Progress In Electromagnetics Research, PIER 98, 207–219, 2009

MAGNETIC FIELD PRODUCED BY A PARALLELEPIPEDIC MAGNET OF VARIOUS AND UNIFORM POLARIZATION

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Abstract—This paper deals with the modeling of parallelepipedic magnets of various polarization directions. For this purpose, we use the coulombian model of a magnet for calculating the magnetic potential in all points in space. Then, we determine the three components of the magnetic field created by a parallepiped magnet of various polarization direction. These three components and the scalar magnetic potential are also expressed in terms of fully analytical terms. It is to be noted that the formulas determined in this paper are more general that the ones established in the literature and can be used for optimization purposes. Moreover, our study is carried out without using any simplifying assumptions. Consequently, these expressions are accurate whatever the magnet dimensions. This analytical formulation is suitable for the design of unconventional magnetic couplings, electric machines and wigglers.

1. INTRODUCTION

Permanent magnets are widely used in many engineering and industrial applications. Their utilization requires calculation methods based on the fundamental laws of the magnetostatics [1, 2]. Two great kinds of applications can be identified. The first ones use parallelepipedic permanent magnets while the second ones use arc-shaped permanent magnets. This paper deals only with parallelepipedic permanent magnets. However, this study can also be extended to the case of cylindrical permanent magnet topologies.

The first analytical studies dealing with the modeling of parallelepiped magnets were studied by Akoun [3] and Yonnet [4].

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Then, several analytical studies were carried out by using the coulombian model of a magnet [5, 6]. The interest of using fully analytical approaches lies in the fact that they have generally a lower computational cost than finite element methods [7–10]. Moreover, analytical approaches are suitable for parametric optimizations using permanent magnets [10], or coils carrying producing magnetic fields [14, 15]. The magnetic field created by parallelepipedic magnets can be expressed in fully analytical parts whereas the magnetic field produced by arc-shaped permanent magnets is generally based on special functions [16–25].

This paper presents 3D analytical expressions of the magnetic field created by a parallelepipedic permanent magnet of various polarization direction. Indeed, its polarization can be along the x, y and z direction, in the (x-y), (x-z) and (y-z) planes but also in any direction in the coordinate system. Such a study is clearly justified by the progress in manufacturing permanent magnets with more complicated magnetizations. Moreover, permanent magnets with various polarization directions allow us to confine the magnetic flux in ironless structures [26, 27] and to optimize the magnetic field shape in electric machines [28].

We present first the 3D analytical expression of the magnetic scalar potential created by a paralellepipedic magnet of various polarization direction: Such an expression is useful for the study of ferrofluids used with permanent magnets [29, 30]. Indeed, the magnetic pressure of the ferrofluid seal requires the accurate knowledge of the magnetic potential in all points in space.

The second part of this paper presents the analytical expressions of the three components of the magnetic field created by a parallelepipedic magnet of various polarization direction.

2. ANALYTICAL EXPRESSION OF THE MAGNETIC POTENTIAL PRODUCED BY A PARALLELEPIPED MAGNET WITH A UNIFORM AND ARBITRARY POLARIZATION

2.1. Notation and Geometry

We present in this section the 3D analytical expression of the magnetic potential created by a parallelepiped magnet of various polarization direction. To do so, let us first consider the representation shown in Fig. 1.

Its dimensions are given by $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$ and its polarization is denoted $\vec{J}$. By using the notations shown in Fig. 1, this

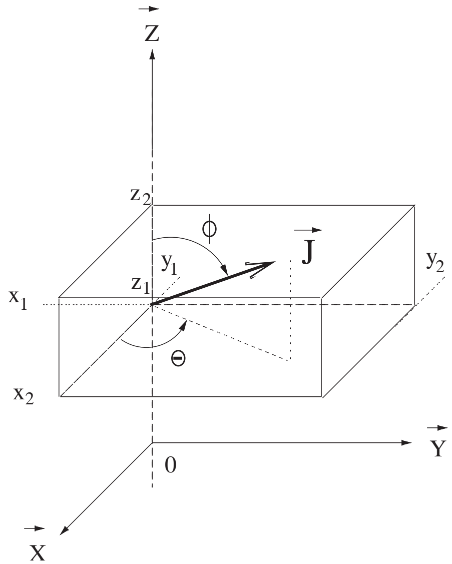

Figure 1. Representation of a parallelepiped magnet of various polarization direction.

