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A UNIFIED FDTD APPROACH FOR ELECTROMAGNETIC ANALYSIS OF DISPERSIVE OBJECTS

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Abstract—In order to obtain a unified approach for the Finite-Difference Time-Domain (FDTD) analysis of dispersive media described by a variety of models, the coordinate stretched Maxwell's curl equation in time domain is firstly deduced. Then the FDTD update formulas combined with the semi-analytical recursive convolution (SARC) in Digital Signal Process (DSP) technique for general dispersive media are obtained. In this method, the flexibility of FDTD in dealing with complicated object is retained; the advantages of absolute stability, high accuracy, less storage and high effectiveness of SARC in treating the linear system problem are introduced, and a more unified update formulation for a variety of dispersion media model including Convolution Perfectly Matched Layers (CPML) absorbing boundary is possessed. Therefore it can be applied to analysis of general dispersive media provided that the poles and corresponding residues in dispersive medium model of interest are given. Finally, three typical kinds of dispersive model, i.e., Debye, Drude and Lorentz medium are tested to demonstrate the feasibility of presented approach.

1. INTRODUCTION

The vast majority of medium in nature is frequency dependent in a certain extent when it interacts with electromagnetic waves (or light), such as water, soil, human tissue, plasma and metals. For different media, its dispersive property in frequency domain can be characterized by different model, such as Debye model, Drude model, Lorentz model. When FDTD method has been applied to the analysis

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of electromagnetic properties of this media, the constitutive relations of dispersive media must be specially treated in which the convolution of electric field with causal susceptibility in time domain usually requires large storage and time consuming computation. Nowadays several frequency-dependent FDTD methods have been developed, such as Recursive Convolution (RC), Auxiliary Differential Equation (ADE), and Z transform techniques. However, these approaches need deduce the corresponding convolution formulation for different dispersive models in order to obtain the corresponding FDTD update equation. This makes the lack of common procedures, and causes inconvenience to the application [1–5].

On the other hand, it is known that the simulation of FDTD is only restricted in a finite domain. The absorbing boundary condition must be given on the truncated boundary of computation domain for an open region problems. Therefore, the study of absorbing boundary has been one of the focus problems, since FDTD scheme was proposed [6–11]. At present, the Convolutionally perfectly Matched Layer (CPML) is one of the most commonly used absorbing boundary because its formulation is more accurate than the classical UPML, more efficient, and better suited for the application of domains with general materials [12–15]. However, due to the field updates related with dispersive media in computation region, the different CPML field updates are also needed for different kinds of dispersive model and thus the poor versatility is possessed.

In this paper, a unified time-domain form for different dispersive model is firstly given in response to these issues, then the stretched-coordinate Maxwell's curl equations in time-domain are deduced, in which the stretched-coordinate variables are Complex Frequency-Shifted (CFS) tensor. Combining with the Semi-Analytical Recursive Convolution (SARC) algorithm [16, 17] in DSP techniques, a unified FDTD algorithm for analysis of electromagnetic characteristic of dispersive objects is proposed, namely SARC-FDTD algorithm. In this scheme, the absolute stability, high accuracy, less storage and high effectiveness are retained, and a unified update formulation for general dispersive media, i.e., Debye, Drude, Lorentz and hybrids model is possessed. The SARC FDTD can therefore be applied to the analysis of general dispersive media provided that the poles and corresponding residues in dispersive medium model of interest are given. It is for this reason also, the FDTD update formulation in CPML region also has the unified form no matter what kind of dispersive media is truncated by CPML. Finally, the validity of proposed SARC FDTD approach is tested with three typical kinds of dispersive model, i.e., Debye, Drude and Lorentz medium.

2. A TIME-DOMAIN UNIFIED MODEL FOR DISPERSIVE MEDIA

For linear dispersive media, some model such as Debye, Drude, Lorentz, etc. are frequently used. The electrical susceptibility function of three typical kinds of models in frequency domain can be expressed, respectively, as the following form:

(1) Debye model

$$\chi(\omega) = \sum_{q=1}^{Q} \frac{\Delta\varepsilon_q}{1 + j\omega\tau_q} = \sum_{q=1}^{Q} \frac{\Delta\varepsilon_q/\tau_q}{j\omega + 1/\tau_q} \tag{1}$$

where $\Delta\varepsilon_q = \varepsilon_{s,q} - \varepsilon_{\infty,q}$, $\varepsilon_{s,q}$ is the static permittivity, $\varepsilon_{\infty,q}$ is relative permittivity at infinite frequency, and τ_q is the relaxation time constant.

(2) Lorentz model

$$\chi(\omega) = \sum_{q=1}^{Q} \frac{\Delta\varepsilon_q \omega_q^2}{\omega_q^2 + 2j\omega\delta_q - \omega^2} = \sum_{q=1}^{Q} \frac{\Delta\varepsilon_q \omega_q^2}{(j\omega)^2 + 2\delta_q (j\omega) + \omega_q^2} \tag{2}$$

where $\Delta\varepsilon_q = \varepsilon_{s,q} - \varepsilon_{\infty,q}$ as above, ω_q is the resonant angular frequency, δ_q is the damping constant.

