2. The diffraction problem.

From SEE (11) one can derive the rest of the operators that relates mode amplitudes at any given z:

$$\begin{array}{ll} I(z,t) = \Omega\,(t) * \partial_z I(z,t), & \Omega\,(t) = -ce^{-\gamma t} J_0\,(\tilde{\omega} t); \\ e(z,t) = \Psi\,(t) * I(z,t), & \Psi\,(t) = -ce^{-2\gamma t}; \\ V(z,t) = \Psi\,(t) * \partial_z I(z,t) & \end{array} \tag{16}$$

3.5. Diffraction Problem

Now we can proceed with the diffraction problem (see Fig. 2). The boundary conditions require continuity of the tangential components of electric and magnetic fields at the medium interface:

$$\left\{ \begin{array}{l} \vec{E}_1 = \vec{E}_2, \\ \vec{H}_1 = \vec{H}_{2,} \end{array} \right. \tag{17}$$

Substituting mode expansions (8) into (17) we arrive to the boundary conditions for the mode amplitudes

$$\begin{array}{l} I_i + I_r = I_t \\ V_i + V_r = V_t/\sqrt{\varepsilon} \end{array} \tag{18}$$

indexes i, r, and t denote incident, reflected, and transmitted wave correspondingly.

We are looking for the solution in the form of diffraction operators $R(t)$ and $T(t)$, which relate derivatives of the reflected and transmitted waves to the incident wave amplitude at the boundary (assumed to be at $z = 0$) [4]:

$$\begin{array}{l} \partial_z I_r(0,t) = R(t) * I_i(0,t) \\ \partial_z I_t(0,t) = T(t) * I_i(0,t) \end{array} \tag{19}$$

The plus sign in the expression for the reflected wave is due to the inverse propagation direction.

Detailed derivation and the resulting expressions for $R(t)$ and $T(t)$ operators are given in Appendix A.

Thus having found the diffraction operators we are able to calculate the transmitted and reflected wave amplitudes at any z, t by applying transport operators (15) to the wave derivatives at the boundary calculated with (19).

3.6. Calculation of the Charge Distribution

Now we need to relate the found field mode amplitudes with mode amplitudes for the charges (9). Accounting for

$$j_{\sigma z} = \sigma E_z \tag{20}$$

and substituting field expansion (8) into (5) the following relation can be obtained:

$$\partial_t P'(z,t) = -2\gamma \partial_z e(z,t); \tag{21}$$

Now we can combine (16), (19), and (21) in a single chain of operators that relates the volume charge mode amplitude with the mode amplitude of the incident wave. $\partial_z e_t$ can be expressed via $\partial_z I_t$ as

$$\partial_z e_t(z,t) = \Psi(t) * \partial_z I_t(z,t), \quad \Psi(t) = -ce^{-2\gamma t}$$

Applying propagation operator (15) we can relate this amplitude with that at the boundary

$$\partial_z e_t(z,t) = \Psi(t) * Z(z,t) * \partial_z I_t(0,t),$$

Finally, applying transmission operator (19) yields the expression for the volume charge amplitude

$$P'(z,t) = -2\gamma \eta(t) * \Psi(t) * Z(z,t) * T(t) * I_i(0,t) \tag{22}$$

It should be noted that since the convolution is associative and commutative operation this chain can be applied in any sequence.

The surface charge amplitude can be found by substituting (8) and (9) into (3):

$$S(t) = \sqrt{\varepsilon} e_t(0,t) - (e_i(0,t) + e_r(0,t))$$

Now we are fully equipped to calculate the sought volume and surface charges for any incident wave.

4. SOLUTION BY FDTD

In order to verify the obtained closed-form solution we solve the problem by the well known FDTD method (Finite Difference in Time Domain). Today FDTD [20] is among the most reliable and universal methods for solving electromagnetic problems. That is why we choose it as a reference solver to compare with our METD results.

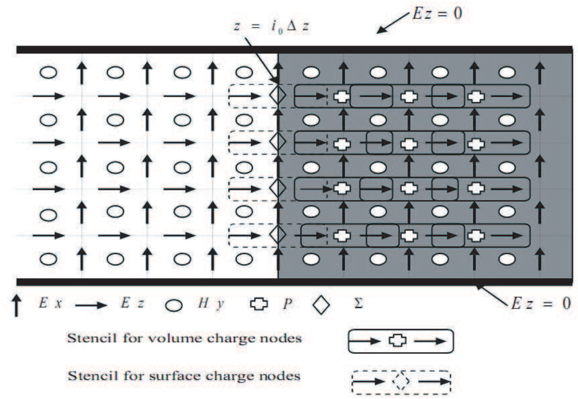


Figure 3. Spatial grid for the field components.

