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BEAMPATTERN SYNTHESIS WITH LINEAR MATRIX INEQUALITIES USING MINIMUM ARRAY SENSORS

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Abstract—A new beampattern synthesis formulation is proposed to compute the minimum number of array sensors required. In order to satisfy all the prescribed specifications of the beampattern, the proposed method imposes linear matrix inequality (LMI) constraints on the beampattern as developed by Davidson et al., which remove the need to discretize the beampattern region. As the proposed formulation is quasi-convex, an iterative procedure is used to decompose it into a systematic sequence of convex feasibility problems, in order to find the minimum number of sensors. The proposed method guarantees convergence if the globally optimum solution lies in the search interval, which is easy to ensure at the start of the search.

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1. INTRODUCTION

Sensor arrays are used ubiquitously in radar, sonar, cellular systems, and more recently, medical imaging [1–4]. One fundamental array processing task is beampattern synthesis, which designs complex weights for the array sensors to achieve a high directive gain or to spatially filter impinging signals by their directions-of-arrival. The aims for developing a beampattern synthesis method to determine the minimum number of array sensors required are twofold from an implementation point of view. First, the physical dimension of the array is normally constrained by its supporting structure, for example, a radio tower, the fuselage of an aeroplane or a towed array. Second, the cost per sensor can be significant in many applications due to the sensor itself and the associated electronics [1]. Hence, we modify the method of Davidson et al. in [5] and propose a new formulation that can find the minimum number of array sensors required to achieve prescribed beampattern specifications and the corresponding beamforming weights, systematically. A related problem arises in minimum order filter design [6].

Classical beampattern synthesis methods such as that of Woodward and Lawson [7] are capable of synthesizing an arbitrary beampattern. However, the ripples between the sample points are uncontrollable and may possibly violate the beampattern requirements [1]. In contrast, the use of the LMI constraint in the proposed method represents the semi-infinite magnitude response constraint on the beampattern in a finite and convex manner. Due to this finite and precise representation of the semi-infinite constraint, there is no need to discretize the beampattern region, which are typically required by the methods of [7–9] so as to approximate the semi-infinite constraint. As such, the synthesized beampattern by the proposed method conforms to the prescribed specifications of an arbitrary beampattern strictly, unlike [7]. An alternative LMI representation of the beampattern constraints is developed in [10].

The proposed formulation is quasi-convex which means that it can have stationary points that are suboptimum solutions, thus an iterative procedure is used to decompose the formulation into a systematic sequence of convex feasibility problems. At each step, a feasibility problem is solved efficiently via the interior-point method (IPM) solver such as [11], which determines if a set of beamforming weights exists, for a particular number of array sensors. As such, the proposed method is guaranteed to find the minimum number of sensors to achieve the prescribed beampattern specifications so long as this solution is located in the search interval. This is easy to ensure at the start of the search.

2. DATA MODEL

Consider a uniform linear array (ULA) of $N+1$ isotropic sensors with d inter-element spacing. A plane wave of λ wavelength impinges on it at θ degrees from the array axis. The array beampattern can be expressed as $G(\theta) = \sum_{k=0}^{N} w_k e^{j\frac{2\pi}{\lambda}kd\cos\theta}$ with complex array weights w_i for $i = 0, \ldots, N$. A typical beampattern design constraint can be expressed as $L(\theta) \leq |G(\theta)| \leq U(\theta)$ where $\theta \in [0, \pi)$, $|\cdot|$ refers to absolute operator, $L(\theta)$ and $U(\theta)$ denote the lower and upper magnitude response limits, respectively. The lower bound of this constraint is non-convex thus, it is necessary to convert it into a convex constraint by a transformation of variables [5, 8, 10].