polarization vector is expressed as follows:

$$\vec{J} = J\cos(\theta)\sin(\phi)\vec{u}_x + J\sin(\theta)\sin(\phi)\vec{u}_y + J\cos(\phi)\vec{u}_z \tag{1}$$

We define the permanent magnet surface as follows:

$$\begin{aligned} \vec{dS}_1 &= +dydz\vec{u}_x \\ \vec{dS}_2 &= -dxdz\vec{u}_y \\ \vec{dS}_3 &= -dydz\vec{u}_x \\ \vec{dS}_4 &= +dxdz\vec{u}_y \\ \vec{dS}_5 &= -dxdy\vec{u}_z \\ \vec{dS}_6 &= +dxdy\vec{u}_z \end{aligned} \tag{2}$$

2.2. Analytical Formulation

In the coulombian approach, the magnetic potential created by the parallelepipedic magnet is given by:

$$\Phi(x, y, z) = \sum_{i=1}^{6} \left(\frac{1}{4\pi\mu_0} \iint_{S_i} \frac{\vec{J} \cdot \vec{dS}_i}{|\vec{r} - \vec{r}_i|} \right) \tag{3}$$

For the rest of this paper, we adopt the following notation:

$$\vartheta^{(ijk)}\left\{\bullet\right\}=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{i+j+k}\left(\bullet\right) \tag{4}$$

By using the egality $\xi_{ijk} = \sqrt{(x-x_i)^2+(y-y_j)^2+(z-z_k)^2}$, the magnetic potential created by a parallelepiped magnet of various polarization direction is expressed as follows:

$$\Phi(x,y,z)=\frac{J}{4\pi\mu_0}\vartheta^{(ijk)}\left\{(\sin(\phi)\cos(\theta)\Phi_1+\sin(\phi)\sin(\theta)\Phi_2+\cos(\phi)\Phi_3\right\} \tag{5}$$

with

$$\begin{aligned}
\Phi_1 =& z_k+(x-x_i)\arctan\left[\frac{z-z_k}{x-x_i}\right]+(y-y_j)\log\left[z-z_k+\xi_{ijk}\right]\\
&-(x-x_i)\arctan\left[\frac{(y-y_j)(z-z_k)}{(x-x_i)\xi_{i,j,k}}\right]+(z-z_k)\log\left[y-y_j+\xi_{ijk}\right]\\
\Phi_2 =& z_k+(y-y_j)\arctan\left[\frac{z-z_k}{y-y_j}\right]+(x-x_i)\log\left[z-z_k+\xi_{ijk}\right]\\
&-(y-y_j)\arctan\left[\frac{(x-x_i)(z-z_k)}{(y-y_j)\xi_{ijk}}\right]+(z-z_k)\log\left[x-x_i+\xi_{ijk}\right]\\
\Phi_3 =& y_j+(z-z_k)\arctan\left[\frac{y-y_j}{z-z_k}\right]+(x-x_i)\log\left[y-y_j+\xi_{ijk}\right]\\
&-(z-z_k)\arctan\left[\frac{(x-x_i)(y-y_j)}{(z-z_k)\xi_{ijk}}\right]+(y-y_j)\log\left[x-x_i+\xi_{ijk}\right]
\end{aligned} \tag{6}$$

As stated previously, the magnetic scalar potential $\Phi(x,y,z)$ is fully analytical and does not require any numerical treatment for its determination. Its expression is suitable for representing its iso-potentials inside the magnet as well as outside it.