(3) Drude model

$$\chi(\omega) = \sum_{q=1}^{Q} \frac{\omega_q^2}{j\omega v_{c,q} - \omega^2} = \sum_{q=1}^{Q} \frac{\omega_q^2/v_{c,q}}{j\omega} - \sum_{q=1}^{Q} \frac{\omega_q^2/v_{c,q}}{j\omega + v_{c,q}} \tag{3}$$

where ω_q is the angular plasma frequency, $v_{c,q}$ is the collision frequency.

Let $s = j\omega$, the time-domain expression corresponding to (1) can be obtained by Laplace transform as,

$$\chi(t) = \sum_{q=1}^{Q} H_q e^{-\alpha_q t} u(t) \tag{4}$$

where, $u(t)$ is the unit step function, $\alpha_q = 1/\tau_q$, $H_q = \Delta\varepsilon_q/\tau_q$ is poles and corresponding residues of (1), respectively. When poles are real in (2), it can be transformed into Debye model as (1). When poles are complex in (2), it will be a pair of complex conjugate poles. By taking Laplace transform, the corresponding time-domain expression of (2) can be obtained as

$$\chi(t) = \sum_{q=1}^{Q} H_q e^{-\alpha_q t} \sin(\beta_q t) u(t) \tag{5}$$

where $\alpha_q = \delta_q$, $\beta_q = \sqrt{\omega_q^2 - \delta_q^2}$, $H_q = \Delta\varepsilon_q \omega_q^2 / \beta_q$. Furthermore we may rewriting (5) in a complex exponential function form and then obtain

$$\chi(t) = \sum_{q=1}^{Q} \operatorname{Im}\left[G_q e^{-\gamma_q t} u(t)\right] \tag{6}$$

where Im is the imaginary operator, and $\gamma_q = \alpha_q - j\beta_q$, $G_q = H_q$. Comparing (4) with (6), it can be observed that the time-domain relation of Debye model as shown in (4) can also be rewritten as (6), if letting $\gamma_q = \alpha_q$, $G_q = H_q$.

For Drude model shown in (3), it can be considered as two additive Debye models, that the pole is $\gamma_1 = 0$, $\gamma_2 = -v_{c,2}$ and the corresponding residue is $G_1 = G_2 = \omega_1^2 / v_{c,1}$, respectively. It can also be expressed as the following form:

$$\chi_{Drude}(\omega) = \frac{\sigma_p}{j\omega\varepsilon_0} - \sum_{q=1}^{Q} \frac{\omega_q^2 / v_{c,q}}{s + v_{c,q}} \tag{7}$$

where $\sigma_p = \sum_{q=1}^{Q} \left(\omega_q^2 \varepsilon_0\right) / v_{c,q}$. The second term on the right-hand side of (7) can be considered as the Debye model of $\gamma_q = -v_{c,q}$ and $G_q = \omega_q^2 / v_{c,q}$, respectively. In a word, Drude model also can be expressed as the exponential function in time domain as shown in (6).

For the dispersion model expressed as a rational fraction, it can always be expressed as the sum of (1) and (2) by rewriting it in the fractional fraction form. So (6) can be viewed as time-domain unified form of susceptibility function for dispersion medium. It can be applied to the analysis of general dispersive media including Debye, Drude and Lorentz provided that the poles and corresponding residues in dispersive medium model of interest are designated.

3. STRETCHED COORDINATE MAXWELL'S EQUATION IN TIME-DOMAIN

In order to obtain a unified FDTD update formulation in the entire computation region including CPML, we first consider the stretched-coordinate Maxwell's curl equation.

For an isotropic, conductive, inhomogeneous, linear permittivity dispersive medium, the stretched-coordinate Maxwell's curl equations can be expressed as

$$\partial\vec{D}\Big/\partial t+\sigma\vec{E} = \tilde{\nabla}\times\vec{H} \tag{8}$$

$$-\partial\vec{B}\Big/\partial t-\sigma_m\vec{H} = \tilde{\nabla}\times\vec{E} \tag{9}$$

The operator $\tilde{\nabla}$ expressed in terms of the complex coordinate-stretching variables $s_w\,(w=x,y,z)$ is [3]

$$\tilde{\nabla}=\hat{x}\frac{1}{s_x}\frac{\partial}{\partial x}+\hat{y}\frac{1}{s_y}\frac{\partial}{\partial x}+\hat{z}\frac{1}{s_z}\frac{\partial}{\partial x} \tag{10}$$

For a general medium, by using Complex Frequency-shifted (CFS) tensor coefficient the complex coordinate-stretching variable is chosen as [11]

$$s_w=\kappa_w+\frac{\sigma_w}{\alpha_w+j\omega\varepsilon_0} \qquad (w=x,y,z) \tag{11}$$

For simplicity, suppose the permeability μ does not change with frequency, thus

$$\vec{B} = \mu\vec{H} \tag{12}$$

$$\vec{D}(\omega) = \varepsilon_0\left(\varepsilon_\infty+\chi(\omega)\right)\vec{E}(\omega) \tag{13}$$

where ε_0 is permittivity of free space, ε_∞ is the infinite frequency relative permittivity, and χ is the electric susceptibility, respectively. To convert (8) and (9) into time-domain, it is necessary to derive the time-domain expression of (13) and $1/s_w$ in (11).