4.1. FDTD Modeling of A Parallel-Plate Waveguide

We consider an E-wave in a parallel-plate waveguide (Fig. 3). Maxwell equations (6) for this structure can be written as:

$$\begin{cases} \partial_{ct}\tilde{H}_y = \partial E_z/\partial x - \partial E_x/\partial z \\ \partial_{ct}E_z + 2\gamma c^{-1}E_z = \partial\tilde{H}_y/\partial x \\ \partial_{ct}E_x + 2\gamma c^{-1}E_x = -\partial\tilde{H}_y/\partial z \end{cases} \tag{23}$$

with $\tilde{H}_y$ normalized as $\tilde{H}_y = H_y\sqrt{\mu\mu_0/\varepsilon\varepsilon_0}$, γ is the decay constant introduced in (11).

Approximating time and spatial derivatives with central differences and using leapfrog time-stepping scheme [20] the following explicit march-in-time formulae can be derived:

$$\begin{cases} \tilde{H}_y\Big|_{i,j}^{k+0.5} = \tilde{H}_y\Big|_{i,j}^{k-0.5} + S_x\left(E_z|_{i,j+0.5}^{k} - E_z|_{i,j-0.5}^{k}\right) \\ \qquad -S_z\left(E_x|_{i+0.5,j}^{k} - E_x|_{i-0.5,j}^{k}\right), \\ E_z|_{i,j+0.5}^{k+1} = \frac{1-\gamma\Delta t}{1+\gamma\Delta t}\, E_z|_{i,j+0.5}^{k} \\ \qquad +\frac{S_x}{1+\gamma\Delta t}\left(\tilde{H}_y\Big|_{i,j+1}^{k+0.5} - \tilde{H}_y\Big|_{i,j}^{k+0.5}\right), \\ E_x|_{i+0.5,j}^{k+1} = \frac{1-\gamma\Delta t}{1+\gamma\Delta t}\, E_x|_{i+0.5,j}^{k} \\ \qquad -\frac{S_z}{1+\gamma\Delta t}\left(\tilde{H}_y\Big|_{i+0.5,j}^{k+0.5} - \tilde{H}_y\Big|_{i,j}^{k+0.5}\right) \end{cases} \tag{24}$$

where $S_x = c\Delta t/\Delta x$, $S_z = c\Delta t/\Delta z$. The staggered grid of the nodes where the field components are taken is shown in Fig. 3. Such a numerical scheme implicitly takes into account Gauss Law relations for electric and magnetic fields, thus ensuring charge conservation.

In the calculations spatial steps were chosen as $\Delta z = \Delta x = L/20$, where L is the cross-section size of the waveguide. It provides 40 points per wavelength at the cutoff frequency. The time step is then chosen in accordance with the stability criterion [20]

$$\Delta t = c_0^{-1} \left(\Delta x^{-2} + \Delta z^{-2}\right)^{-1/2}$$

4.2. Calculation of the Charge Distribution

The scheme for calculating surface and volume charge is shown in Fig. 3. Density of the surface charge can be found from the discontinuity in the longitudinal component of the electric flux density at the boundary (see Eq. (4))

$$\Sigma|_j = Ex|_{i_0-0.5,j} - \varepsilon \, Ex|_{i_0+0.5,j}\,, \tag{25}$$

Eqs. (5) and (20) yield the following time-stepping formula for the volume charge calculation

$$\rho|_{i+0.5,j+0.5}^{n+0.5} = \rho|_{i+0.5,j+0.5}^{n-0.5} - 2\sigma \frac{\Delta t}{\Delta x} \left(Ex|_{i+1,j+0.5}^{n} - Ex|_{i,j+0.5}^{n} \right), \tag{26}$$

In this way using formulae (25) and (26) the volume and surface charges were calculated.

4.3. Comparison with METD Results

In the FDTD calculations we used Total Field/Scattered Field (TF/SF) method for exciting the waveguide with a traveling wave [20]. Within TF/SF one need to set the incident fields at two adjacent cross-sections $\Delta z/2$ apart. The waveform of the longitudinal E-fields in the first cross-section was set to be a bipolar pulse shown in Fig. 6(a) (Eq. (27)). Then the cross-components of the fields at $\Delta z/2$ apart were calculated exactly using propagation operators (15). Setting the fields at two adjacent cross-sections in this way launches a traveling wave towards the boundary (see [20] for details on TF/SF implementation).