2.3. Representation of the Magnetic Scalar Potential

We illustrate our previous analytical expression with two configurations. The first one allows us to verify the accuracy of our expression and allows us to compare it with the ones published in the literature. The second configuration is less usual as the first one: It is an academic illustration of the usefulness of our 3D analytical expression.

2.3.1. Case of a Parallelepipedic Magnet Whose Polarization Is Directed along the z Direction

The first configuration is well known and corresponds to the case when the polarization is directed along the z-direction. We take the following dimensions: $x_2 - x_1 = 0.005\,\mathrm{m}$, $y_2 - y_1 = 0.01\,\mathrm{m}$, $z_2 - z_1 = 0.02\,\mathrm{m}$, $J = 1\,\mathrm{T}$. We represent in Fig. 2 three 2D cross-sections of the iso-potentials created by the parallelepipedic magnet whose polarization is directed along the z direction.

Figures 2 show that the iso-potentials are circles in the (x-y) plane, which is consistent with the polarization direction of the parallelepipedic permanent magnet. Moreover, we see the iso-potentials in the (x-y) and (y-z) planes are the same: It is still consistent with the polarization direction of the parallelepipedic permanent magnet.

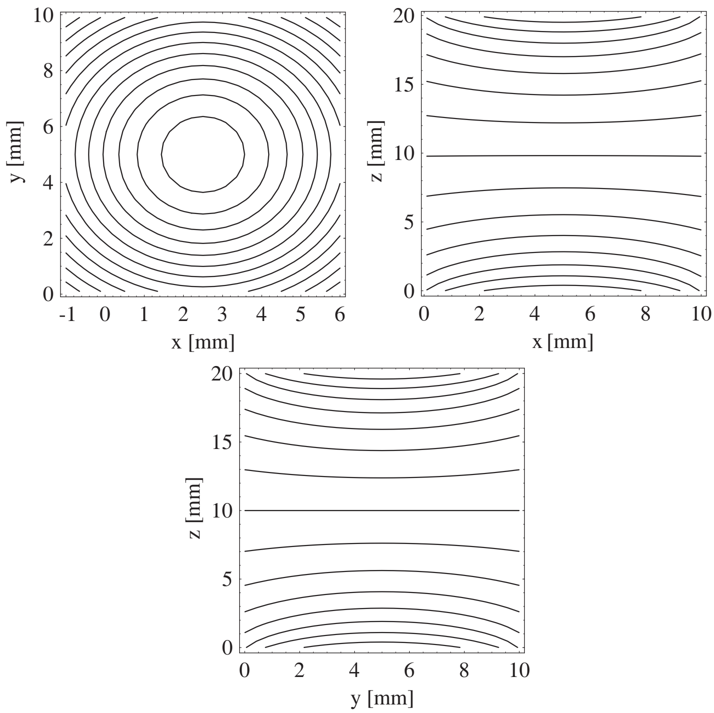

Figure 2. 2D representation of the iso-potentials created by a parallelepiped magnet whose polarization is directed along the z direction; (x-y)-plane: $z = 15\,\mathrm{mm}$, (x-z)-plane: $y = 5\,\mathrm{mm}$, (y-z)-plane: $x = 5\,\mathrm{mm}$.

2.3.2. Case of a Parallelepipedic Magnet of Various Polarization Direction

The second configuration we consider is a parallelepipedic permanent magnet with the polarization shown in Fig. 3: This polarization is expressed as follows:

$$\vec{J} = \frac{J}{\sqrt{2}}\vec{u}_y + \frac{J}{\sqrt{2}}\vec{u}_z \tag{7}$$

We take the following dimensions: $x_2 - x_1 = 0.005\,\text{m}$, $y_2 - y_1 = 0.01\,\text{m}$, $z_2 - z_1 = 0.02\,\text{m}$, $J = 1\,\text{T}$. We represent in Fig. 4 three 2D cross-sections of the iso-potentials created by the parallelepiped magnet with the polarization $\vec{J} = \frac{J}{\sqrt{2}}\vec{u}_y + \frac{J}{\sqrt{2}}\vec{u}_z$. In this configuration, we take $\theta = \frac{\pi}{2}$ rad and $\phi = \frac{\pi}{4}$ rad.