By taking the Fourier transform, (13) becomes

$$\vec{D}(t)=\varepsilon_0\varepsilon_\infty\vec{E}(t)+\varepsilon_0\chi(t)*\vec{E}(t) \tag{14}$$

where $*$ denotes convolution operator. The time-domain expression corresponding to $1/s_w$ can be obtained by Fourier transform as in [11]

$$\bar{s}_w(t)=\frac{\delta(t)}{\kappa_w}-\frac{\sigma_w}{\varepsilon_0\kappa_w^2}e^{-\left(\frac{\sigma_w}{\varepsilon_0\kappa_w}+\frac{\alpha_w}{\varepsilon_0}\right)t}u(t) \tag{15}$$

Finally, substituting (14) and (15) into (8) and (9), we obtain (for the sake of brevity, we will only consider x component)

$$\varepsilon_0\varepsilon_\infty\frac{\partial E_x}{\partial t}+\varepsilon_0\chi*\frac{\partial E_x}{\partial t}+\sigma E_x=\frac{1}{\kappa_y}\frac{\partial H_z}{\partial y}-\frac{1}{\kappa_z}\frac{\partial H_y}{\partial z}+\zeta_z*\frac{\partial H_y}{\partial z}-\zeta_y*\frac{\partial H_z}{\partial y} \tag{16}$$

$$-\mu\frac{\partial H_x}{\partial t}-\sigma_m H_x=\frac{1}{\kappa_y}\frac{\partial E_z}{\partial y}-\frac{1}{\kappa_z}\frac{\partial E_y}{\partial z}+\zeta_z*\frac{\partial E_y}{\partial z}-\zeta_y*\frac{\partial E_z}{\partial y} \tag{17}$$

where

$$\zeta_w(t) = \frac{\sigma_w}{\varepsilon_0 \kappa_w^2} e^{-\left(\frac{\sigma_w}{\varepsilon_0 \kappa_w} + \frac{\alpha_w}{\varepsilon_0}\right)t} u(t) \tag{18}$$

These are the stretched-coordinate formulations of Maxwell's curl equation in time-domain for general dispersive media that can be applied to FDTD calculation in CPML region. The FDTD update formulations in computational region can be obtained by considering it as a special case of CPML region and letting $\kappa_w = 1$, $\sigma_w = 0$. On the other hand, it can be seen from (16) and (17) that the convolution of spatial derivative of E, H with $\zeta_w(t)$, and E with $\chi(t)$ must be effectively solved in order to obtain the stretched-coordinate FDTD update formulations for general dispersive media.

4. SARC FDTD UPDATE FORMULATION FOR GENERAL DISPERSIVE MEDIA

Next, the FDTD update formulations corresponding (16) and (17) are derived based on the semi-analytical recursive convolution (SARC) algorithm in Digital Signal Processing (DSP) technique.

4.1. SARC Algorithm

Based on knowledge of DSP technique, for a linear, time-invariant and causal system, suppose the impulse response can be written as

$$h(t) = \sum_{q=1}^{Q} H_q e^{-\alpha_q t} u(t) \tag{19}$$

The system response for arbitrary input signal $x(t)$ can then written as

$$y(t) = h(t) * x(t) = \int_0^t h(t-\tau) x(\tau)\, d\tau \tag{20}$$

Substituting (19) into (20), and introducing discrete time steps $t = n\Delta t$, $y^n = y(n\Delta t)$, $x^n = x(n\Delta t)$, the system response can be written in the discretized form as follows

$$y^n = \sum_{q=1}^{Q} y_q^n \tag{21}$$

where

$$y_q^n = \int_0^{n\Delta t} H_q e^{-\alpha_q (n\Delta t - \tau)} x(\tau)\, d\tau \tag{22}$$

Decomposing the integral over the time interval $[0, n\Delta t]$ into integral over $[0, (n-1)\Delta t]$ and $[(n-1)\Delta t, n\Delta t]$, we can obtain

$$\begin{aligned} y_q^n &= \int_0^{(n-1)\Delta t} H_q e^{-\alpha_q(n\Delta t-\tau)} x(\tau)\, d\tau + \int_{(n-1)\Delta t}^{n\Delta t} H_q e^{-\alpha_q(n\Delta t-\tau)} x(\tau)\, d\tau \\ &= e^{-\alpha_q\Delta t} y_q^{n-1} + I_q^n \end{aligned} \tag{23}$$

where

$$I_q^n = H_q \int_{(n-1)\Delta t}^{n\Delta t} e^{-\alpha_q(n\Delta t-\tau)} x(\tau)\, d\tau \tag{24}$$

