APPLICATION OF QUASI MONTE CARLO INTEGRATION TECHNIQUE IN EM SCATTERING FROM FINITE CYLINDERS

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Abstract—In this work, a Quasi Monte Carlo (QMC) Integration Technique using Halton Sequence is proposed for the Electric Field Integral Equation (EFIE) in the Method of Moments (MoM) solution for scattering problems. It is found that the Halton Sequence used in QMC integration scheme is capable of handling the singularity issue in the EFIE automatically and at the same time provides solution to the scattering problems very easily. Finally the proposed technique is applied to solve the scattering problem from a finite cylinder employing the entire domain basis function expansions. The results obtained show a good agreement between the proposed and conventional technique.

1. INTRODUCTION

Multidimensional numerical quadratures are of great importance in many practical areas, ranging from radiation/scattering problems in computational electromagnetics to atomic physics. The EFIE in solution of MoM for scattering problems involves multidimensional integrals especially when the Galerkin's technique for solution is employed. It is well known that the (N)-Dimensional scattering problem using Galerkin's technique involves solution of a (2N)-Dimensional Integral Equation. Gaussian Quadrature methods, on one hand, yield precise results with relatively few integrand evaluations, but they are not too robust and work best for very smooth functions and the time complexity in numerical quadrature methods increases as the dimension of the problem increases. Monte Carlo methods [1–3], on the other hand, impose few requirements on the integrand, but are known to converge slowly. It is an integration approach that is

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well suited for irregular or singular integrands and requires no analytic knowledge about the form of the integrand. The conventional Monte Carlo integration (MCI) method is independent of the dimension of the integral, and that is why MCI is the only practical method for many high-dimensional problems.

QMCI methods are based on the idea that random Monte Carlo techniques can often be improved by replacing the underlying source of random numbers with a more uniformly distributed deterministic sequence. The fundamental feature underlying all QMCIs, however, is the use of a quasi-random number (QRN) sequences in place of the usual pseudorandom numbers which often improves the convergence of the numerical integration.

One of the key issues in the solution of the EFIE using Galerkin's technique is the singularity appearing the Green's function kernel of the Integral Equation. The type of the singularity is weak in nature. Several techniques [4–6] have been used in the past to deal with the issue of singularity in order to solve the problem. The conventional MCI takes care of the singularity aspect without employing any analytical techniques such as Singularity subtraction/removal, polar co-ordinate transformation, etc. and implements the idea just by restricting the random points to fall in the singular region by including a simple statement in the program code used for the simulation purpose [7, 8]. However, the proposed Halton sequence in QMCI takes care of the singularity issue automatically without even modification or inclusion of any condition in the program code and provides solution to the problem more accurately and faster than the conventional MCI with randomly generated point sequences. It is also proved that the use of Halton sequences due to their inherent property automatically makes the kernel non-singular. The mathematical concept of the generation of Halton sequences is elaborated in [9, 10].

2. FORMULATION OF SCATTERING PROBLEM

The EFIE is formulated for the case of finite open-ended cylinders and solved using Galerkin's MoM solution implementing QMCI.

The electric field is given by $\mathbf{E} = \frac{1}{j4\pi\varepsilon_0\omega}(\nabla(\nabla \bullet \mathbf{A}) + k^2\mathbf{A})$ where $\mathbf{A} = \int_{R'} \mathbf{J}(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')d\mathbf{r}'$, where the integration is over the entire source region R'. $G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ is the free space Green's function. Consider the TM scattering of a plane wave incident upon an open-ended cylinder of length l and radius ρ as shown in the Figure 1. The axis of the cylinder is along the z-direction and the wave is traveling along the x-direction.
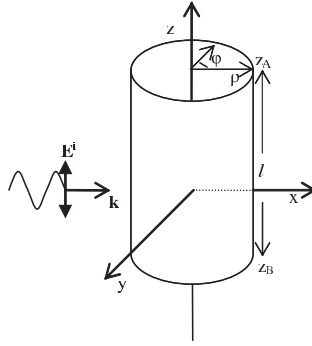
Figure 1. A finite PEC cylinder illuminated by a TM plane wave.