The computational cost is 1 s for representing the magnetic potential in the three previous illustrations.

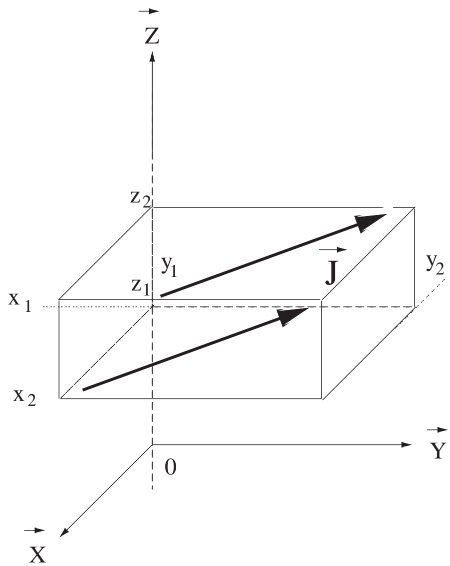

Figure 3. Representation of a parallelepiped magnet with the following polarization vector: $\vec{J} = \frac{J}{\sqrt{2}}\vec{u}_y + \frac{J}{\sqrt{2}}\vec{u}_z$.

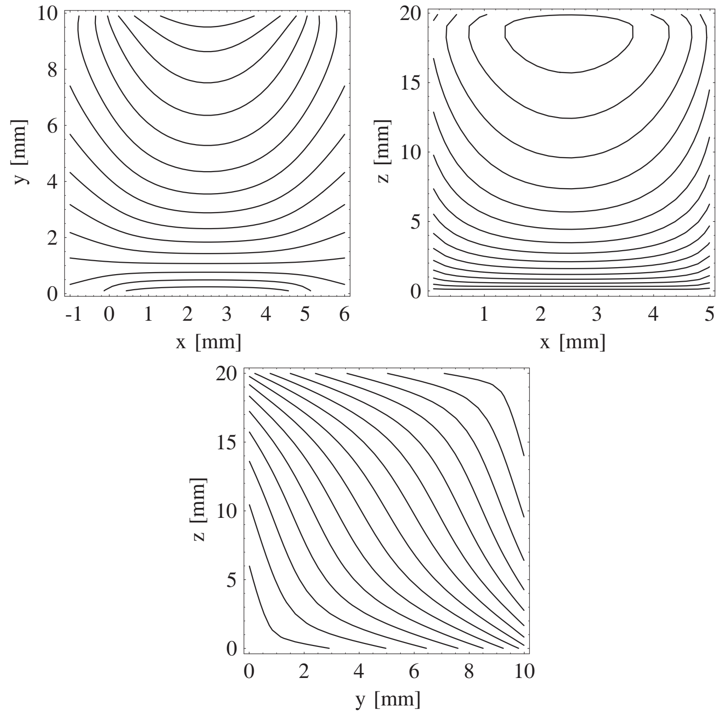

Figure 4. 2D representation of the iso-potentials created by a parallelepiped magnet whose polarization is $\vec{J} = \frac{J}{\sqrt{2}}\vec{u}_y + \frac{J}{\sqrt{2}}\vec{u}_z$; ($x$-$y$)-plane: $z = 19\,\mathrm{mm}$, (x-z)-plane: $y = 10\,\mathrm{mm}$, (y-z)-plane: $x = 5\,\mathrm{mm}$.