Replacing the input signal $x(\tau)$ over the time interval $\tau \in ((n-1)\Delta t, n\Delta t)$ with an interpolation polynomial, namely

$$x(\tau) = \sum_{r=0}^{R} A_r \tau^r \tag{25}$$

And substituting (25) into (24), then the integral I_q^n can be analytically calculated as follows

$$I_q^n = \sum_{r=0}^{R} C_{r,q}\, x^{n-r} \tag{26}$$

where R stands for the order of approximation, x^{n-r} is the $(n-r)$th sample value of input signal, A_r and $C_{r,q}$ is related to the applied interpolation scheme. For example, we replace $x(\tau)$ over the time interval $\tau \in [(n-1)\Delta t, n\Delta t]$ with the Newton polynomial as follows:

$$x(\tau) = x_{ne} + \frac{x^n - x^{n-1}}{\Delta t}(\tau - (n-1)\Delta t) + \cdots \tag{27}$$

where $x_{ne} = x^{n-1}$. However, in practices, it may also take x^n, or $\left(x^n + x^{n-1}\right)/2$.

If we take the zero order approximation for (27), namely $R = 0$, then

$$x(\tau) \approx x_{ne} \tag{28}$$

Substituting it into (24), we obtain

$$I_q^n = c_{0,q} x_{ne} \tag{29}$$

where

$$c_{0,q} = \begin{cases} H_q \Delta t & \alpha_q = 0 \\ (H_q/\alpha_q)\left(1 - e^{-\alpha_a \Delta t}\right) & \alpha_q \neq 0 \end{cases} \tag{30}$$

If we take the first order approximation for (27), namely $R = 1$, then

$$x(\tau) \approx x^{n-1} + \frac{x^n - x^{n-1}}{\Delta t}(\tau - (n-1)\Delta t) \tag{31}$$

Substituting it into (24), and transforming integral variable τ into $\tau = \tau + (n-1)\Delta t$, we obtain

$$I_q^n = c_{0,q}x^n + c_{1,q}x^{n-1} \tag{32}$$

where

$$c_{0,q} = \begin{cases} (H_q\Delta t)/2 & \alpha_q = 0 \\ \dfrac{H_q}{\alpha_q}\left(1 - \left(1 - e^{-\alpha_q\Delta t}\right)/(\alpha_q\Delta t)\right) & \alpha_q \neq 0 \end{cases} \tag{33}$$

$$c_{1,q} = \begin{cases} (H_q\Delta t)/2 & \alpha_q = 0 \\ \dfrac{H_q}{\alpha_q}\left(\left(1 - e^{-\alpha_q\Delta t}\right)/(\alpha_q\Delta t) - e^{-\alpha_q\Delta t}\right) & \alpha_q \neq 0 \end{cases} \tag{34}$$

In practice, the approximation degree of I_q^n is dependent on the applied different interpolation schemes and order of approximation. However, the discrete system response in time-domain y^n as (19) always can be solved with semi-analytical recursive form no matter what kind of interpolation scheme is applied.

4.2. SARC FDTD Update Formulations for General Dispersive Media

As mentioned earlier, the electric susceptibility function χ in time-domain can always be expressed in the form of (6) for the typical dispersive model including Debye, Drude, Lorentz and etc. It can be seen that the form of $\chi(t)$ and $\zeta_w(t)$ in (6) are similar to (19) which is the impulse response of the linear, time-invariant and causal system. Therefore the convolution in (16) and (17) can be solved by SARC algorithm. For generality, the SARC FDTD update formulations in CPML region are considered.

Let $\psi_{w,v}^n = \zeta_w(t) * \frac{\partial}{\partial w}F_v(t)\big|_{t=n\Delta t}$ ($F = H$ or E; $w, v = y$ or z, and $w \neq v$). Replacing $F_v(\tau)$ over the time interval $\tau \in [n\Delta t, (n+1)\Delta t]$ with the zero order Newton polynomial, we obtain

$$\frac{\partial}{\partial w}F_v(\tau) = \frac{\partial}{\partial w}F_v^n \tag{35}$$

Making use of (18), we have $\alpha_1 = \frac{\sigma_\omega}{\varepsilon_0\kappa_\omega} + \frac{\alpha_\omega}{\varepsilon_0}$, $H_1 = \frac{\sigma_\omega}{\varepsilon_0\kappa_\omega^2}$ in $\zeta_w(t)$. Substituting (23) and (29) into(35), we obtain

$$\psi_{w,v}^{n+1} = e^{-\left(\frac{\sigma_w}{\varepsilon_0\kappa_w} + \frac{\alpha_w}{\varepsilon_0}\right)\Delta t}\psi_{w,v}^n + c_w\frac{\partial}{\partial w}F_v^{n+1} \tag{36}$$

where

$$c_w = \frac{\sigma_w}{\sigma_w\kappa_w + \kappa_w^2\alpha_w}\left[1 - e^{-\left(\frac{\sigma_w}{\varepsilon_0\kappa_w} + \frac{\alpha_w}{\varepsilon_0}\right)\Delta t}\right] \tag{37}$$

which is the same as CPML formulation given in reference [11]. However, it can be directly obtained by providing only pole and its corresponding residue, because the SARC formulations have a unified form.