The autocorrelation sequence of the complex weights w_i is defined as $r_w(k) = \sum_{i=-N}^{N} w_i w_{i+k}^*$ where $k = -N, \ldots, 0, \ldots, N$, $(\cdot)^*$ represents the complex conjugate operator and $w_i = 0$ for $i < 0$ and $i > N$ [8]. The Fourier Transform of $r_w(k)$ is

$$R_w(\theta) = \sum_{k=-N}^{N} r_w(k) e^{jk\frac{2\pi}{\lambda}d\cos\theta} \tag{1}$$

and it is shown in [8] that

$$R_w(\theta) = \mathbf{a}^T(\theta)\mathbf{r}_w = |G(\theta)|^2, \tag{2}$$

where $\mathbf{a}(\theta) = [e^{-jN\frac{2\pi}{\lambda}d\cos\theta} \ \ldots \ 1 \ \ldots \ e^{jN\frac{2\pi}{\lambda}d\cos\theta}]^T$, $\mathbf{r}_w = [r_w(-N) \ \ldots \ r_w(0) \ \ldots \ r_w(N)]^T$ and $(\cdot)^T$ refers to the transpose operator. Thus, the original non-convex constraint is equivalent to

$$L^2(\theta) \leq \mathbf{a}^T(\theta)\mathbf{r}_w \leq U^2(\theta), \quad \theta \in [0, \pi) \tag{3}$$

which is convex. As there are two linear inequalities in $\mathbf{r}_w$ for every $\theta \in [0, \pi)$, (3) is a semi-infinite constraint.

3. THE PROPOSED METHOD

We aim to find the minimum number of array sensors $N+1$ and the corresponding weights w_i for beampattern design. For notational convenience, let $M = N+1$. The semi-infinite constraint (3) is used to specify beampattern requirements explicitly on the mainlobe (Θ_{ML}), sidelobe (Θ_{SL}) and null (Θ_{N}) regions, respectively via (4b)–(4c). The proposed problem is formulated as

$$\min_{M,\mathbf{r}_w} \quad M \tag{4a}$$

$$\text{s.t.} \quad L^2(\theta) \le \mathbf{a}^T(\theta)\mathbf{r}_w \le U^2(\theta), \quad \theta \in \Theta_{\text{ML}} \tag{4b}$$

$$\mathbf{a}^T(\theta)\mathbf{r}_w \le U^2(\theta), \quad \theta \in \Theta_{\text{SL}} \cup \Theta_{\text{N}} \tag{4c}$$

$$\mathbf{a}^T(\theta)\mathbf{r}_w \ge 0, \quad \theta \in \Theta \tag{4d}$$

$$r_w(-k) = r_w^*(k), \quad k = 0, \ldots, M-1 \tag{4e}$$

$$M \in \mathrm{Z}_+, \tag{4f}$$

where the abbreviation s.t. stands for "subject to", Z_+ denotes the set of positive integers and $\Theta_{(\cdot)}$ represents the set of θ in a particular angular region designated by the subscript. The optimization variable in (4) is the array weight autocorrelation sequence $\mathbf{r}_w$. The constraint (4c) requires that the response in Θ_{SL} (or Θ_{N}) is at most $U^2(\theta)$. The constraint (4d) is sufficient to ensure that the complex weights w_i can be extracted (though not uniquely) from the obtained $\mathbf{r}_w$ by spectral factorization [8]. Here, minimum phase spectral factor is used†.

4. REFORMULATION OF THE PROPOSED METHOD VIA LMI CONSTRAINTS

It is a common practice to approximate the semi-infinite constraints (4b)–(4d) by discretizing θ, like in [8, 9]. This does not affect the convexity of the resulting constraints. Though the discretization grid can be made very fine, unfortunately the discretized version of problem (4) can have numerical difficulties and still, the resulting constraints are approximations of (4b)–(4d) at best. To avoid these drawbacks, (4b)–(4d) are expressed as LMI constraints by rewriting the beampattern expression in (3) as

$$\mathbf{a}^T(\theta)\mathbf{r}_w = \Re\left\{\tilde{\mathbf{s}}^H(\Omega)\tilde{\mathbf{r}}_w\right\} \tag{5}$$

