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INTRODUCING FICTITIOUS CURRENTS FOR CALCULATING ANALYTICALLY THE ELECTRIC FIELD IN CYLINDRICAL CAPACITORS

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Abstract—The aim of this paper is to show the interest of using equivalence models for calculating the electric field produced by cylindrical capacitors with dielectrics. To do so, we use an equivalent model, based on the dual Maxwell's Equations for calculating the two electric field components created inside the capacitor and outside it. This equivalent model uses fictitious currents generating a electric vector potential that allows us to determine the electric field components in all points in space. The electric field produced by charge distributions as capacitor with dielectrics is generally determined by using the coulombian model. Indeed, it is well known that the electric field derives from a scalar potential. By using the Maxwell's equations, this scalar potential is in fact linked to the existence of charge distributions that are generally located on the faces of the capacitors. However, this last model does not allow us to obtain reduced analytical expressions since it involves the calculation of charge volume density appearing in the dielectric material for arc-shaped cylindrical topologies. Consequently, it is interesting to look for another approach that gives analytical expressions with a lower computational cost. In this paper, we show that the use of fictitious currents instead of charges allow us to obtain 3D analytical reduced expressions with a lower computational cost. This analytical approach is compared to the coulombian model for showing the equivalence between the two approaches.

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1. INTRODUCTION

The Maxwell's Equations are widely used for solving magnetostatics or electrostatics problems. These local equations characterize the magnetic or electric fields produced by charge or current distributions. In magnetostatics or electrostatics problems, the Maxwell's equations can be divided in two equation groups (Eqs. (1) and (2)). This separation is interesting for calculations involving dielectrics or magnetized materials.

$$\nabla\vec{D} = \rho_f \quad \nabla\times\vec{E} = \vec{0} \tag{1}$$

$$\nabla\vec{B} = 0 \quad \nabla\times\vec{H} = \vec{j} \tag{2}$$

where $\vec{D}$ is the electric displacement vector, ρ_f is the free electric charge volume density, $\vec{E}$ is the electric field vector, $\vec{B}$ is the magnetic induction field vector, $\vec{H}$ is the magnetic field vector and $\vec{j}$ is the current density.

The previous equations involve that the magnetic field derives from a vector potential though the electric field derives from a scalar potential [1]. Consequently, the magnetic field created by currents are often modelled by using the Biot-Savart Law [2, 3] or by using the amperian current model [4–8]. In the other hand, the electric field is often determined by using the coulombian model, the equivalent surface charges and dipoles [9–14]. However, in some cases, it is more interesting to use equivalent charge distributions yielding the same field as the one produced by source distributions. For example, the magnetic field created by permanent magnets has been largely studied by using the coulombian model [15–18]. For arc-shaped permanent magnets, the coulombian model is suitable for obtaining analytical expressions based on special functions [19–25]. Consequently, it seems to be more judicious to introduce in some cases equivalent models for calculating the magnetic field created by arc-shaped permanent magnets. In short, the dual Maxwell's Equations involving the coulombian model of a magnet are the following:

$$\begin{aligned} \nabla\vec{B} &= \rho^* \quad \nabla\times\vec{H} = \vec{0} \\ \nabla\vec{D} &= 0 \quad\;\; \nabla\times\vec{E} = \vec{j}^* \end{aligned} \tag{3}$$

where ρ^* is a magnetic charge volume density and $\vec{j}^*$ is a fictitious current density. The coulombian model of a magnet corresponds to the case when $\nabla\vec{B} = \rho^*$. Furthermore, for linear media, the two following equations are useful to define the main parameters.

$$\vec{B} = \mu_0\vec{H} + \vec{J} \quad \vec{D} = \epsilon_0\vec{E} + \vec{P} \tag{4}$$

By using the analogy between the classical Maxwell's equations and the dual Maxwell's equations, we can introduce equivalent currents for calculating the electric field in all points in space. These equivalent currents are linked to the existence of a vector potential.