Furthermore, let $\vec{P} = \chi(t) * E(t)$ and replace $E_x(\tau)$ over the time interval $\tau \in [n\Delta t, (n+1)\Delta t]$ with the first order Newton interpolation as (31),

$$E_x(\tau) = E_x^n + \frac{E_x^{n+1} - E_x^n}{\Delta t}(\tau - n\Delta t) \tag{38}$$

Making use of (23) and (32), we obtain

$$P_x^{n+1} = \sum_{q=1}^{Q} \mathrm{Im}\left[e^{-\gamma_q \Delta t} P_{x,q}^n + c_{0,q} E_x^{n+1} + c_{1,q} E_x^n\right] \tag{39}$$

where $c_{0,q}$ and $c_{1,q}$ can be calculated by (33) and (34).

Finally, substituting (36) and (39) into (16), and taking the average approximation to $E_x^{n+1/2} = \left(E_x^{n+1} + E_x^n\right)/2$, we obtain

$$\begin{aligned} E_x^{n+1} &= CA \cdot \vec{E}_x^n + CB \cdot \left[\frac{1}{\kappa_y}\frac{\partial H_z}{\partial y} - \frac{1}{\kappa_z}\frac{\partial H_y}{\partial z} + \psi_{E_{x,z}}^n - \psi_{E_{x,y}}^n\right]\Delta t \\ &\quad + CB \cdot \mathrm{Im}\sum_{q=1}^{Q}\left[\varepsilon_0\left(1 - e^{-\gamma_q \Delta t}\right) P_{x,q}^n\right] \end{aligned} \tag{40}$$

where

$$CA = \frac{\varepsilon_0\left(\varepsilon_\infty - \mathrm{Im}\sum\limits_{q=1}^{Q} c_{1,q}\right) - \dfrac{\sigma\Delta t}{2}}{\varepsilon_0\left(\varepsilon_\infty + \mathrm{Im}\sum\limits_{q=1}^{Q} c_{0,q}\right) + \dfrac{\sigma\Delta t}{2}} \tag{41}$$

$$CB = \frac{1}{\varepsilon_0\left(\varepsilon_\infty + \mathrm{Im}\sum\limits_{q=1}^{Q} c_{0,q}\right) + \dfrac{\sigma\Delta t}{2}} \tag{42}$$

$\psi_{E_{x,z}}^n$ and $\psi_{E_{x,y}}^n$ is given in (36) when $F = H$, $w = z$ and y, respectively. Substituting (36) into (17) and taking the average approximation to $H_x^n = \left(H_x^{n+1/2} + H_x^{n-1/2}\right)\Big/2$, we obtain

$$\begin{aligned} H_x^{n+1/2} &= DA \cdot H_x^{n-1/2} \\ &\quad - DB \cdot \left[\frac{1}{\kappa_y}\frac{\partial E_z}{\partial y} - \frac{1}{\kappa_z}\frac{\partial E_y}{\partial z} + \psi_{H_{x,z}}^n - \psi_{H_{x,y}}^n\right] \end{aligned} \tag{43}$$

where

$$DA = \left[1 - \frac{\sigma_m \Delta t}{2\mu}\right] \Big/ \left[1 + \frac{\sigma_m \Delta t}{2\mu}\right] \quad (44)$$

$$DB = \left[\frac{\sigma_m \Delta t}{2\mu}\right] \Big/ \left[1 + \frac{\sigma_m \Delta t}{2\mu}\right] \quad (45)$$

where $\psi^n_{H_{x,z}}$ and $\psi^n_{H_{x,y}}$ denote the expression of (36) when $F = E$, $w = z$ and y, respectively. In this way, the stretched-coordinate SARC FDTD update formulations for general dispersive media are consisted of (40), (39), (43), and (36), which suits for CPML region. It can be degenerated to FDTD computation region when $\psi^n_{w,v} = 0$.

It is seen from the SARC FDTD update formulations that it are related only to temporal discrete step Δt, poles γ_q and corresponding residues G_q of susceptibility function $\chi(t)$. The update formulations always have unified form and FDTD computation can be performed providing the poles and corresponding residues in dispersive medium model of interest are given no matter what kind of dispersive medium model is used. The calculation of integral χ^m, ζ^m, $\Delta\chi^m$ and $\Delta\zeta^m$ in RC or PLRC approach are no longer necessary. The SARC FDTD algorithms require only Q and 2Q additional real variables per electric field components and 2Q and 5Q extra real multiplications, respectively, for Q-order Debye and Lorentz media. It is equivalent to PLRC in terms of memory and CPU time. One more advantage is that the FDTD update formulations for CPML will be applicable to truncation of any kind of dispersive medium, because the update formulation for dispersive medium has been described by a unified form.

On the other hand, it can be seen by substituting (29) and (32) into (23) that the SARC FDTD formulations may come back to the Recursive Convolution (RC) FDTD, or Trapezoidal Recursive Convolution (TRC) and Piecewise Linear Recursive Convolution (PLRC), if we take the zero order approximation and $x_{ne} = x^n$, $x_{ne} = \left(x^n + x^{n-1}\right)/2$, or the first order approximation, respectively. In practical usage, SARC FDTD may also use other type of interpolation scheme and high order approximation to improve the accuracy.