where $\tilde{\mathbf{s}}(\Omega) = [1 \;\; e^{j\Omega} \;\; \ldots \;\; e^{jN\Omega}]^T$, $\Omega = \frac{2\pi}{\lambda} d\cos\theta$, $\tilde{\mathbf{r}}_w = [r_w(0)\; 2r_w(1)\; \ldots\; 2r_w(N)]^T$, $(\cdot)^H$ defines Hermitian transpose operator and $\Re\{\cdot\}$ denotes a real operator [12]. With (5), the constraints (4b)–(4d) can be written as (6a)–(6c) according to the specific beampattern region:

$$\Re\left\{\tilde{\mathbf{s}}^H(\Omega)(-\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1)\right\} \ge 0, \Omega \in \boldsymbol{\Omega}_{\text{ML}} \cup \boldsymbol{\Omega}_{\text{SL}} \cup \boldsymbol{\Omega}_{\text{N}} \tag{6a}$$

$$\Re\left\{\tilde{\mathbf{s}}^H(\Omega)(\tilde{\mathbf{r}}_w - L^2(\Omega)\mathbf{e}_1)\right\} \ge 0, \Omega \in \boldsymbol{\Omega}_{\text{ML}} \tag{6b}$$

$$\Re\left\{\tilde{\mathbf{s}}^H(\Omega)\tilde{\mathbf{r}}_w\right\} \ge 0, \forall\boldsymbol{\Omega} \tag{6c}$$

† Many choices exist for the spectral factor and the minimum phase spectral factor may not always be the most appropriate choice.

where $\mathbf{\Omega}_{(\cdot)}$ denotes the set of Ω in an angular region defined by the subscript and the unit vector $\mathbf{e}_1$ is the first column of a $M \times M$ identity matrix. Note the change in the argument of $L^2(\cdot)$ and $U^2(\cdot)$. Next, Lemma 1 is applied to transform the constraints in (6) into their equivalent LMI forms‡.

Lemma 1 *With a* $\mathbf{p} \in \mathrm{R} \times \mathrm{C}^M$ *where* R *and* C^M *denote the sets of real numbers and complex* $M \times 1$ *vectors, respectively and that* $0 \leq \Omega_l < \Omega_u < 2\pi$*, the following sets*

$$\mathcal{K}(\Omega_l, \Omega_u) = \left\{\mathbf{p} | \Re\{\tilde{\mathbf{s}}^H(\Omega)\mathbf{p}\} \geq 0, \Omega \in [\Omega_l, \Omega_u]\right\} \tag{7a}$$

$$\bar{\mathcal{K}}(\Omega_l, \Omega_u) = \left\{\mathbf{p} | \Re\{\tilde{\mathbf{s}}^H(\Omega)\mathbf{p}\} \geq 0, \Omega \in [0, 2\pi) \backslash (\Omega_l, \Omega_u)\right\} \tag{7b}$$

$$\mathcal{K}(0, 2\pi) = \left\{\mathbf{p} | \Re\{\tilde{\mathbf{s}}^H(\Omega)\mathbf{p}\} \geq 0, \Omega \in [0, 2\pi)\right\} \tag{7c}$$

describe trigonometric polynomials that are non-negative over a segment of the unit circle $[\Omega_l, \Omega_u]$*, the complement of that segment* $[0, 2\pi) \backslash (\Omega_l, \Omega_u)$ *and on the unit circle* $[0, 2\pi)$*, respectively. By a generalized Positive Real Lemma [5], (7a)–(7c) can be converted into equivalent LMI forms as*

$$\mathcal{K}(\Omega_l, \Omega_u) = \{\mathbf{p} | \mathbf{p} + j\xi \boldsymbol{e}_1 = \bar{L}(\mathbf{X}) + \bar{\Lambda}(\mathbf{Z}; \Omega_l, \Omega_u), \exists \mathbf{X}, \mathbf{Z} \succeq 0\}, \tag{8a}$$