1.1. Introducing the Vector Potential from the Fictitious Currents

The previous section introduces fictitious currents for calculating the electric field created by capacitors with dielectrics by using the analogy between the classical Maxwell's equations and the dual Maxwell's equations. The use of fictitious currents can be implemented to all kinds of geometries (parallelepipedic geometries, cylindrical geometries, spherical geometries). However, the way of treating each configuration is directly linked to its topology. For example, if we take a parallelepipedic capacitor, the coulombian model or the fictitious currents lead to fully analytical expressions of the electric field. Consequently, the choice of the approach is not of great importance for parallelepipedic topologies. For cylindrical topologies, the problem is different.

As we have introduced equivalent fictitious currents, we can define a vector potential $\vec{A}$ that verifies:

$$\vec{E} = \nabla \times \vec{A} \quad \nabla \vec{E} = 0 \tag{5}$$

In magnetostatics, the vector potential $\vec{A}_m$ is defined by the following relation [26].

$$\vec{A}_m = \frac{1}{4\pi} \int_V \vec{J} \times \nabla \left(\frac{1}{\left| \vec{r} - \vec{r'} \right|} \right) d\tilde{V} \tag{6}$$

It is noted that this vector potential verifies $\vec{H} = \nabla \times \vec{A}_m$. Moreover, is is emphasized that the Hamiltonian operator is with respect to the source coordinates in this paper, as in [26].

By using the analogy between the vector potential in magnetostatics and electrostatics, we can define a vector potential in electrostatics as follows:

$$\vec{A} = \frac{1}{4\pi} \int_V \vec{P} \times \nabla \left(\frac{1}{\left| \vec{r} - \vec{r'} \right|} \right) d\tilde{V} \tag{7}$$

As stated in [26], we can use the following identity:

$$\nabla \times \left(\frac{\vec{J}}{\left| \vec{r} - \vec{r'} \right|} \right) = \frac{1}{\left| \vec{r} - \vec{r'} \right|} \left(\nabla \times \vec{J} \right) - \vec{J} \times \nabla \left(\frac{1}{\left| \vec{r} - \vec{r'} \right|} \right) \tag{8}$$

Therefore, the vector potential in magnetostatics can be written as follows:

$$\vec{A}_m = \frac{1}{4\pi}\left[\int_V -\nabla\times\left(\frac{\vec{J}}{\left|\vec{r}-\vec{r'}\right|}d\tilde{V}\right)+\int_V \frac{\nabla\times\vec{J}}{\left|\vec{r}-\vec{r'}\right|}d\tilde{V}\right] \tag{9}$$

By applying the previous relation to the case of the vector potential in electrostatics, we obtain:

$$\vec{A} = \frac{1}{4\pi}\left[\int_V -\nabla\times\left(\frac{\vec{P}}{\left|\vec{r}-\vec{r'}\right|}d\tilde{V}\right)+\int_V \frac{\nabla\times\vec{P}}{\left|\vec{r}-\vec{r'}\right|}d\tilde{V}\right] \tag{10}$$

By using the Stokes theorem, we obtain the two final expressions:

$$\vec{A}_m = \frac{\epsilon_0}{4\pi}\left[\int_S \frac{\frac{\vec{J}}{\epsilon_0}\times\vec{n}}{\left|\vec{r}-\vec{r'}\right|}d\tilde{S}+\int_V \frac{\nabla\times\left(\frac{\vec{J}}{\epsilon_0}\right)}{\left|\vec{r}-\vec{r'}\right|}d\tilde{V}\right] \tag{11}$$

$$\vec{A} = \frac{\epsilon_0}{4\pi}\left[\int_S \frac{\frac{\vec{P}}{\epsilon_0}\times\vec{n}}{\left|\vec{r}-\vec{r'}\right|}d\tilde{S}+\int_V \frac{\nabla\times\left(\frac{\vec{P}}{\epsilon_0}\right)}{\left|\vec{r}-\vec{r'}\right|}d\tilde{V}\right] \tag{12}$$

Basically, we find the two equivalent surface and volume currents distributions giving the same electric field as the one obtained in the coulombian approach.