Figure 4 compares the waveform calculated with FDTD after propagating $5\Delta z$ off the excitation boundary against the exact waveform calculated as a convolution of the excitation waveform with propagation operator (15). A good agreement between the curves confirms validity of both FDTD and METD calculations. It was revealed that when the time step is chosen at the very stability limit

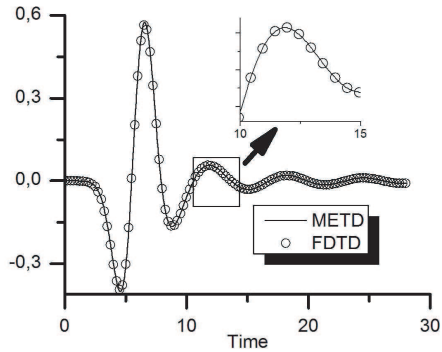
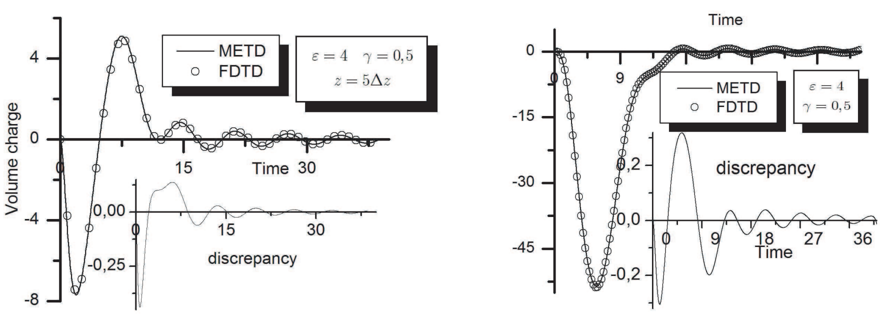


Figure 4. Comparison of the results obtained by FDTD (at 40 points per wavelength) and METD for signal propagation in an empty waveguide (distance $5\Delta z$).



(a) (b)

Figure 5. Comparison of FDTD and METD results for volume (a) and surface (b) charge densities.

then the intrinsic FDTD dispersion error is very low (this is an analogue to the 'magic' time step for the wave equation [20]).

Finally Fig. 5 shows the results of FDTD and METD calculations of volume (a) and surface (b) charges. The curves coincide perfectly well, thus proving the validity of the propagation operators derived with METD.

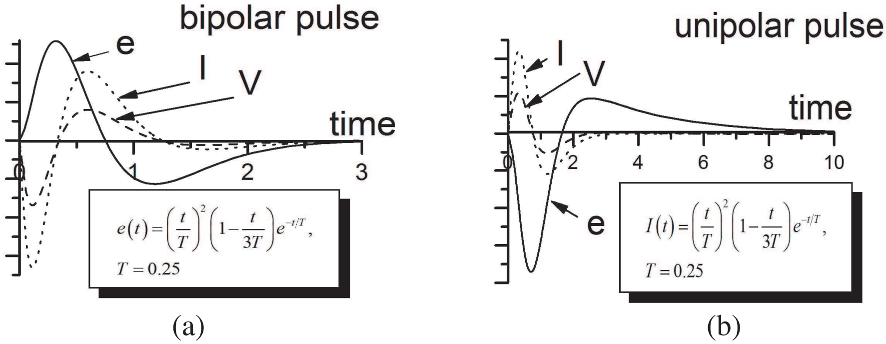


Figure 6. Excitation signals.

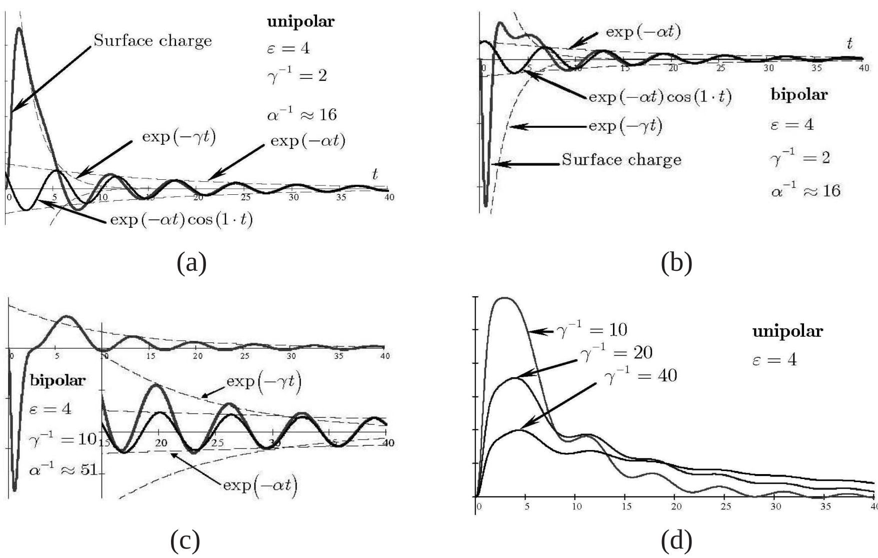


Figure 7. Surface charges.