$$\bar{\mathcal{K}}(\Omega_l, \Omega_u) = \{\mathbf{p} | \mathbf{p} + j\xi \boldsymbol{e}_1 = \bar{L}(\mathbf{X}) - \bar{\Lambda}(\mathbf{Z}; \Omega_l, \Omega_u), \exists \mathbf{X}, \mathbf{Z} \succeq 0\}, \tag{8b}$$

$$\mathcal{K}(0, 2\pi) = \{\mathbf{p} | \mathbf{p} = \bar{L}(\mathbf{X}), \exists \mathbf{X} \succeq 0\}, \tag{8c}$$

respectively where ξ is an arbitrary real scalar and $\mathbf{X}, \mathbf{Z} \succeq 0$ denotes positive semi-definite Hermitian matrices. $\bar{L}(\cdot)$ and $\bar{\Lambda}(\cdot)$ are linear operators where their definitions have been deferred to the Appendix.

Applying Lemma 1 to our beamforming problem of interest, the proposed formulation (4) is transformed into

$$\begin{aligned}
\min_{M, \tilde{\mathbf{r}}_w} \quad & M \qquad\qquad (9) \\
\text{s.t.} \quad & -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_1\mathbf{e}_1 = \bar{L}(\mathbf{X}_1) - \bar{\Lambda}(\mathbf{Z}_1; \Omega_{ml}, \Omega_{mu}), \Omega \in \mathbf{\Omega}_{\mathrm{ML}} \\
& \tilde{\mathbf{r}}_w - L^2(\Omega)\mathbf{e}_1 + j\xi_2\mathbf{e}_1 = \bar{L}(\mathbf{X}_2) - \bar{\Lambda}(\mathbf{Z}_2; \Omega_{ml}, \Omega_{mu}), \Omega \in \mathbf{\Omega}_{\mathrm{ML}} \\
& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_3\mathbf{e}_1 = \bar{L}(\mathbf{X}_3) + \bar{\Lambda}(\mathbf{Z}_3; \Omega_{sl}, \Omega_{su}), \Omega \in \mathbf{\Omega}_{\mathrm{SL}} \\
& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_4\mathbf{e}_1 = \bar{L}(\mathbf{X}_4) + \bar{\Lambda}(\mathbf{Z}_4; \Omega_{nl}, \Omega_{nu}), \Omega \in \mathbf{\Omega}_{\mathrm{N}} \\
& \tilde{\mathbf{r}}_w = \bar{L}(\mathbf{X}_5), \forall \mathbf{\Omega} \\
& M \in \mathrm{Z}_+, \forall \mathbf{X} \succeq 0, \forall \mathbf{Z} \succeq 0
\end{aligned}$$

where Ω_{ml} and Ω_{mu} define the lower and upper boundaries of $\mathbf{\Omega}_{\mathrm{ML}}$, respectively. This notational convention applies to $\mathbf{\Omega}_{\mathrm{SL}}$ and $\mathbf{\Omega}_{\mathrm{N}}$ as well. The mainlobe constraints in (9) obtain a broad mainlobe and assume it is centred at 90° hence (8b) is used. Since $\mathbf{\Omega}_{\mathrm{SL}}$ and $\mathbf{\Omega}_{\mathrm{N}}$

‡ For related development on trigonometric polynomials, please refer to [10, 13, 14].

fall outside this region, then (8a) is used. The sets of $\mathbf{X}$, $\mathbf{Z}$ and ξ with different subscripts are used to differentiate the constraints in (9). The optimization variable here is $\tilde{\mathbf{r}}_w$ and not $\mathbf{r}_w$ as in (4), so the constraint (4e) can be omitted from (9). After (9) is solved, the solution $\tilde{\mathbf{r}}_w$ is rearranged to obtain $\mathbf{r}_w$ according to (4e) and (5), after which spectral factorization is applied on $\mathbf{r}_w$ to derive the beamforming weights w_i.