For the rest of this paper, we use the notation X_{magn} when we consider a charged distribution related to magnetostatics and X_{elec} for electrostatics. According to the coulombian model, a permanent magnet can be represented by fictitious magnetic pole surface and volume densities that are expressed as follows:

$$\sigma^*_{magn} = \vec{J}\cdot\vec{n} \quad \rho^*_{magn} = -\nabla\cdot\vec{J} \tag{13}$$

Applying the coulombian model in the case of dielectric capacitors, we obtain:

$$\sigma^*_{elec} = \vec{P}\cdot\vec{n} \quad \rho^*_{elec} = -\nabla\cdot\vec{P} \tag{14}$$

The amperian current model used for calculating the magnetic field created by permanent magnets implies the determination of fictitious currents of surface and volume densities:

$$\vec{K}^*_{magn} = \frac{\vec{J}\times\vec{n}}{\mu_0} \quad \vec{j}^*_{magn} = \frac{\nabla\times\vec{J}}{\mu_0} \tag{15}$$

It is emphasized here that the previous relations are commonly used in the literature. In this paper, we choose to introduce fictitious currents in electrostatics for calculating the electric field produced in cylindrical capacitors with dielectrics.

$$\vec{K}^*_{elec} = \frac{\vec{P} \times \vec{n}}{\epsilon_0} \quad \vec{j}^*_{elec} = \frac{\nabla \times \vec{P}}{\epsilon_0} \tag{16}$$

The previous relations must verify the following equations:

$$\begin{aligned}
\int_{S_1} \sigma^*_{elec} d\tilde{S} + \int_{S_2} \sigma^*_{elec} d\tilde{S} + \int_V \rho^*_{elec} d\tilde{V} = 0 \\
\int_{S_1} \sigma^*_{magn} d\tilde{S} + \int_{S_2} \sigma^*_{magn} d\tilde{S} + \int_V \rho^*_{magn} d\tilde{V} = 0 \\
\int_{S_1} \vec{K}^*_{elec} d\tilde{S} + \int_{S_2} \vec{K}^*_{elec} d\tilde{S} + \int_V \vec{j}^*_{elec} d\tilde{V} = 0 \\
\int_{S_1} \vec{K}^*_{magn} d\tilde{S} + \int_{S_2} \vec{K}^*_{magn} d\tilde{S} + \int_V \vec{j}^*_{magn} d\tilde{V} = 0
\end{aligned} \tag{17}$$

2. APPLICATION OF THE CASE OF A CYLINDRICAL CAPACITOR WITH DIELECTRIC MATERIAL

2.1. Notation and Geometry

We present in this section the 3D analytical expression of the electric field created by a cylindrical capacitor with dielectric material. This is typically an academic example for showing the properties of the Maxwell's equations and the interest of using equivalent models. To do so, let us first consider the representation shown in Figure 1. The outer face is charged with the surface density $+\sigma = \frac{q}{S_2}$ and the inner face is charged with the surface density $\sigma = \frac{-q}{S_1}$. The dielectric has also a charge volume density σ_v that is given by

$$\sigma_v = -\nabla \vec{P} \tag{18}$$

By using the vector potential, the calculation of σ_v is not required because it does not appear in the basic equations describing fictitious currents generating the same electric field.

2.2. Radial Component Determined with the Amperian Current Model

The radial component $E_r(r, z)$ can be determined by calculating the projection of $\vec{H}(\vec{r})$ along $\vec{u}_r$:

$$E_r(r, z) = \vec{E}(\vec{r}) \bullet \vec{u}_r = \left(\frac{1}{\epsilon_0} \nabla \times \vec{A}(\vec{r})\right) \bullet \vec{u}_r \tag{19}$$

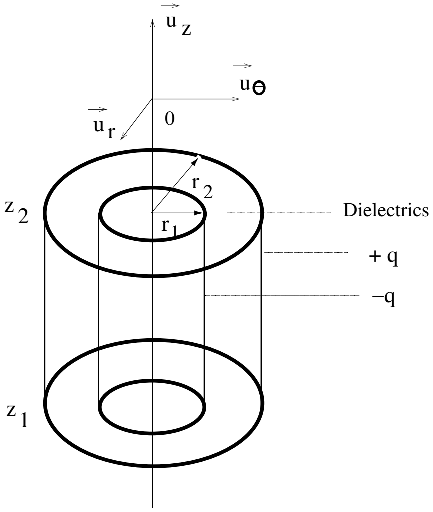

Figure 1. Representation of a cylindrical capacitor charged with two arc-shaped planes.