5. ANALYSIS

In further analysis we will use two kinds of excitation shown in Fig. 6. In the first pulse the longitudinal field component varies in time as

$$f(t) = t^2(1 - t/3)e^{-t} \tag{27}$$

It is a bipolar pulse, which have no content at zero frequency. The second pulse is calculated as an integral of the former:

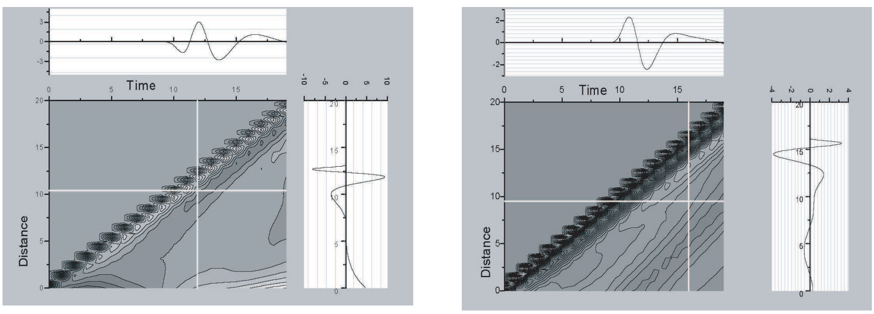
$$f(t) = -t^3 e^{-t} \tag{28}$$

Here the longitudinal E-field component spectrum extends up to zero frequency so it can be said to have some static content. Though since the transverse field components ($I \sim H_y$, $V \sim E_x$) are calculated as derivatives of the longitudinal one their spectra vanish at zero frequency.

These waveforms are set on the very boundary at $z = 0$. Such fields can be created by some sources (impressed currents and charges) located at $z < 0$.

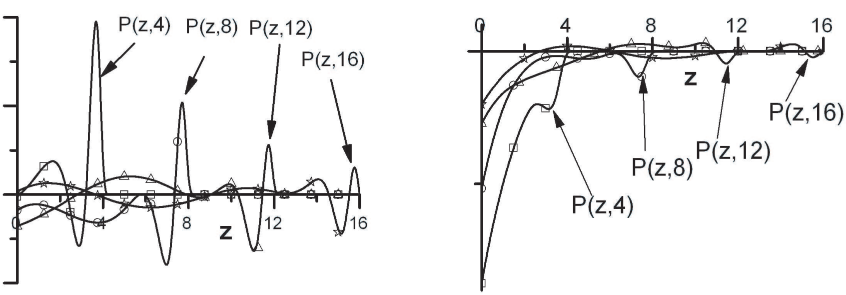
The diffracted fields look similar to the incident ones. The distortion is higher for the second signal due to larger low frequency content.

The volume charge calculated by formula (21) as a function of the normalized coordinate $(z/\sqrt{\varepsilon})$ for several time instants is plotted in



(a) (b)

Figure 8. Space-time diagrams of the volume charge distribution in the medium ((a) bipolar pulse for $\varepsilon = 4,\ \gamma = 0.01$, (b) unipolar pulse for $\varepsilon = 2,\ \gamma = 0.01$).



(a) (b)

Figure 9. The volume charge distribution in the medium with $\varepsilon = 4,\ \gamma = 0.01$ ((a) bipolar pulse, (b) unipolar pulse).

Fig. 8 and Fig. 9 ((a) bipolar pulse, (b) unipolar pulse). In Fig. 10 one can compare the spatial distributions of the electric field longitudinal component for lossy and lossless medium. The "static" (potential) electric field created due to charges separation is shown as well. It is calculated as

$$\partial_z E_{zst} = \rho, \quad \Rightarrow \quad E_{zst}\left(z,t\right) = -\int_{z_{front}}^{z} \rho\left(z,t\right) dz$$

It is due to this field the charges return back to the surface and the system relaxes to the uncharged neutral state. From Fig. 9(a), it is clearly seen that the pulse wave does move a charge wave ahead. This pulse wave decays with depth leaving small oscillating charges behind. After the pulse completely decays the charges are moved back to the surface.

From Fig. 9(b) for unipolar pulse, we can conclude that a unipolar character of the charge wave can be observed only at small distances from the surface since low frequency content of the wave decays heavily and the pulse becomes bipolar after propagating some distance off the surface. Thought it should be noted that the unipolar wave results in much higher volume charges left near the surface.