As is clear from the geometry, the induced current on the surface will have two components; $J_\phi(\phi', z')$ and $J_z(\phi', z')$. As a result, $\mathbf{A}(\phi, z) = \boldsymbol{\varphi} A_\phi(\phi, z) + \mathbf{z} A_z(\phi, z)$ where

$$A_\phi(\phi, z) = \int\limits_{z_A}^{z_B} \int\limits_{\phi'=0}^{\phi'=2\pi} J_\phi(\phi' z') G(\phi', z'; \phi, z) \rho d\phi' dz' \tag{1}$$

and

$$A_z(\phi, z) = \int\limits_{z_A}^{z_B} \int\limits_{\phi'=0}^{\phi'=2\pi} J_z(\phi' z') G(\phi', z'; \phi, z) \rho d\phi' dz' \tag{2}$$

The free space Green's function is given by

$$G(\phi', z'; \phi, z) = \frac{e^{-jk\sqrt{2a^2(1-\cos(\phi-\phi'))+(z-z')^2}}}{\sqrt{2a^2(1 - \cos(\phi - \phi')) + (z - z')^2}} \tag{3}$$

The EFIE $-\mathbf{E}^{inc} = \mathbf{E}^s$ on the surface of the cylinder takes the form

$$-j4\pi\varepsilon_0\omega E^{inc}(\phi, z) = (\nabla(\nabla \bullet \mathbf{A}(\phi, z)) + k^2\mathbf{A}(\phi, z)) \tag{4}$$

where $\nabla \equiv \boldsymbol{\varphi}\dfrac{1}{\rho}\dfrac{\partial}{\partial\phi} + \mathbf{z}\dfrac{\partial}{\partial z}$. Explicitly for the TM wave propagating along the x axis the equation is

$$
\begin{aligned}
-\mathbf{z}j4\pi\varepsilon_0\omega e^{jk\rho\cos\phi} &= \boldsymbol{\varphi}\left[\left(\frac{1}{\rho^2}\frac{\partial^2 A_\phi}{\partial\phi^2} + k^2 A_\phi\right) + \frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial z}\right] \\
&\quad + \mathbf{z}\left[\frac{1}{\rho}\frac{\partial^2 A_\phi}{\partial\phi\partial z} + \left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right)\right]
\end{aligned} \tag{5}
$$

Taking the z component of the equation,

$$-j4\pi\varepsilon_0\omega e^{jk\rho\cos\phi} = \frac{1}{\rho}\frac{\partial^2 A_\phi}{\partial\phi\partial z} + \left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right) \tag{6}$$

where A_ϕ and A_z are given by (1) and (2) respectively.

2.1. Basis Function Expansion and MoM Solution

The unknown current components are expanded in terms of known basis functions

$$J_\phi(\phi',z') = \alpha_{\phi 1}f_{\phi 1}(\phi',z') + \ldots + \alpha_{\phi M_\phi}f_{\phi M_\phi}(\phi',z'); \ \ 0\le\phi'\le 2\pi \text{ and}$$
$$J_z(\phi',z') = \alpha_{z1}f_{z1}(\phi',z') + \ldots + \alpha_{zMz}f_{zM_z}(\phi',z'); \ \ z_A\le z'\le z_B. \tag{7}$$

Here, $|z_A| = |z_B|$. The total number of unknowns in the expansion is thus $M_\phi\cdot M_z$.

Substituting the expansion of the current into (6) and applying Galerkin's approach, the equation reduces into the matrix equation

$$-j4\pi\varepsilon_0\omega \begin{bmatrix} < e^{jk\rho\cos\phi}\bullet f_{\phi 1}(\phi,z) > \\ . \\ . \\ . \\ < e^{jk\rho\cos\phi}\bullet f_{\phi M_\phi}(\phi,z) > \\ < e^{jk\rho\cos\phi}\bullet f_{z1}(\phi,z) > \\ . \\ . \\ . \\ < e^{jk\rho\cos\phi}\bullet f_{zM_z}(\phi,z) > \end{bmatrix} = \begin{bmatrix} A_{\phi\phi} & A_{z\phi} \\ A_{\phi z} & A_{zz} \end{bmatrix}\begin{bmatrix} \alpha_{\phi 1} \\ . \\ . \\ \alpha_{\phi M\phi} \\ \alpha_{z1} \\ . \\ . \\ . \\ \alpha_{zMz} \end{bmatrix} \tag{8}$$