3. ANALYTICAL EXPRESSION OF THE MAGNETIC FIELD PRODUCED BY A PARALLELEPIPED MAGNET WITH A UNIFORM AND ARBITRARY POLARIZATION

The three components of the magnetic field created by one parallelepipedic magnet of various polarization direction can be determined by using the following expression:

$$\begin{aligned} H_x(x,y,z) &= -\vec{\nabla}\left(\phi(x,y,z)\right)\cdot\vec{u}_x \\ H_y(x,y,z) &= -\vec{\nabla}\left(\phi(x,y,z)\right)\cdot\vec{u}_y \\ H_z(x,y,z) &= -\vec{\nabla}\left(\phi(x,y,z)\right)\cdot\vec{u}_z \end{aligned} \tag{8}$$

After mathematical manipulations, we obtain the three components $H_x(x,y,z)$, $H_y(x,y,z)$ and $H_z(x,y,z)$ that are expressed as follows:

$$\begin{aligned}
H_x(x,y,z) =& \frac{J\sin(\phi)\cos(\theta)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{\arctan\left[\frac{(y-y_j)(z-z_k)}{(x-x_i)\xi_{ijk}}\right]\right\} \\
&+\frac{J\sin(\phi)\sin(\theta)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{-\log\left[z-z_k+\xi_{ijk}\right]\right\} \\
&+\frac{J\cos(\phi)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{-\log\left[y-y_j+\xi_{ijk}\right]\right\} \\
H_y(x,y,z) =& \frac{J\sin(\phi)\cos(\theta)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{-\log\left[z-z_k+\xi_{ijk}\right]\right\} \\
&+\frac{J\sin(\phi)\sin(\theta)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{\arctan\left[\frac{(x-x_i)(z-z_k)}{(y-y_j)\xi_{ijk}}\right]\right\} \\
&+\frac{J\cos(\phi)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{-\log\left[x-x_i+\xi_{ijk}\right]\right\} \\
H_z(x,y,z) =& \frac{J\sin(\phi)\cos(\theta)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{-\log\left[y-y_j+\xi_{ijk}\right]\right\} \\
&+\frac{J\sin(\phi)\sin(\theta)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{-\log\left[x-x_i+\xi_{ijk}\right]\right\} \\
&+\frac{J\cos(\phi)}{4\pi\mu_0}\vartheta^{(ijk)}\left\{\arctan\left[\frac{(x-x_i)(y-y_j)}{(z-z_k)\xi_{ijk}}\right]\right\}
\end{aligned} \tag{9}$$

If we take $\theta = 0$ and $\phi = 0$, we obtain the same expression as Bancel [5].

3.1. Representation of the Magnetic Field

We illustrate now the use of our three-dimensional analytical expression by calculating the magnetic field modulus **H**. Then, we represent the iso-lines in two configurations corresponding to the previous ones.

3.1.1. Case of a Parallelepipedic Magnet Whose Polarization Is Directed along the z Direction

We take the following dimensions: $x_2 - x_1 = 0.005\,\text{m}$, $y_2 - y_1 = 0.01\,\text{m}$, $z_2 - z_1 = 0.02\,\text{m}$, $J = 1\,\text{T}$. We represent in Fig. 5 three 2D cross-sections of the iso-lines created by the parallelepiped magnet whose polarization is directed along the axial direction. In this configuration, we take $\theta = 0\,\text{rad}$ and $0\,\text{rad}$.

The computational cost for representing the iso-lines is lower than 1 s: This shows the interest of using a fully analytical approach for calculating the magnetic field produced by a parallelepipedic magnet of various and uniform polarization.

3.1.2. Case of a Parallelepipedic Magnet of Various Polarization Direction

We take the following dimensions: $x_2 - x_1 = 0.005\,\text{m}$, $y_2 - y_1 = 0.01\,\text{m}$, $z_2 - z_1 = 0.02\,\text{m}$, $J = 1\,\text{T}$. We represent in Fig. 6 three 2D cross-sections of the iso-lines created by the parallelepiped magnet with the polarization $\vec{J} = \frac{J}{\sqrt{2}}\vec{u}_y + \frac{J}{\sqrt{2}}\vec{u}_z$. In this configuration, we take $\theta = \frac{\pi}{2}$ rad and $\phi = \frac{\pi}{4}$ rad.