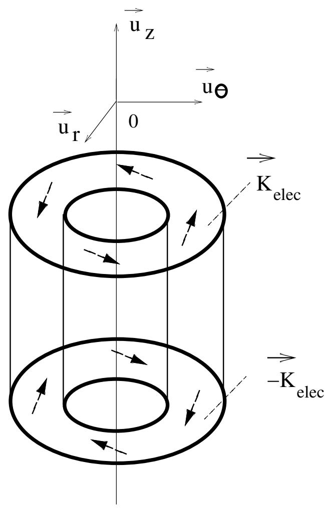

Figure 2. Representation of a cylindrical capacitor modelled by fictitious currents flowing on the upper and lower faces.

By using the notations in Figures 1 and 2, and by using (16) we have:

$$E_r(r,z) = \frac{P}{4\pi\epsilon_0}\int_{r_1}^{r_2}\int_0^{2\pi}\frac{(z-z_2)\cos(\tilde{\theta})}{\left|\vec{r}-\vec{r'}\right|^3}\tilde{r}d\tilde{r}d\tilde{\theta} - \frac{P}{4\pi\epsilon_0}\int_{r_1}^{r_2}\int_0^{2\pi}\frac{(z-z_1)\cos(\tilde{\theta})}{\left|\vec{r}-\vec{r'}\right|^3}\tilde{r}d\tilde{r}d\tilde{\theta} \quad (20)$$

We also use the following relation:

$$\frac{1}{\left|\vec{r}-\vec{r'}\right|^3} = \frac{1}{\left(r^2+\tilde{r}^2-2r\tilde{r}\cos(\tilde{\theta})+(z-\tilde{z})^2\right)^{\frac{3}{2}}} \quad (21)$$

It is emphasized here that the interest of using the amperian current model for this configuration lies in the fact that only two integrals must be calculated. Consequently, we obtain an analytical expression of the radial field component based on elliptic integrals.

After mathematical manipulations and by using Mathematica, the arguments of the elliptic integrals used are defined as follows:

$$\begin{aligned}\phi_1^{+,-} &= \frac{(b+2e)x}{bx \pm \sqrt{2}\sqrt{xe^2(x-c)}} \\ \phi_2 &= i\sinh^{-1}\left(\sqrt{\frac{-1}{b+2e}}\sqrt{b-2e\cos(\tilde{\theta})}\right) \\ \phi_3 &= \frac{b+2e}{b-2e}\end{aligned} \tag{22}$$

It is noted that we mainly use the elliptic integrals of the second and third kind for the calculation of the radial component. These special functions have also been used in [19, 20]. However, no further numerical integrations are required here. In short, the radial component $E_r(r,z)$ can be expressed as follows:

$$E_r(r,z) = \frac{P}{4\pi\epsilon_0}\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}\left(g(i,k,2\pi) - g(i,k,0)\right) \tag{23}$$

with

$$g(i,k,\tilde{\theta}) = 2(z-z_k)f(r^2+(z-z_k)^2, r^2+r_i^2+(z-z_k)^2, rr_i, -r^2-2(z-z_k)^2, \tilde{\theta}) \tag{24}$$