The unknowns can be obtained by matrix inversion. The entries of the matrix are given by the four dimensional integrals

$$(A_{\phi\phi})_{mn} = \int_{\phi'=0}^{\phi'=2\pi}\int_{z'=z_A}^{z'=z_B}\int_{\phi=0}^{\phi=2\pi}\int_{z=z_A}^{z=z_B} f_{\phi n}(\phi',z')G(\phi',z';\phi,z)$$
$$\left(\frac{1}{\rho}\frac{\partial^2 f_{\phi m}(\phi,z)}{\partial\phi\partial z}\right)\rho^2 d\phi'dz'd\phi dz \tag{9a}$$

$$(A_{z\phi})_{mn} = \int_{\phi'=0}^{\phi'=2\pi}\int_{z'=z_A}^{z'=z_B}\int_{\phi=0}^{\phi=2\pi}\int_{z=z_A}^{z=z_B} f_{zn}(\phi',z')G(\phi',z';\phi,z)$$
$$\left(\frac{\partial^2 f_{\phi m}(\phi,z)}{\partial z^2} + k^2 f_{\phi m}(\phi,z)\right)\rho^2 d\phi'dz'd\phi dz \tag{9b}$$

$$(A_{\phi z})_{mn} = \int\limits_{\phi'=0}^{\phi'=2\pi} \int\limits_{z'=z_A}^{z'=z_B} \int\limits_{\phi=0}^{\phi=2\pi} \int\limits_{z=z_A}^{z=z_B} f_{\phi n}(\phi', z')G(\phi', z'; \phi, z)$$

$$\left(\frac{1}{\rho}\frac{\partial^2 f_{zm}(\phi, z)}{\partial\phi\partial z}\right)\rho^2 d\phi' dz' d\phi dz \qquad (9c)$$

$$(A_{zz})_{mn} = \int\limits_{\phi'=0}^{\phi'=2\pi} \int\limits_{z'=z_A}^{z'=z_B} \int\limits_{\phi=0}^{\phi=2\pi} \int\limits_{z=z_A}^{z=z_B} f_{zn}(\phi', z')G(\phi', z'; \phi, z)$$

$$\left(\frac{\partial^2 f_{zm}(\phi, z)}{\partial z^2} + k^2 f_{zm}(\phi, z)\right)\rho^2 d\phi' dz' d\phi dz \qquad (9d)$$

From the geometry of the problem as shown in Figure 1, it is clear that

a) J_ϕ is antisymmetric with respect to the variables ϕ' and z', i.e., $J_\phi(-\phi', z') = -J_\phi(\phi', z')$ and $J_\phi(\phi', -z') = -J_\phi(\phi', z')$. Also, J_ϕ will show diverging effect at the edges $z' = z_A$ and $z' = z_B$.

b) J_z is symmetric with respect to the variables ϕ' and z', i.e., $J_z(-\phi', z') = J_z(\phi', z')$ and $J_z(\phi', -z') = J_z(\phi', z')$. Also, J_ϕ will show diverging effect at the edges $z' = z_A$ and $z' = z_B$. Also, J_z is zero at the edges $z' = z_A$ and $z' = z_B$, i.e., $J_z(\phi', z_A) = J_z(\phi', z_B) = 0$.

Keeping the above facts in mind, the basis functions taken are:

$$J_\phi(\phi', z') = \sum_{n_{\phi\phi}=1 \; odd}^{M_{\phi\phi}} \sum_{n_{z\phi}=3}^{M_{z\phi}} \alpha_{n_{\phi\phi}n_{z\phi}} \sin(n_{\phi\phi}\phi')\frac{z'^{n_{z\phi}}}{\sqrt{z_A^2 - z'^2}}$$

$$0 \leq \phi' \leq 2\pi, \; z_A \leq z' \leq z_B \qquad (10a)$$

and

$$J_z(\phi', z') = \sum_{n_{\phi z}=1}^{M_{\phi z}} \sum_{even \; n_{zz}=2}^{M_{zz}} \alpha_{n_{\phi z}n_{zz}} \phi'^{n_{\phi z}}(z'^{n_{zz}} - z_A^{n_{zz}})$$

$$0 \leq \phi' \leq 2\pi, \; z_A \leq z' \leq z_B. \qquad (10b)$$

Here, $M_{\phi\phi} + M_{z\phi} = M_\phi$ and $M_{\phi z} + M_{zz} = M_z$.