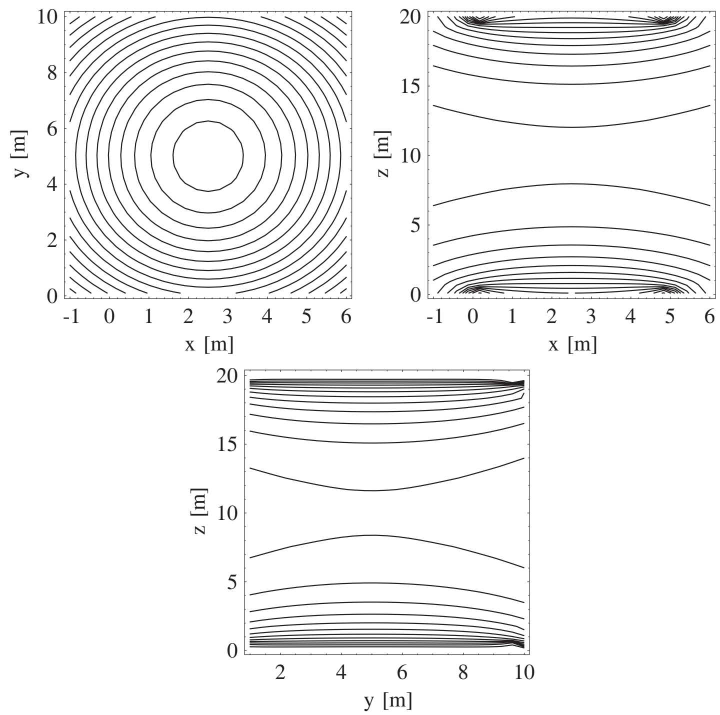

Figure 5. 2D representation of the iso-lines created by a parallelepiped magnet whose polarization is directed along the z direction; (x-y)-plane: $z = 19.9\,\text{mm}$, (x-z)-plane: $y = 5\,\text{mm}$, (y-z)-plane: $x = 4.5\,\text{mm}$.

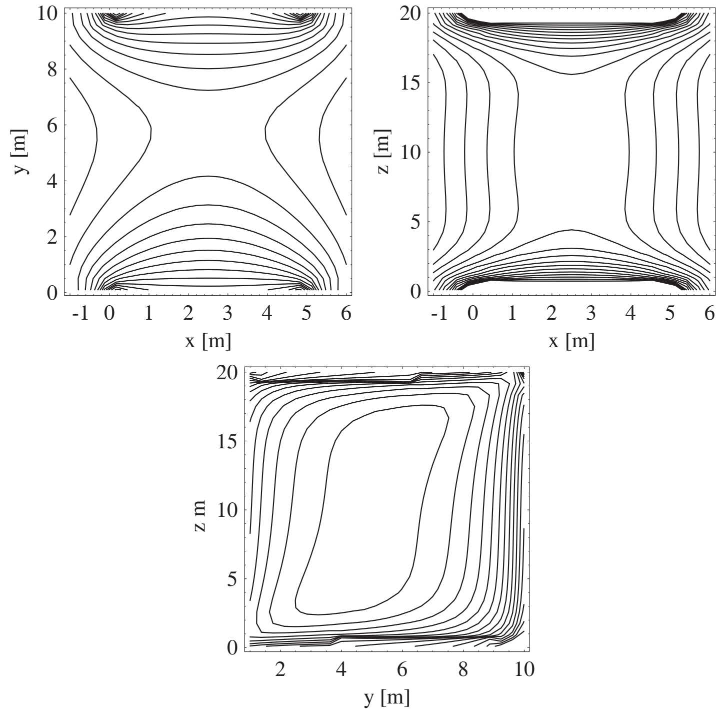

Figure 6. 2D representation of the iso-potentials created by a parallelepiped magnet whose polarization is $\vec{J} = \frac{J}{\sqrt{2}}\vec{u}_y + \frac{J}{\sqrt{2}}\vec{u}_z$; ($x$-$y$)-plane: $z = 19\,\mathrm{mm}$, (x-z)-plane: $y = 5\,\mathrm{mm}$, (y-z)-plane: $x = 4.5\,\mathrm{mm}$.