Figure 7 shows dependence on time of the surface charges for several loss parameters. The reciprocal of parameter γ introduced in (11) determines relaxation time for the conductive medium. It can be seen from the figures that at the very early time, after the pulse has just passed, the surface charges, left by the pulse, decays exponentially with decay parameter equal to γ. This process is determined by charges moved by the pulse into medium, these charges return back to the surface under potential (irrotational) electric field created due to charge separation.

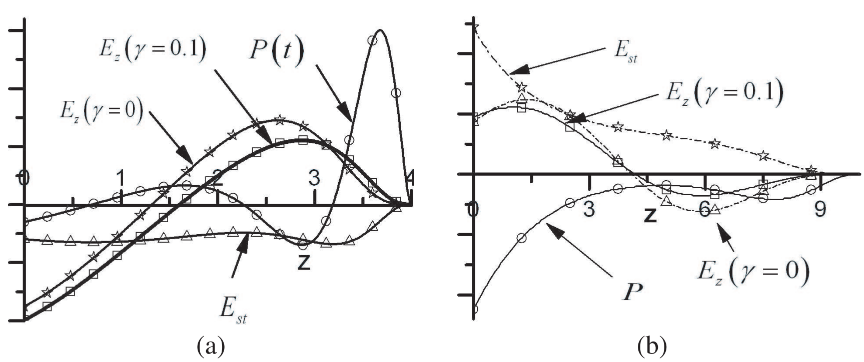


Figure 10. "Static" potential longitudinal electric field component for bipolar (a) and unipolar (b) pulses.

There is another periodic process in the late time that decays much slowly. This process is determined by a surface wave resonance. It is well known that at any surface of conductive medium a surface wave exists that decays off the surface. Such a wave has the following dispersion relation [7, 18]:

$$k(\omega) = \sqrt{\frac{\varepsilon^*(\omega)}{1+\varepsilon^*(\omega)}}\frac{\omega}{c_0} \tag{29}$$

The waveguide walls form resonator boundaries so that a surface wave rebounds between the walls. The decay time can be determined from the imaginary part of the eigenfrequency obtained from the boundary problem for the transverse resonator with dispersive properties (29) and bounded by the waveguide walls. The resulting decay parameter is $\alpha = 1/16$, it is shown in Fig. 7.

The incident short pulse has a very broad spectrum. It contains some energy at near the cut-off frequency of the mode. These spectral components propagate with Brillouin angles close to $\pi/2$ and thus they effectively excite the surface wave.

After we considered the phenomena on an example of a specific waveform, we can proceed with determining transient responses that correspond to excitation with a Heaviside step function. Such transient responses can be then convolved with time-derivative of any incident waveform in order to obtain the resulting signals:

$$\text{Resulting Signal}(t) = \int_0^t \text{Transient Response}(t-\tau)\frac{d}{d\tau}\text{Incident}(\tau)\,d\tau \tag{30}$$

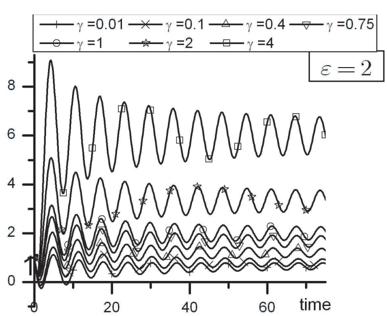


Figure 11. The transient response of the surface charge for different values of the conductivity.

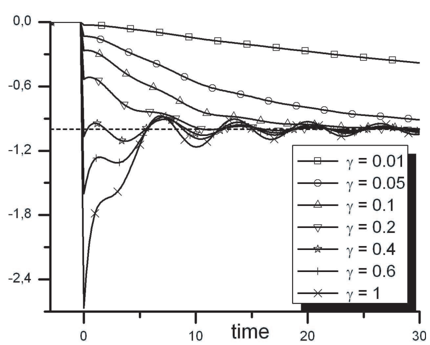


Figure 12. The transient response of the volume charge for different values of the conductivity.

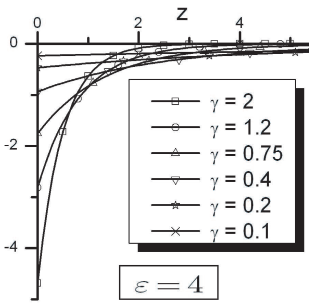


Figure 13. Dependence of the front impulse amplitude on the longitudinal coordinate.

So the transient response can be considered as a characteristic of the phenomena itself.