2.2. QMCI Technique Implementation

It can be seen from (9) and the expression for Green's function in (3) that the matrix entries are four dimensional integrals in variables

ϕ', z', ϕ and z due to the application of Galerkin's technique for solution. These contain line singularities at all values of variables where $z = z'$ and $\phi = \phi'$ for the region $0 \leq \phi'$, $\phi \leq 2\pi$ and $z_A \leq z'$, $z \leq z_B$. There can be added point singularities if the derivatives of the basis function contain singularities. The technique that has been used in integrating functions in (9) is the QMCI. Here, Halton sequences with four different bases 2, 3, 5 and 7 are generated for the four variables ϕ', z', ϕ and z respectively. Since Halton sequence points are self-avoiding and no two Halton sequences are same, the QMCI tackles the singularity problem effectively for integrals in (9). This can be further elaborated with the help of Table 1 [10]. As evident, for a three dimensional Halton point, the x, y and z-coordinates are generated using three different bases such as base 2, base 3 and base 5 respectively. Therefore, if two separate quasi-random sequences spread over the domain are chosen for the source and the field points for each coordinate, then they are never equal to each other. Therefore the condition for the integrand to become singular will not arise. On the other hand, in Faure sequence same base is used for all the dimensions. Therefore even if it has the advantage that it can be generated recursively using one dimension sequence for a multidimensional problem; some points that will be generated for other dimensions will be the same. Thus the same sampling points will be obtained for the source and the field coordinates leading to the singularity condition in the integrand. Besides, the numerical effort required to generate Halton sequences for different dimensions using different bases is less than Faure sequences for different dimensions employing scrambled sequences. Moreover, there is no strong evidence that when the dimension of the problem is moderate (e.g., $d \leq 15$) it makes a great deal of difference whether one uses Halton, Faure or Sobol sequence. The suitability of the proposed technique is demonstrated for charge density problem for several examples in [10].

3. TEST CASES

As an example, scattering of a TM plane wave of frequency $10\,\text{GHz}$ by a finite open metallic cylinder of length 1λ and circumference 1λ is considered. In this case, in the proposed entire domain expansion of the current distribution, $M_{\phi\phi} = M_{\phi z} = 1$, $M_{z\phi} = 3$ and $M_{zz} = 2$. It is observed that increasing the number of terms in the expansion does not change the result. For this case, the magnitude of the normalized current distribution along the length of the cylinder for $\phi = \pi/2$ is shown in Figure 2. It is compared with the result obtained by using conventional sub domain method. A good agreement between the two is observed.
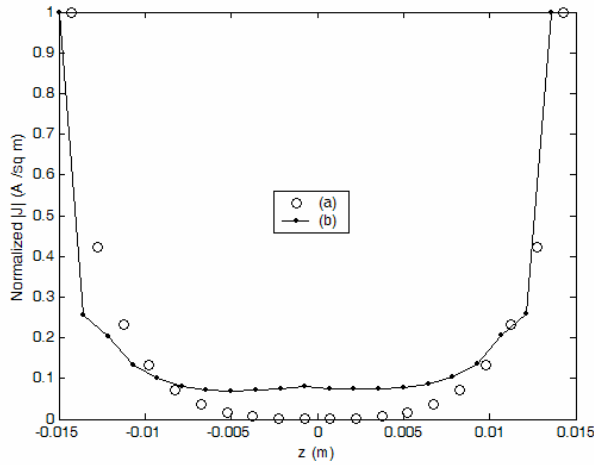
Figure 2. Current distribution along the length of the cylinder for $\phi = \pi/2$. (a) Proposed entire domain. (b) Sub domain.
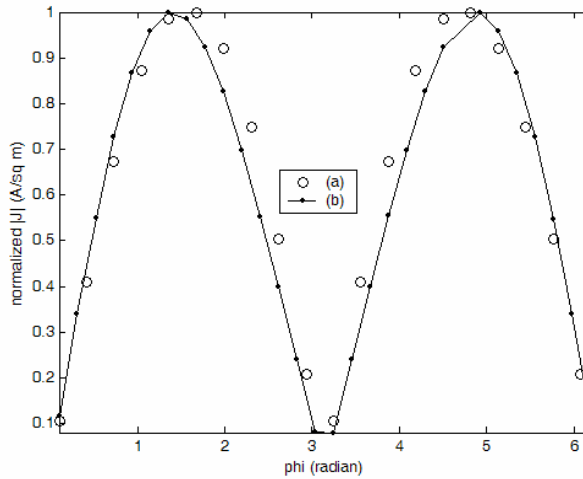

Figure 3. Current distribution along the circumference of the cylinder at the edge. (a) Proposed entire domain. (b) Sub domain.