4. CONCLUSION

This paper has presented three-dimensional analytical expressions for calculating the magnetic scalar potential and the magnetic field produced by a parallelepiped magnet of various polarization direction. In particular, the polarizations considered are entirely uniform, as it is generally the case in practice. By using the coulombian model of a magnet, we have expressed the the three components of the magnetic field in terms of fully analytical parts whose computational cost is very low. Such expressions have been compared to the ones published in the literature when the magnet polarization is directed along the axial direction. From an academic point of view, these expressions are an extension to the case of parallelepiped permanent magnet whose polarizations are directed along the x, y or z directions.

REFERENCES

1. Babic, S. I., C. Akyel, and M. M. Gavrilovic, "Calculation improvement of 3D linear magnetostatic field based on fictitious magnetic surface charge," *IEEE Trans. Magn.*, Vol. 36, No. 5, 3125–3127, 2000.
2. Babic, S. I. and C. Akyel, "Improvement in the analytical calculation of the magnetic field produced by permanent magnet rings," *Progress In Electromagnetics Research C*, Vol. 5, 71–82, 2008.
3. Akoun, G. and J. P. Yonnet, "3D analytical calculation of the forces exerted between two cuboidal magnets," *IEEE Trans. Magn.*, Vol. 20, No. 5, 1962–1964, 1984.
4. Yonnet, J. P., *Rare-earth Iron Permanent Magnets*, Ch. Magnetomechanical devices, Oxford science publications, 1996.
5. Ravaud, R., G. Lemarquand, V. Lemarquand, and C. Depollier, "Discussion about the analytical calculation of the magnetic field created by permanent magnets," *Progress In Electromagnetics Research B*, Vol. 11, 281–297, 2009.
6. Furlani, E. P., "Field analysis and optimization of ndfeb axial field permanent magnet motors," *IEEE Trans. Magn.*, Vol. 33, No. 5, 3883–3885, 1997.
7. Furlani, E. P., *Permanent Magnet and Electromechanical Devices: Materials, Analysis and Applications*, Academic Press, 2001.
8. Furlani, E. P. and M. Knewston, "A three-dimensional field solution for permanent-magnet axial-field motors," *IEEE Trans. Magn.*, Vol. 33, No. 3, 2322–2325, 1997.
9. Furlani, E. P., S. Reznik, and A. Kroll, "A three-dimensonal field solution for radially polarized cylinders," *IEEE Trans. Magn.*, Vol. 31, No. 1, 844–851, 1995.
10. Jian, L. and K. T. Chau, "Analytical calculation of magnetic field distribution in coaxial magnetic gears," *Progress In Electromagnetics Research*, PIER 92, 1–16, 2009.
11. Huang, S. M. and C. K. Sung, "Analytical analysis of magnetic couplings with parallelepiped magnets," *Journal of Magnetism and Magnetic Materials*, Vol. 239, 614–616, 2002.
12. Lemarquand, V., J. F. Charpentier, and G. Lemarquand, "Nonsinusoidal torque of permanent-magnet couplings," *IEEE Trans. Magn.*, Vol. 35, No. 5, 4200–4205, 1999.
13. Yonnet, J. P., et al., "Analytical calculation of permanent magnet couplings," *IEEE Trans. Magn.*, Vol. 29, No. 6, 2932–2934, 1993.