5. NUMERICAL RESULTS

To validate the proposed algorithm, examples involving three typical kinds of dispersive model, i.e., Debye, Drude and Lorentz are taken into consideration [18–21].

At first example, we consider the performance of CPML absorbing boundary truncating dispersive medium. The FDTD computation region is shown in Fig. 1. The spatial cell size and temporal step are $\delta = 5\,\text{cm}$ and $\Delta t = \delta/2c$, respectively. The entire computational region is divided into $40\times40\times40$ cells, where the thickness of CPML absorbing boundary are 8 cells. The electric dipole is centered in computational region and surrounded by dispersive medium besides 64 cells near the dipole that are in free space. The modulated Gauss pulse used in calculation can be expressed as

$$P_i(t) = -10^{-10}\cos(\omega t)\exp\left[-\frac{4\pi(t-t_0)^2}{\tau^2}\right] \tag{46}$$

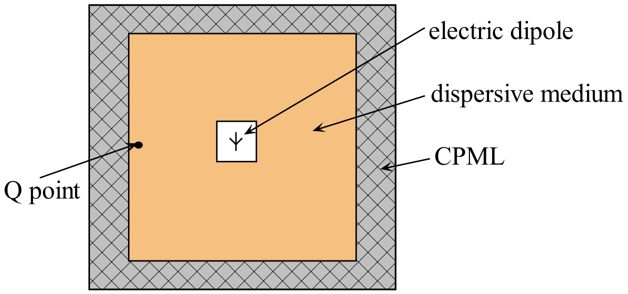


Figure 1. Diagram of absorbing boundary for performance test.

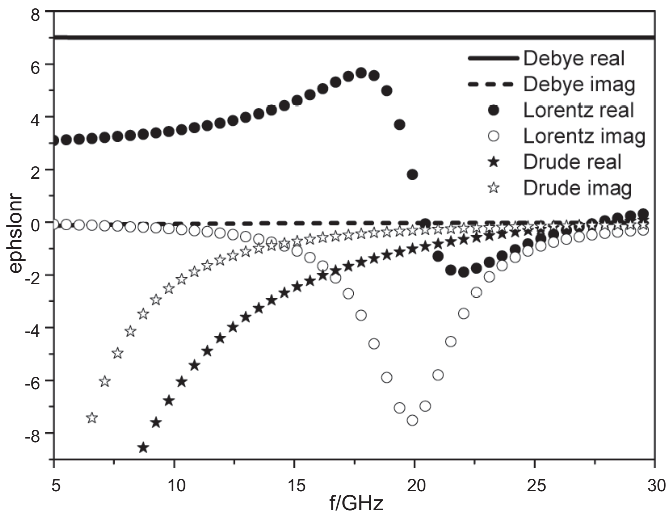
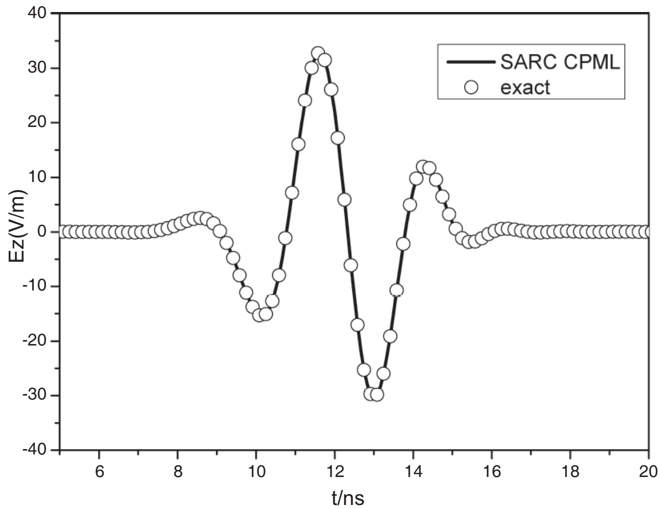
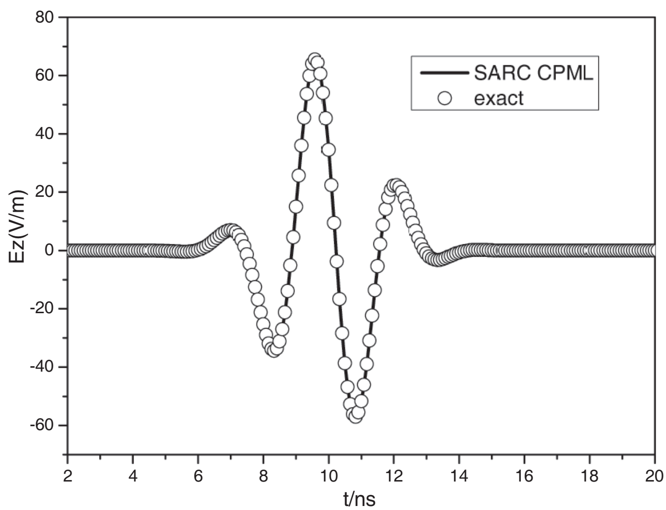


Figure 2. Real and imaginary part of the relative complex permittivity verse frequency for three typical kinds of dispersive medium.