Next, the magnitude of the normalized current distribution along the circumference of the cylinder at the edge is shown in Figure 3. It is also compared with the result obtained by using conventional sub
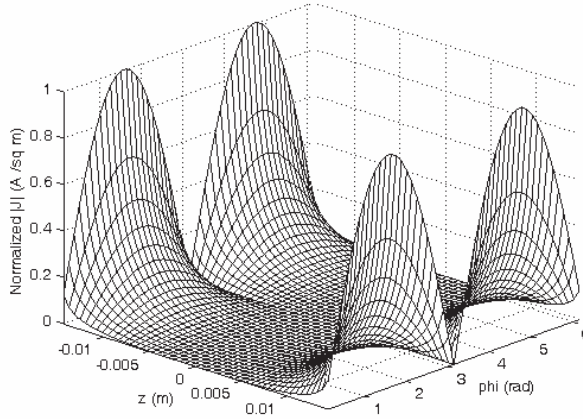
Figure 4. Normalized surface current density variation with z and ϕ for $l = \lambda$, circumference $= \lambda$.
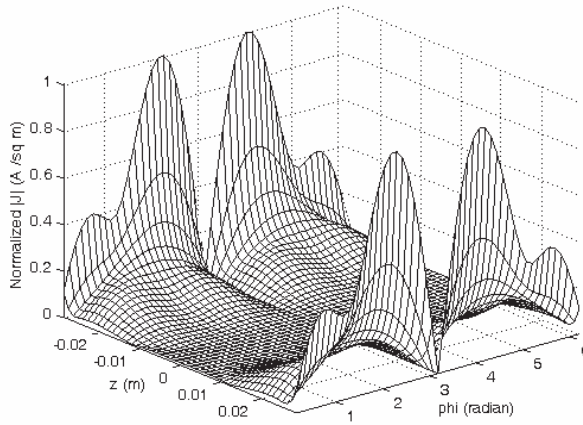

Figure 5. Normalized surface current density variation with z and ϕ for $l = 2\lambda$, circumference $= 2\lambda$.

domain method. Again a good agreement between the two is observed. Two-dimensional surface views depicting ϕ and z variation of the normalized current distributions for the two cases $l = \lambda$, circumference $= \lambda$, and $l = 2\lambda$, circumference $= 2\lambda$ are shown in Figure 4 and Figure 5 respectively. Finally, the RCS for the above two cases have been plotted in Figure 6 and Figure 7.
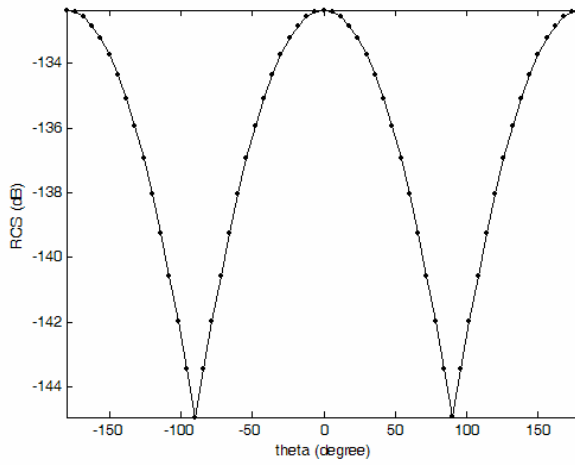
Figure 6. RCS for $l = \lambda$, circumference $= \lambda$.
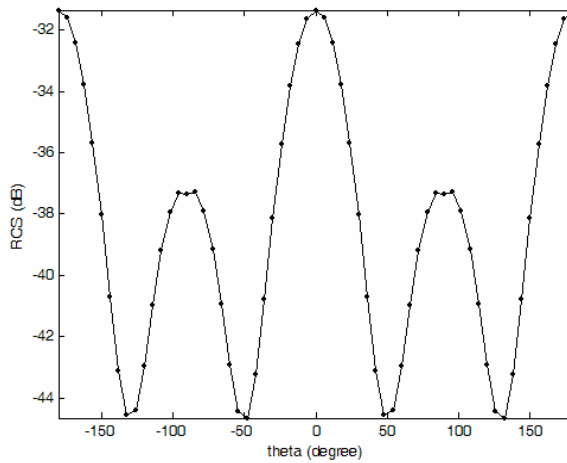
Figure 7. RCS for $l = 2\lambda$, circumference $= 2\lambda$.

4. CONCLUSION

The QMCI technique using Halton sequence is proposed in the MoM solution of the EFIE. As an example, TM scattering of a plane wave by a finite PEC cylinder is investigated. It is found that the

proposed technique not only solves the scattering problem efficiently but also removes the singularity problem appearing in the kernel of the integrand due to the inherent property of the Halton sequence. In addition to this, exact form of the kernel is retained without any approximation or analytical effort.

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