14. Blache, C. and G. Lemarquand, "New structures for linear displacement sensor with hight magnetic field gradient," *IEEE Trans. Magn.*, Vol. 28, No. 5, 2196–2198, 1992.
15. Conway, J., "Noncoaxial inductance calculations without the vector potential for axisymmetric coils and planar coils," *IEEE Trans. Magn.*, Vol. 44, No. 4, 453–462, 2008.
16. Babic, S. I., F. Sirois, and C. Akyel, "Validity check of mutual inductance formulas for circular filaments with lateral and angular misalignments," *Progress In Electromagnetics Research M*, Vol. 8, 15–26, 2009.
17. Ravaud, R., G. Lemarquand, V. Lemarquand, and C. Depollier, "The three exact components of the magnetic field created by a radially magnetized tile permanent magnet," *Progress In Electromagnetics Research*, PIER 88, 307–319, 2008.
18. Perigo, E., R. Faria, and C. Motta, "General expressions for the magnetic flux density produced by axially magnetized toroidal permanent magnets," *IEEE Trans. Magn.*, Vol. 43, No. 10, 3826–3832, 2007.
19. Rakotoarison, H. L., J. P. Yonnet, and B. Delinchant, "Using coulombian approach for modeling scalar potential and magnetic field of a permanent magnet with radial polarization," *IEEE Trans. Magn.*, Vol. 43, No. 4, 1261–1264, 2007.
20. Varga, E. and A. Beyer, "Magnetic field of a uniformly magnetized hollow cylinder," *IEEE Trans. Magn.*, Vol. 34, No. 3, 613–618, 1998.
21. Zhilichev, Y., "Calculation of magnetic field of tubular permanent magnet assemblies in cylindrical bipolar coordinates," *IEEE Trans. Magn.*, Vol. 43, No. 7, 3189–3195, 2007.
22. Selvaggi, J. P., et al., "Computation of the three-dimensional magnetic field from solid permanent-magnet bipolar cylinders by employing toroidal harmonics," *IEEE Trans. Magn.*, Vol. 43, No. 10, 3833–3839, 2007.
23. Selvaggi, J. P., et al., "Calculating the external magnetic field from permanent magnets in permanent-magnet motors — An alternative method," *IEEE Trans. Magn.*, Vol. 40, No. 5, 3278–3285, 2004.
24. Selvaggi, J. P., et al., "Computation of the external magnetic field, near-field or far-field from a circular cylindrical magnetic source using toroidal functions," *IEEE Trans. Magn.*, Vol. 43, No. 4, 1153–1156, 2007.
25. Ravaud, R. and G. Lemarquand, "Comparison of the coulombian

and amperian current models for calculating the magnetic field produced by arc-shaped permanent magnets radially magnetized," *Progress In Electromagnetics Research*, PIER 95, 309–327, 2009.
26. Xia, Z., Z. Q. Zhu, and D. Howe, "Analytical magnetic field analysis of Halbach magnetized permanent-magnet machines," *IEEE Trans. Magn.*, Vol. 40, No. 4, 1864–1872, 2004.
27. Wang, J., G. W. Jewell, and D. Howe, "Design optimisation and comparison of permanent magnet machines topologies," *IEE Proc. Elect. Power Appl.*, Vol. 148, 456–464, 2001.
28. Ravaud, R. and G. Lemarquand, "Discussion about the magnetic field produced by cylindrical Halbach structures," *Progress In Electromagnetics Research B*, Vol. 13, 275–308, 2009.
29. Ravaud, R. and G. Lemarquand, "Mechanical properties of a ferrofluid seal: Three-dimensional analytical study based on the coulombian model," *Progress In Electromagnetics Research B*, Vol. 13, 385–407, 2009.
30. Ravaud, R. and G. Lemarquand, "Design of ironless loudspeakers with ferrofluid seals: Analytical study based on the coulombian model," *Progress In Electromagnetics Research B*, Vol. 14, 285–309, 2009.
31. Bancel, F. and G. Lemarquand, "Three-dimensional analytical optimization of permanent magnets alternated structure," *IEEE Trans. Magn.*, Vol. 34, No. 1, 242–247, 1998.