The transient responses of the surface charge (the response to incident field with longitudinal component being a Heaviside unit step function) for different values of the medium conductivity are plotted in Fig. 11. One can see that all the curves start from unit level. This is due to unit amplitude of the incident wave front. After the impulse front has passed the surface charge stabilizes on some level corresponding to the final steady-state static field.

The volume charge is a function of z and t but nevertheless it can be described with a single time response function $TR_{P'}(t)$ while z-dependence is determined by the propagation operator (15), which is common for all the mode amplitudes. From (22) and (16) it follows that

$$\begin{aligned} P'(z,t) &= -2\gamma\,\eta(t) * \Psi(t) * Z(z,t) * T(t) * \left(-c^{-1}\delta'(t) * e_i(0,t)\right) \\ &= Z(z,t) * TR_{P'}(t) * e_i(0,t) \qquad (31) \\ TR_{P'}(t) &= 2\gamma c^{-1}\,\Psi(t) * T(t) \end{aligned}$$

This function $TR_{P'}(t)$ that describes volume charge at the medium boundary for a step function excitation is shown in Fig. 12. Volume charges deeper in the medium may be calculated by applying the propagation operator to this function (31). The transient response starts from $-4\gamma\sqrt{\varepsilon}/(\sqrt{\varepsilon}+1)$ at $t = 0$ and tends to minus unit at the infinity. If γ is small enough ($\gamma < 0.15$) then $TR_{P'}(t)$ becomes non-oscillating function. For $\gamma \approx 0.4$ the function has no exponential part and oscillates from the very beginning around the level -1.

The wave front propagates in the medium with the speed $c = c_0/\sqrt{\varepsilon}$ and its level (the step) decays with depth as is shown in

Fig. 13. It can be seen that in the medium with smaller conductivity the charge wave has smaller front amplitude that decays for a longer distance and vice versa for higher conductivity — the front amplitude is higher but it decays at much shorter distance off the surface.

The dependence of the front amplitude on the longitudinal coordinate can be described as $-4\gamma\sqrt{\varepsilon}/(\sqrt{\varepsilon}+1)\ \exp(-\gamma z)$.

6. CONCLUSION

The transient process of charge transportation in a closed waveguide by a pulse E-wave has been considered. A closed-form solution has been obtained based on transport operators for the waveguide modes. The obtained results have been validated by direct FDTD calculations.

Among others it was shown that a surface wave is excited at the conductive medium boundary due to UWB signal. Relaxation time of the surface wave is much greater than that of the conducting medium that leads to long transients at the surface.

We analyzed the behavior of the charge wave for a unipolar incident pulse and for a bipolar one, and finally introduce some excitation independent impulse characteristics that can be convolved with any incident waveform.

The same consideration can be applied not only to the waveguide problem but also to the problem of a plane E-wave oblique incidence onto a lossy half-space.

Among possible applications in science and technology where free (conductivity) charges need to be accounted and the considered effect may be important we should mention studying effects of ultrashort electromagnetic pulses on biological tissues, semiconductor structures, processing metal surface with laser pulses, improving alloys in metallurgy with electromagnetic pulses [21].

APPENDIX A.

This appendix details derivation of the diffraction operators (19).

Using (13) the following relations can be found for incident, reflected, and transmitted waves

$$c_0\partial_z I_i = -\sqrt{s^2+\lambda_0^2}\, I_i(s), \tag{A1}$$

$$c_0\partial_z I_r = +\sqrt{s^2+\lambda_0^2}\, I_r(s), \tag{A2}$$

$$\frac{c_0}{\sqrt{\varepsilon}}\partial_z I_t = -\sqrt{s^2+2\gamma s+\lambda^2}\, I_t(s) \tag{A3}$$

$\lambda_0 = c_0\kappa$, $\lambda = c_0\kappa/\sqrt{\varepsilon}$, $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in free space.

Substituting expressions (A1), (A2) and (A3) into the boundary conditions (18) one can obtain the transmission operator in Laplace domain

$$T(s) = -2\left(\frac{s}{\sqrt{s^2+\lambda_0^2}\sqrt{\varepsilon}\,(s+2\gamma)} + \frac{1}{\sqrt{s^2+2\gamma s+\lambda^2}}\right)^{-1} \tag{A4}$$

Now this expression should be converted into TD. To this aim we transform it to the following form

$$T(s) = \frac{-2s\left(s^2+2\gamma s+\lambda^2\right)\left(s^2+\lambda_0^2\right)(s+2\gamma)^2}{s^2(s^2+2\gamma s+\lambda^2)-\varepsilon\left(s^2+\lambda_0^2\right)(s+2\gamma)^2} \times\left(\frac{\sqrt{\varepsilon}}{(s+2\gamma)\sqrt{s^2+\lambda_0^2}} - \frac{\varepsilon}{s\sqrt{s^2+2\gamma s+\lambda^2}}\right) \tag{A5}$$