If beampattern requirements specify a mainlobe with unity gain in a desired direction θ_0 (or Ω_0), then the mainlobe constraints in (9) are not applicable. They should be replaced by $\Re\{\tilde{\mathbf{s}}^H(\Omega_0)\tilde{\mathbf{r}}_w\} = 1$ as

$$\begin{aligned}
\min_{M,\tilde{\mathbf{r}}_w} \quad & M \qquad (10)\\
\text{s.t.} \quad & \Re\{\tilde{\mathbf{s}}^H(\Omega_0)\tilde{\mathbf{r}}_w\} = 1,\\
& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_1\mathbf{e}_1 = \bar{L}(\mathbf{X}_1) + \bar{\Lambda}(\mathbf{Z}_1; \Omega_{sl}, \Omega_{su}), \Omega \in \boldsymbol{\Omega}_{\mathrm{SL}}\\
& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_2\mathbf{e}_1 = \bar{L}(\mathbf{X}_2) + \bar{\Lambda}(\mathbf{Z}_2; \Omega_{nl}, \Omega_{nu}), \Omega \in \boldsymbol{\Omega}_{\mathrm{N}}\\
& \tilde{\mathbf{r}}_w = \bar{L}(\mathbf{X}_3), \forall\boldsymbol{\Omega}\\
& M \in \mathrm{Z}_+, \forall\mathbf{X} \succeq 0, \forall\mathbf{Z} \succeq 0.
\end{aligned}$$

Different from (4), the proposed formulations (9) and (10) impose finite LMI constraints on the beampattern. However, (4), (9) and (10) are all quasi-convex optimization problems consisting of convex constraints and a quasi-convex objective function. The discontinuous objective function $f(M) = M$ is quasi-convex since the modified Jensen's inequality is proven to hold:

$$f(\alpha m_1 + (1-\alpha)m_2) \leq \max\{f(m_1), f(m_2)\}, \qquad (11)$$

$$\alpha m_1 + (1-\alpha)m_2 \leq \max\{m_1, m_2\}, \qquad (12)$$

$$\alpha(m_1 - m_2) + m_2 \leq m_2, \qquad (13)$$

where $0 \leq \alpha \leq 1$, $\max\{\cdot,\cdot\}$ is an operator that gives the maximum value of its arguments, m_1 and m_2 are elements in the domain of the objective function. The last two inequalities hold with $m_1 < m_2$.

5. IMPLEMENTATION OF THE PROPOSED METHOD

The sub-level sets of quasi-convex optimization problems are convex [15], thus the globally optimum solutions of the proposed formulations (4), (9) and (10) can be found by decomposing them into a sequence of convex feasibility problems. We use (10) as an example. Suppose M_{opt} is the globally optimum solution of (10). Consider the decomposition of (10) into (14) with the same constraints but at M_c

number of sensors.

$$\begin{aligned}
\text{Find} \quad & \tilde{\mathbf{r}}_w \qquad (14)\\
\text{s.t.} \quad & \Re\left\{\tilde{\mathbf{s}}^H(\Omega_0)\tilde{\mathbf{r}}_w\right\} = 1,\\
& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_1\mathbf{e}_1 = \bar{L}(\mathbf{X}_1) + \bar{\Lambda}(\mathbf{Z}_1; \Omega_{sl}, \Omega_{su}), \Omega \in \mathbf{\Omega}_{\mathrm{SL}}\\
& -\tilde{\mathbf{r}}_w + U^2(\Omega)\mathbf{e}_1 + j\xi_2\mathbf{e}_1 = \bar{L}(\mathbf{X}_2) + \bar{\Lambda}(\mathbf{Z}_2; \Omega_{nl}, \Omega_{nu}), \Omega \in \mathbf{\Omega}_{\mathrm{N}}\\
& \tilde{\mathbf{r}}_w = \bar{L}(\mathbf{X}_3), \forall\mathbf{\Omega}\\
& \forall\mathbf{X} \succeq 0, \forall\mathbf{Z} \succeq 0.
\end{aligned}$$