where

$$\begin{aligned}f(a,b,e,c,x,\tilde{\theta}) = {} & \eta\left(2\xi_1(2ce^2+\xi_2)\right)\mathbf{F}^*\left[\phi_2,\phi_3\right] \\ & +\eta\left(-e^2(c-x)(bx\sqrt{2}+2\xi_1)\right)\mathbf{\Pi}^*\left[\phi_1^+,\phi_2,\phi_3\right] \\ & +\eta\left(e^2(c-x)(bx\sqrt{2}-2\xi_1)\right)\mathbf{\Pi}^*\left[\phi_1^-,\phi_2,\phi_3\right] \\ & -2\eta ax\left(xe^2-ce^2\sqrt{2}+b\xi_1\right)\mathbf{\Pi}^*\left[\phi_1^+,\phi_2,\phi_3\right] \\ & -2\eta ax\left(-xe^2+ce^2\sqrt{2}+b\xi_1\right)\mathbf{\Pi}^*\left[\phi_1^-,\phi_2,\phi_3\right]\end{aligned} \tag{25}$$

where $\mathbf{F}^*[x,y]$ and $\mathbf{\Pi}^*[x,y,z]$ are the incomplete elliptic integrals of the second and third kind that have been used in previous papers [19, 20]. In addition, the parameters ξ_1, ξ_2, η are defined as follows:

$$\begin{aligned}\xi_1 &= \sqrt{e^2x(x-c)} \\ \xi_2 &= x(b^2-2e^2) \\ \eta &= \frac{i\sqrt{\frac{-e^2\sin(\tilde{\theta})^2}{(b-2e)^2}}\csc(\tilde{\theta})}{2\sqrt{\frac{-1}{b+2e}}x\xi_1(2ce^2+\xi_2)}\end{aligned} \tag{26}$$

2.3. Axial Component $E_z(r,z)$

The axial component $E_z(r,z)$ can be determined by calculating the projection of $\vec{E}(\vec{r})$ along $\vec{u}_z$:

$$E_z(r,z) = \vec{E}(\vec{r}) \bullet \vec{u}_z = \left(\frac{1}{\epsilon_0}\nabla \times \vec{A}(\vec{r})\right) \bullet \vec{u}_z \tag{27}$$

By using the notations in Figures 1 and 2, and by using (16) we have:

$$\begin{aligned} E_z(r,z) &= \frac{P}{4\pi\epsilon_0}\int_{r_1}^{r_2}\int_0^{2\pi} \frac{r\cos(\tilde{\theta})}{\left|\vec{r}-\vec{\tilde{r}}(z_2)\right|^3}\tilde{r}d\tilde{r}d\tilde{\theta} \\ &\quad -\frac{P}{4\pi\epsilon_0}\int_{r_1}^{r_2}\int_0^{2\pi} \frac{r\cos(\tilde{\theta})}{\left|\vec{r}-\vec{\tilde{r}}(z_1)\right|^3}\tilde{r}d\tilde{r}d\tilde{\theta} \end{aligned} \tag{28}$$

where

$$\frac{1}{\left|\vec{r}-\vec{\tilde{r}}(z_j)\right|^3} = \frac{1}{\left(r^2+\tilde{r}^2-2r\tilde{r}\cos(\tilde{\theta})+(z-z_j)^2\right)^{\frac{3}{2}}} \tag{29}$$

After mathematical manipulations, we find the following reduced semi-analytical expression of the axial component $E_z(r,z)$:

By using the relation $a_{ik} = (r-r_i)^2 + (z-z_k)^2$, this axial component $H_z(r,z)$ is thus given by:

$$E_z(r,z) = \frac{P}{4\pi\epsilon_0}\sum_{i=1}^{2}\sum_{k=1}^{2}(-1)^{i+k}\left(I_1(r,z)+I_2(r,z)\right) \tag{30}$$

with

$$I_1(r,z) = \frac{-4r_i}{\sqrt{a_{ik}}}\mathbf{K}^*\left[\frac{-4rr_i}{a_{ik}}\right] \tag{31}$$

where $\mathbf{K}^*[x]$ is the complete elliptic integral of the first kind [19].

$$I_2(r,z) = \int_0^{2\pi}\ln\left[r_i - r\cos(\tilde{\theta}) + \sqrt{r^2+r_i^2+(z-z_k)^2-2rr_i\cos(\tilde{\theta})}\right]d\tilde{\theta} \tag{32}$$

2.4. Comparison between the Classical Coulombian Model and the Fictitious Current Model

We illustrate in Figures 3 and 4 the equivalence between the two approaches (Amperian current model and coulombian model) in the case of the electric field created by the capacitor with dielectric material. For this purpose, we use the following parameters in Figure 3: $r_{in} = 0.025\,\mathrm{m}$, $r_{out} = 0.026\,\mathrm{m}$, $z = 0.003\,\mathrm{m}$, $P = 1.1\ 10^{-4}\,\mathrm{C/m^2}$, $z = 0.02\,\mathrm{m}$.