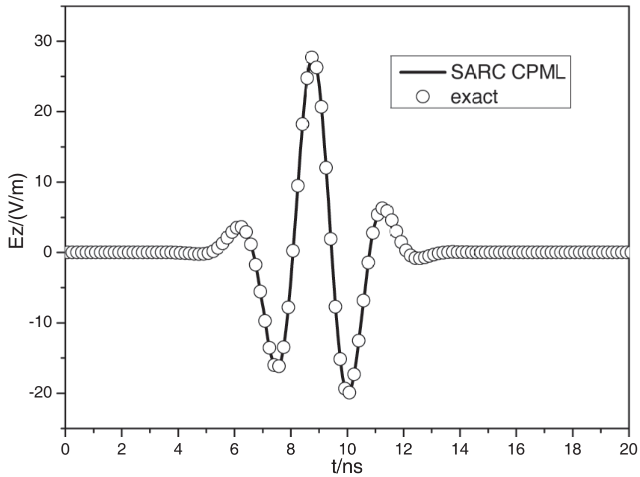
where t_0 is the time when the peak value of envelop of Gauss pulse emerge, τ is the pulse width, $\omega = 2\pi \times 0.3 \times 10^9$ rad/s, $t_0 = 9\pi/2\omega$, $\tau = 80\Delta t$. The dispersive medium chosen in test include Debye, Lorentz and Drude, respectively, where the parameters of Debye medium are: $\varepsilon_\infty = 7$, $\Delta\varepsilon_1 = 3$, $\tau_1 = 7.0 \times 10^{-10}$ s; the parameters of Lorentz medium are: $\varepsilon_\infty = 1.5$, $\Delta\varepsilon_1 = 1.5$, $\omega_p = 40\pi$ GHz, $\delta_p = 4\pi$ GHz; the parameters of Drude are $\varepsilon_\infty = 1$, $f_p = 2.87$ GHz, $\omega_p = 2\pi f_p$, $v_c = 200$ GHz. Fig. 2 shows the relative permittivity of three dispersive



(a)



(b)



(c)

Figure 3. Time-domain waveform at point Q when computational region is filled with three typical kinds of dispersive medium, i.e., Debye (a), Lorentz (b) and Drude (c) medium.

medium changes with the frequency. The solid line in Figs. 3(a), 3(b) and 3(c) denotes the time-domain waveform at observation point Q when the computational region is filled with Debye, Lorentz and Drude medium, respectively. For comparison, the calculation results that the FDTD computational region is expanded so that the electromagnetic wave pulse does not reach the absorbing boundary in view time are also given in Fig. 3 (denoted as a reference solution). It can be seen from Fig. 3 that the excellent agreement is obtained.

Next, we consider the backscattering RCS of three typical kinds of dispersive medium sphere, respectively. The foam is similar to a Debye dielectric material with a nonzero conductivity $\sigma = 2.95 \times 10^{-4}$ s/m. It can be obtained from (1) that the pole and residue of first order Debye model is respectively

$$\gamma = \frac{1}{\tau} - j0, \quad G = 0.0 + j\frac{\varepsilon_s - \varepsilon_\infty}{\tau}, \tag{47}$$

where the parameters for this material are $\varepsilon_s = 1.16$, $\varepsilon_\infty = 1.01$, $\tau = 6.497 \times 10^{-10}$ s. The radius of dielectric sphere $r = 3.75 \times 10^{-3}$ m, the cell size $\delta = 0.005$ m. The backscattering RCS of the sphere calculated by SARC FDTD is shown in Fig. 4. And the Mie's solution [22] is also given for comparison. Fig. 4 shows excellent agreement between the SARC FDTD calculation and the exact Mie series solution. Nonmagnetic plasma is a typical Drude model dispersive medium and

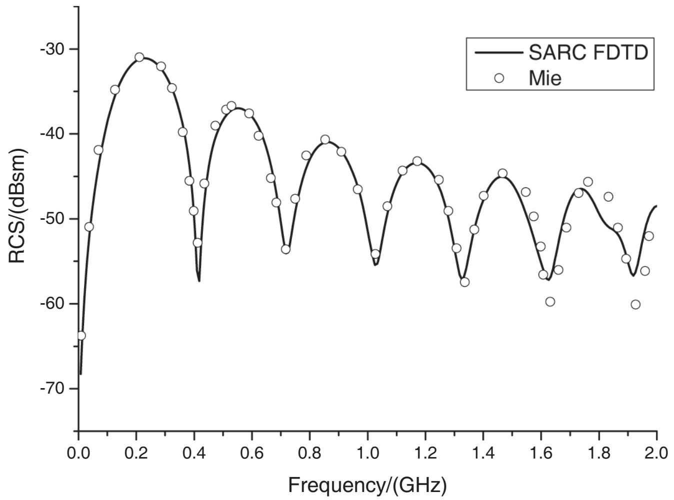


Figure 4. Backscattering RCS of a foam sphere.