The problem (14) is convex which can be solved via a IPM solver to determine if the constraints are feasible at M_c number of sensors. If so, there exists a non-empty feasible set and it finds a solution point $\tilde{\mathbf{r}}_w$ in the set, implying that $M_{opt} \leq M_c$. Feasible beamforming weights w_i can be found after spectral factorization is performed on $\mathbf{r}_w$ (by transforming $\tilde{\mathbf{r}}_w$). Otherwise, the feasible set is empty and the solver issues a certificate of infeasibility implying that $M_{opt} > M_c$.

As such, the proposed formulations can be solved by an iterative procedure known as bisection search. It starts with a search interval $[M_l, M_u]$ assumed to contain M_{opt} where $M_l < M_u$. The feasibility problem (14) is solved at $M_c = \frac{(M_l+M_u)}{2}$, to determine if M_{opt} resides in the lower or upper half of the interval, after which the search interval is updated accordingly. This produces a new interval containing M_{opt} at half of the previous interval width. The above steps are repeated until the stop criterion is reached where the search interval converges to a globally optimum M_{opt} value, i.e., $M_u - M_l < 1$.

6. CONVERGENCE OF THE PROPOSED METHOD

In the previous section, it is assumed that M_{opt} exists in the search interval $[M_l, M_u]$. This insinuates that the problem (14) has to be feasible at M_u sensors. Otherwise, M_{opt} is not located in $[M_l, M_u]$ and (14) will be infeasible at all the tested values in $[M_l, M_u]$.

To prevent the proposed method from searching through $[M_l, M_u]$ which does not contain M_{opt}, a simple suggestion is to check the feasibility of (14) at M_u number of sensors at the start of the search. If (14) is infeasible at M_u sensors, this infeasible M_u value is assigned to M_l and M_u is re-assigned with M_u' ($M_u < M_u'$) so that the new search interval $[M_l, M_u]$ does not widen unnecessarily. Given the nature of bisection search, an appropriate choice of M_u' can be twice the value of M_u. This is done until the feasibility check is passed at M_u number of sensors. In so doing, the proposed method ensures that it is searching through an interval which contains M_{opt} and thereby guarantees convergence. The implementation procedure of the proposed method is summarized here.

(i) Choose a search interval $[M_l, M_u]$. M_l can be set to 1.

(ii) Check the feasibility of Equation (14) at M_u number of sensors. If it is feasible, proceed to step (iii). Otherwise, update M_l with the value of M_u. Update M_u with M'_u whose value is greater than M_u. Repeat step (ii). This ensures that the proposed method is searching through $[M_l, M_u]$ where $M_l < M_{opt} \leq M_u$ and that the search interval does not widen unnecessarily.

(iii) Let $M_c = \lceil \frac{M_l + M_u}{2} \rceil$. Since the number of sensors has to be an integer, a $\lceil \cdot \rceil$ operator is used to round up $(\frac{M_l + M_u}{2})$, which can be a non-integer.

(iv) Check the feasibility of Equation (14) at M_c number of sensors.

(v) If Equation (14) is reported feasible, update M_u with the value of M_c. Otherwise, update M_l with the value of M_c.

(vi) Repeat steps (iii)–(v) until the stop criterion $(M_u - M_l < 1)$ is reached.

Suppose the proposed method begins with $[M_l, M_u]$ known in advance to contain M_{opt}, then the number of iterations before it stops is $\lceil \log_2(M_u - M_l) \rceil$ [15].