The first factor in formula (A5) can be presented in the form of partial fractions

$$\frac{s\left(s^2+2\gamma s+\lambda^2\right)\left(s^2+\lambda_0^2\right)(s+2\gamma)^2}{s^2\left(s^2+2\gamma s+\lambda^2\right)-\varepsilon\left(s^2+\lambda_0^2\right)(s+2\gamma)^2} = \frac{1}{1-\varepsilon}\left(s^3+\alpha(2-\varepsilon)s^2+\alpha^2 s+\alpha^3\varepsilon+\sum_{i=1}^{4}\frac{A_i}{s-\beta_i}\right) \tag{A6}$$

where $\alpha = 2\gamma/(1-\varepsilon)$, β_i are the roots of the following polynomial in s

$$(1-\varepsilon)s^4+2\gamma(1-2\varepsilon)s^3+\left(\lambda^2-\varepsilon\lambda_0^2-\varepsilon(2\gamma)^2\right)s^2-4\gamma\varepsilon\lambda_0^2 s-(2\gamma)^2\varepsilon\lambda_0^2 = 0$$

The coefficients A_i can be obtained as a solution to the following SLAE

$$\mathbf{M}\cdot\mathbf{a} = \mathbf{b}$$

$$\mathbf{M}_{i,1} = 1, \quad \mathbf{M}_{i,2} = \sum_{\substack{j=1\\ j\neq i}}^{4}\beta_j, \quad \mathbf{M}_{i,4} = \prod_{\substack{j=1\\ j\neq i}}^{4}\beta_j$$

$$\mathbf{M}_{i,3} = \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{4}\sum_{\substack{j=1\\ j\neq i}}^{4}\beta_j\beta_k - \sum_{\substack{j=1\\ j\neq i}}^{4}\beta_j^2, \qquad i = 1,\ldots,4$$

$$\mathbf{a} = \begin{pmatrix} A_1\\ A_2\\ A_3\\ A_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \varepsilon^2\alpha^4+\varepsilon\lambda^4\\ -2\gamma\varepsilon\left(\varepsilon\alpha^4+2\lambda^4\right)\\ \varepsilon\lambda^2\left(\alpha^4\varepsilon(\varepsilon+1)+4\gamma^2\lambda^2\right)\\ -2\gamma\alpha^4\varepsilon^3\lambda^2 \end{pmatrix}$$

The partial fractions in (A6) correspond to exponential functions in TD. The second factor in formula (A5) can also be directly transformed into TD:

$$\frac{1}{s\sqrt{s^2+2\gamma s+\lambda^2}} \quad \Rightarrow \quad \eta(t)*e^{-\gamma t}J_0(\tilde{\omega}t) \tag{A7}$$

$$\frac{1}{(s+2\gamma)\sqrt{s^2+\lambda_0^2}} \quad \Rightarrow \quad e^{-2\gamma t}*J_0(\lambda_0 t) \tag{A8}$$

Now gathering (A6), (A7) and (A8) all together the complete TD presentation of the transmission operator can be found. It can be separated into the singular and regular parts:

$$T = T_s + T_{reg} \tag{A9}$$

The singular part of the operator contains Dirac function $\delta(t)$ and its first derivative $\delta'(t)$, these functions describe the prompt response:

$$\begin{gathered} T_s = T_{s1}\delta'(t) + T_{s0}\delta(t) \\ T_{s0} = \gamma\frac{2\varepsilon\left(1+\varepsilon-2/\sqrt{\varepsilon}\right)}{(1-\varepsilon)^2}, \quad T_{s1} = -\frac{2\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}}. \end{gathered}, \tag{A10}$$

The regular part of the operator contains regular functions and describes the late-time (resonance) response.

$$\begin{aligned} T_{reg}(t) = {} & \tfrac{\sqrt{\varepsilon}}{1-\varepsilon}\left(\lambda_0^2\tilde{J}_1\left(\lambda_0 t\right)+\left(\varepsilon\alpha^2-\lambda_0^2\right)J_0\left(\lambda_0 t\right)+\lambda_0 J_1\left(\lambda_0 t\right)\right) \\ & +\tfrac{\sqrt{\varepsilon}}{1-\varepsilon}F_1\left(t\right)*J_0\left(\lambda_0 t\right)-\tfrac{\varepsilon}{1-\varepsilon}F_2\left(t\right)*e^{-\gamma t}J_0\left(\tilde{\omega}t\right) \\ & -\tfrac{\varepsilon}{1-\varepsilon}\left(\left[\tfrac{4\gamma^2(1+\varepsilon)}{(1-\varepsilon)^2}-\lambda^2\right]J_0\left(\tilde{\omega}t\right)-\tfrac{2\gamma\tilde{\omega}}{(1-\varepsilon)}J_1\left(\tilde{\omega}t\right)+\tilde{\omega}^2\tilde{J}_1\left(\tilde{\omega}t\right)\right)e^{-\gamma t} \end{aligned} \tag{A11}$$