We use the following parameters in Figure 4: $r_{in} = 0.025\,\mathrm{m}$, $r_{out} = 0.026\,\mathrm{m}$, $z = 0.003\,\mathrm{m}$, $P = 1.1\ 10^{-4}\,\mathrm{C/m^2}$, $z = 0.02\,\mathrm{m}$.

2.5. Comparison of the Computational Cost between the Classical Coulombian Model and the Fictitious Current Model

Another important comparison between the amperian current model and the coulombian model is the computational cost for calculating the electric field. We have compared the two models for the two electric field components by using the coulombian model and the amperian

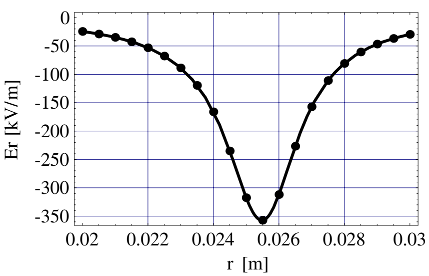

Figure 3. Representation of the radial electric field versus the radial observation point for the following dimensions: $r_{in} = 0.025\,\mathrm{m}$, $r_{out} = 0.026\,\mathrm{m}$, $z = 0.003\,\mathrm{m}$, $P = 1.1\ 10^{-4}\,\mathrm{C/m^2}$, $z = 0.02\,\mathrm{m}$. (Line = amperian current model, Points = coulombian model).

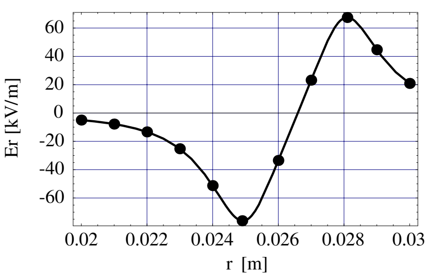

Figure 4. Representation of the axial electric field versus the radial observation point for the following dimensions: $r_{in} = 0.025\,\mathrm{m}$, $r_{out} = 0.028\,\mathrm{m}$, $z = 0.003\,\mathrm{m}$, $P = 1.1\ 10^{-4}\,\mathrm{C/m^2}$, $z = 0.001\,\mathrm{m}$. (Line = amperian current model, Points = coulombian model).

Table 1. Comparison of the computational cost for calculating the two electric field components with the coulombian approach and the amperian current approach; the two components are determined in the following observation points: $E_z(r = 0.015\,\mathrm{m}, z = 0.001\,\mathrm{m})$ and $E_r(r = 0.002\,\mathrm{m}, z = 0.0035\,\mathrm{m})$.

Component	**Coulombian model**	**Amperian model**
E_z	0.015 s	0.0001 s
E_r	0.016 s	0.0002 s

current model. The Table 1 clearly shows that the amperian current approach is more appropriate than the colombian model for calculating the electric field produced by dielectrics.

3. CONCLUSION

This paper presents an analytical method, based on the dual Maxwell's equations for calculating the electric field in capacitors using dielectrics. The interest of such an approach lies in the fact that no volume integrals are required for the determination of the two electric field components though such a volume integral is required with the classical coulombian model. The comparison between two approaches shows the accuracy of the amperian current model used in electrostatics. Furthermore, the comparison of the computational cost for calculating the two electric field components clearly shows that the amperian current model is more appropriate than the coulombian model. More generally, we can say that the proposed technique can be implement to realistic 1D, 2D, 3D structures with dielectrics of low or high permittivity.

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