7. SIMULATION RESULTS

The proposed formulations (9) and (10) are applied to find the minimum number of sensors needed for two different beampattern designs (specifications are shown in dotted lines) and their corresponding beamforming weights. A ULA of M isotropic sensors with a 0.5λ spacing is used. The search interval is set to $[M_l, M_u] = [1, 32]$ where both designs are tested feasible at 32 sensors. The beampatterns for both designs using $M_u = 32$ sensors by the method of [5] are shown.

First, a beampattern with a broad mainlobe and suppressed sidelobes is desired. The sidelobe region is $[0°, 69.4°] \cup [110.6°, 180°]$ to be suppressed by -25 dB. The mainlobe width is set to 22° and its response ripple is to be within 0.48 dB. The proposed formulation (9) (without null constraint) is used to achieve the beampattern in Fig. 1 and the minimum number of sensors required is $M_{opt} = 14$.

Next, a beampattern with a narrow mainlobe, controlled sidelobes and nulls is desired, so the proposed formulation (10) is used. The sidelobe region is $[0°, 79.55°] \cup [100.45°, 180°]$ to be suppressed by -40 dB. The null regions are $[50°, 60°] \cup [120°, 130°]$ with an attenuation

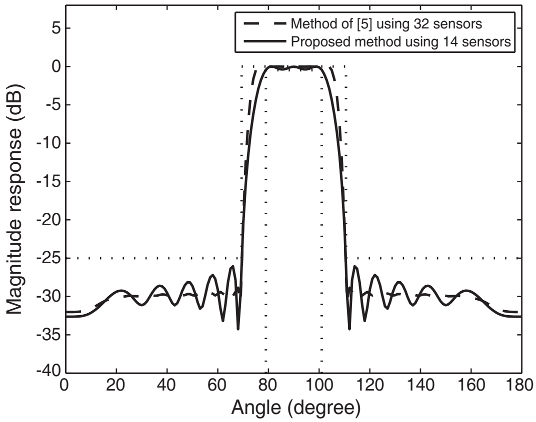

Figure 1. Beampatterns obtained by the method of [5] with 32 sensors and proposed method with 14 sensors. The beampattern has to be lower than outer dotted lines and higher than inner dotted lines.

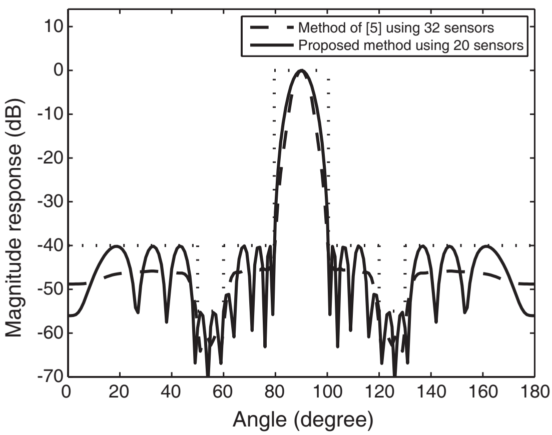

Figure 2. Beampatterns obtained by the method of [5] with 32 sensors and proposed method with 20 sensors. The beampattern has to be lower than the dotted lines.

level of -55 dB. The resulting beampattern is shown in Fig. 2 and $M_{opt} = 20$.

Given the same prior information that the optimum solution M_{opt} lies in $[M_l, M_u] = [1, 32]$ as in the examples, the direct application of the method of [5] to find the minimum number of sensors for beampattern designs would involve either increasing the number of sensors one by one from $M_l = 1$ or reducing the number of sensors one by one from $M_u = 32$. However, this results in a large number of iterations (20 and 14 iterations for the first and second designs, respectively for the latter case). Both ways are not efficient and the number of iterations required before stopping is unknown. In contrast, the advantages of the proposed method are that the minimum number of array sensors for beampattern designs is computed in a systematic way via bisection and the number of iterations before it stops, is known. For the proposed method, only 5 iterations are required in both cases.

Note that in the case of applying the method of [5] using more array sensors than required, the extra degrees of freedom are well-utilized as the generated beampatterns in Figs. 1–2 have exceeded the beampattern specifications. Such a performance is due to the properties of the IPM.