where we have introduced the following functions

$$\begin{aligned} F_1(t) &= \sum_{i=1}^{4}\frac{A_i\left(e^{\beta_i t}-e^{-2\gamma t}\right)}{\beta_i+2\gamma}+\varepsilon^2\alpha^3 e^{-2\gamma t} \\ F_2(t) &= \sum_{i=1}^{4}\frac{A_i\left(e^{\beta_i t}-\eta(t)\right)}{\beta_i}-\varepsilon\alpha^3\eta(t) \\ \tilde{J}_1(x) &= J_1(x)/x \end{aligned}$$

It is important to notice that functions $F_1(t)$ and $F_2(t)$ are real-valued ones.

Now that we have the transmission operator we can express the reflection operator using boundary conditions (19)

$$R(s) = \sqrt{s^2+\lambda_0^2}+\frac{s}{s+2\gamma}\frac{T(s)}{\sqrt{\varepsilon}},$$

It is convenient to split it into the singular and regular parts:

$$R = R_s + R_{reg} \tag{A12}$$

where the regular part is

$$\begin{aligned}R_{reg}(t) =& \tfrac{2\sqrt{\varepsilon}}{1-\varepsilon}\left(2\gamma^2\tfrac{\varepsilon(1+\varepsilon)}{(1-\varepsilon)^2} - \lambda^2\right)e^{-\gamma t}J_0\left(\tilde{\omega}t\right)\\ &-\lambda_0^2\tfrac{1+\varepsilon}{1-\varepsilon}\tilde{J}_1\left(\lambda_0 t\right) - \tfrac{4\lambda_0\varepsilon\gamma}{(1-\varepsilon)^2}J_1\left(\lambda_0 t\right) - 2\left(\tfrac{4\gamma^2\varepsilon^2}{(1-\varepsilon)^3} - \tfrac{\lambda_0^2}{(1-\varepsilon)}\right)J_0\left(\lambda_0 t\right)\\ &+\tfrac{2\left(\lambda^2-\gamma^2\right)\sqrt{\varepsilon}}{1-\varepsilon}e^{-\gamma t}\tilde{J}_1\left(\tilde{\omega}t\right) - \tfrac{4\gamma\varepsilon\tilde{\omega}\sqrt{\varepsilon}}{(1-\varepsilon)^2}e^{-\gamma t}J_1\left(\tilde{\omega}t\right)\\ &-\tfrac{2}{1-\varepsilon}F_3\left(t\right) * J_0\left(\lambda_0 t\right) + \tfrac{2\sqrt{\varepsilon}}{1-\varepsilon}F_4\left(t\right) * e^{-\gamma t}J_0\left(\tilde{\omega}t\right)\end{aligned}$$

where we have introduced the following functions (they are also real ones)

$$\begin{aligned}F_3(t) =& \sum_{i=1}^{4}\frac{(\beta_i + A_i)(e^{\beta_i t} - e^{-2\gamma t}) - (A_i - 2\gamma)(\beta_i + 2\gamma)\,te^{-2\gamma t}}{(\beta_i + 2\gamma)^2}\\ &-\alpha^3\varepsilon^2(\varepsilon - 2\gamma t)e^{-2\gamma t}\end{aligned}$$

$$F_4(t) = \sum_{i=1}^{4}\frac{2\gamma\,(\beta_i + A_i)(e^{\beta_i t} - \eta(t)) + \beta_i(A_i - 2\gamma)\,e^{-2\gamma t}}{2\gamma\beta_i(\beta_i + 2\gamma)} - \varepsilon^2\alpha^3e^{-2\gamma t}$$

The singular part of the reflection operator is

$$R_s(t) = R_{s1}\delta'(t) + R_{s0}\delta(t) \quad R_{s1} = \tfrac{\sqrt{\varepsilon}-1}{\sqrt{\varepsilon}+1}, \quad R_{s0} = \gamma\tfrac{\sqrt{\varepsilon}\left(3\sqrt{\varepsilon}+1\right)}{(\varepsilon-1)\left(\sqrt{\varepsilon}+1\right)}\ ,$$

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