its frequency domain susceptibility function is expressed in the form of (3). It can be seen that poles and the corresponding residues are

$$\gamma_1 = 0 + j0, \quad \gamma_2 = v_c + j0; \quad G_1 = 0.0 - j\frac{\omega_p^2}{v_c}, \quad G_2 = 0.0 + j\frac{\omega_p^2}{v_c}, \tag{48}$$

where the parameters are $\omega_p = 2\pi \times 28.7 \times 10^9\,\text{Hz}$ and $v_c = 2.0 \times 10^{10}\,\text{s}^{-1}$. The radius of the plasma sphere is $r = 3.75 \times 10^{-3}\,\text{m}$, the cell size $\delta = 5.0 \times 10^{-5}\,\text{m}$. Fig. 5 compares the backscattering RCS calculated using SARC FDTD and the exact Mie series solution. It can be seen that the SARC FDTD result agrees with the exact Mie series solution.

The first order Lorentz model in frequency is in the form of (2) and its poles and the corresponding residues are

$$\gamma_q = \delta_q - j\sqrt{\omega_q^2 - \delta_q^2}\,, \; G_q = \frac{\Delta\varepsilon_q \omega_q^2}{\sqrt{\omega_q^2 - \delta_q^2}} - j0.0, \tag{49}$$

where the parameters are $\varepsilon_s = 2.25$, $\varepsilon_\infty = 1.0$, $\delta_q = 0.28 \times 10^{16}\,\text{s}^{-1}$, $\omega_q = 4.0 \times 10^{16}\,\text{Hz}$. The radius of the Lorentz sphere is $r = 15.0 \times 10^{-9}\,\text{m}$ and the cell size $\delta = 3.0 \times 10^{-10}\,\text{m}$. Fig. 6 shows the backscattering RCS calculated using the SARC FDTD and the exact Mie series solution.

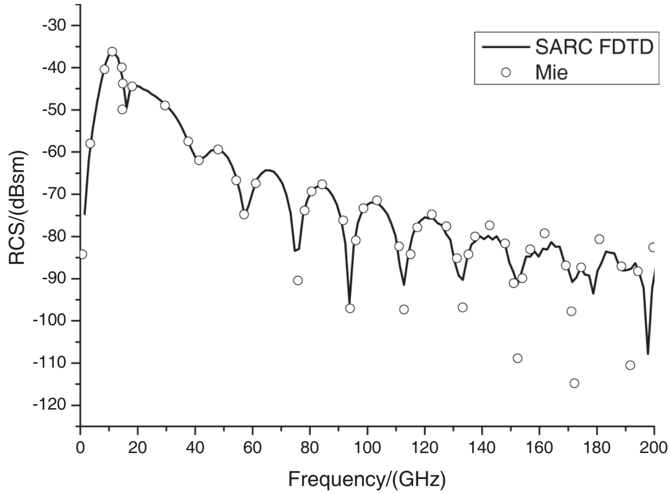


Figure 5. Backscattering RCS of non-magnetized plasma.

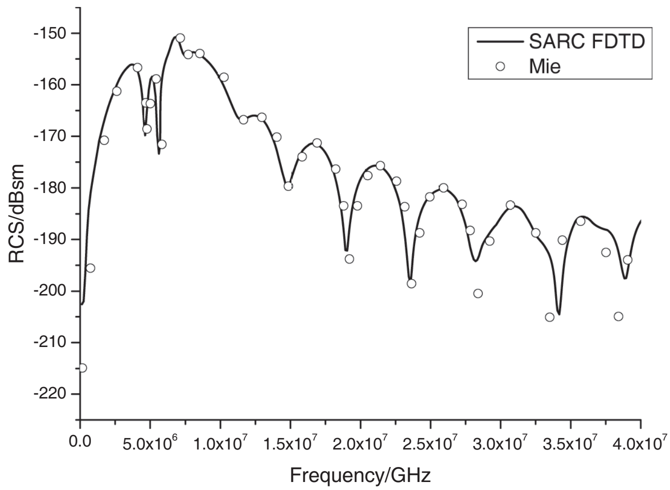


Figure 6. Backscatter RCS of a Lorentz sphere.

6. CONCLUSION

The SARC is a fast recursive convolution algorithm widely used in DSP technique based on the linear interpolation theory to the input signal and has some advantages in computation complexity, stability and accuracy over the other recursive convolution schemes, such as RC/PLRC. In this paper, combining with the SARC algorithm, the stretched-coordinate SARC FDTD formulations for the general dispersive medium are obtained from the stretched-coordinate Maxwell's curl equations. The SARC FDTD update formulations are unified in the computational region and CPML region for the different dispersive medium model. Finally, to validate the \SARC FDTD algorithm, the radiation of dipole when the computational region is filled with dispersive medium and backscattering RCS of three typical kinds of dispersive medium sphere is calculated, respectively. In practice, since the FDTD method suit analysis of the complexity object, the SARC FDTD presented in this paper also suits the analysis of complex object including magnetic dispersion medium, anisotropic dispersive medium and etc.

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