8. CONCLUSION

A beampattern synthesis formulation has been proposed to find the minimum number of array sensors needed and the corresponding beamforming weights, systematically. The proposed method employs LMI constraints on the beampattern so that the prescribed specifications are satisfied precisely. An iterative procedure is used to decompose the proposed quasi-convex formulation into a systematic sequence of convex feasibility problems. The proposed method is guaranteed to find the minimum number of array sensors so long as this solution lies in the search interval, which can be easily ensured at the start of the search. The effectiveness of the proposed method is shown with two design examples via computer simulations.

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APPENDIX A.

To allow a concise LMI description of the cones in (7), a Toeplitz matrix $\mathbf{T}_{k,N}$ is defined as

$$[\mathbf{T}_{k,N}]_{i,j} = \begin{cases} 1, & \text{if } i = j + k \\ 0, & \text{otherwise} \end{cases}$$

for $i, j \in [0, 1, \ldots, N]$. The notation $\langle \mathbf{T}_{k,N}, \mathbf{X} \rangle = \sum_{l=0}^{N-k} X_{l+k,l}$ means the sum of the elements on the kth lower off-diagonal of $\mathbf{X}$. The adjoint operator $\bar{L}(\cdot)$ is expressed as $y = \bar{L}(\mathbf{X})$ with $y_0 = \langle \mathbf{T}_{0,N}, \mathbf{X} \rangle$, $y_k = 2\langle \mathbf{T}_{k,N}, \mathbf{X} \rangle$, for $k = 1, \ldots, N$.

With $0 \leq \Omega_l < \Omega_u < 2\pi$, the vector $\mathbf{d}(\Omega_l, \Omega_u)$ is defined as:

$$\mathbf{d}(\Omega_l, \Omega_u) = \begin{cases} \begin{bmatrix} \cos(\Omega_l) + \cos(\Omega_u) - \cos(\Delta\Omega) - 1 \\ \left(1 - e^{j\Omega_l}\right)\left(e^{j\Omega_u} - 1\right) \end{bmatrix}, & \text{if } \Omega_l > 0 \\ \begin{bmatrix} -\sin(\Omega_u) \\ j(1 - e^{j\Omega_u}) \end{bmatrix}, & \text{if } \Omega_l = 0 \end{cases}$$

where $\Delta\Omega = \Omega_u - \Omega_l$.

With $\mathbf{d}(\Omega_l, \Omega_u)$, the adjoint operator $\bar{\Lambda}(\cdot)$ is expressed by $y = \bar{\Lambda}(\mathbf{X})$:

$$\begin{aligned} y_0 &= d_0(\Omega_l, \Omega_u)\langle \mathbf{T}_{0,N-1}, \mathbf{X} \rangle + d_1^*(\Omega_l, \Omega_u)\langle \mathbf{T}_{1,N-1}, \mathbf{X} \rangle, \\ y_k &= 2d_0(\Omega_l, \Omega_u)\langle \mathbf{T}_{k,N-1}, \mathbf{X} \rangle + d_1(\Omega_l, \Omega_u)\langle \mathbf{T}_{k-1,N-1}, \mathbf{X} \rangle \\ &\quad + d_1^*(\Omega_l, \Omega_u)\langle \mathbf{T}_{k+1,N-1}, \mathbf{X} \rangle, \quad k = 1, \ldots, N-2, \\ y_{N-1} &= 2d_0(\Omega_l, \Omega_u)\langle \mathbf{T}_{N-1,N-1}, \mathbf{X} \rangle + d_1(\Omega_l, \Omega_u)\langle \mathbf{T}_{N-2,N-1}, \mathbf{X} \rangle, \\ y_N &= d_1(\Omega_l, \Omega_u)\langle \mathbf{T}_{N-1,N-1}, \mathbf{X} \rangle. \end{aligned} \tag{A1}$$

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