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AVERAGE INTENSITY AND SPREADING OF PARTIALLY COHERENT STANDARD AND ELEGANT LAGUERRE-GAUSSIAN BEAMS IN TURBULENT ATMOSPHERE

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Abstract—Analytical expressions for the average intensity, mean-squared beam width and angular spread of partially coherent standard and elegant Laguerre-Gaussian (LG) beams propagating in turbulent atmosphere are derived. The properties of the average intensity, spreading and directionality of partially coherent standard and elegant LG beams in turbulent atmosphere are studied numerically and comparatively. It is found that the beam parameters and structure constant of turbulence together determine the properties of the beams in turbulent atmosphere. Partially coherent standard and elegant LG beams with smaller coherence length, larger beam orders and longer wavelength are less affected by the turbulence. A partially coherent elegant LG beam is less affected by turbulence than a partially coherent standard LG beam under the same condition. Furthermore, it is found that there exist equivalent partially coherent standard and elegant LG beams, equivalent fully coherent standard and elegant LG beams, equivalent Gaussian Schell-model beams that may have the same directionality as a fully coherent Gaussian beam both in free space and in turbulent atmosphere. Our results will be useful in long distance free-space optical communications.

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1. INTRODUCTION

As applied to free-space optical communications, laser radar, remote sensing, imaging and radar system, propagation of laser beams in turbulent atmosphere are recently studied in detail [1–24]. Various methods have been proposed to overcome or reduce the turbulence-induced degradation of laser beams [1]. It was shown that partially coherent beams are less affected by turbulence than fully coherent beams [2, 7, 10, 13–18, 20–23]. The use of higher-order beams such as Hermite-Gaussian, Hermite-Sine-Gaussian, Hermite-sinh-Gaussian and Laguerre-Gaussian (LG) beams also can reduce the turbulence-induced degradation [3–6].

Standard LG beams are commonly encountered in laser optics, material processing and atomic optics [25]. Takenaka et al. proposed the elegant LG beam as an extension of standard LG beam [26]. Both standard LG modes and elegant LG modes satisfy the paraxial wave equation, while the elegant LG modes have a more symmetrical form. Various methods have been proposed to generate standard and elegant LG beams [27–32]. Paraxial and non-paraxial propagation properties of standard and elegant LG beams in free space or through paraxial optical system have been studied extensively [33–42]. Propagation properties of a standard LG beam in turbulent atmosphere were investigated in [3]. More recently, partially coherent standard and elegant LG beams of all orders were proposed as an extension of corresponding fully coherent beams, and their paraxial propagations in free space or through ABCD optical system have been studied in detail [43]. In this paper, our aim is to investigate the propagation properties of partially coherent standard and elegant LG beams in turbulent atmosphere comparatively, and to explore the advantage of such beams for application in free-space optical communications. Analytical formulae for the average intensity, mean-squared beam width and angular spread are derived and numerical examples are provided.

2. AVERAGE INTENSITY OF PARTIALLY COHERENT STANDARD AND ELEGANT LG BEAMS PROPAGATING IN TURBULENT ATMOSPHERE

The second-order statistical properties of a partially coherent beam are generally characterized by the cross-spectral density (CSD) $W(x_1, y_1, x_2, y_2; z) = \langle E^*(x_1, y_1; z)E(x_2, y_2; z)\rangle$, where $\langle\rangle$ denotes the ensemble average and "*" is the complex conjugate. In the cylindrical coordinates, the CSD of a partially coherent standard or elegant LG beam of orders p and l generated by a Schell-model source ($z = 0$) can be expressed as [43]

$$W(r_1,\varphi_1,r_2,\varphi_2;0)=\left(\frac{qr_1}{\omega_0}\right)^l\left(\frac{qr_2}{\omega_0}\right)^l L_p^l\left(\frac{q^2r_1^2}{\omega_0^2}\right) L_p^l\left(\frac{q^2r_2^2}{\omega_0^2}\right)\exp\left(-\frac{r_1^2+r_2^2}{\omega_0^2}\right)$$
$$\times\exp\left[-il\left(\varphi_1-\varphi_2\right)\right]\exp\left(-\frac{r_1^2-2r_1r_2\cos(\varphi_1-\varphi_2)+r_2^2}{2\sigma_g^2}\right), \quad (1)$$

where r_i and φ_i ($i = 1, 2$) are the radial and azimuthal (angle) coordinates. ω_0 is the beam width of the fundamental Gaussian mode, σ_g is the transverse coherence length. L_p^l denotes the Laguerre polynomial with mode orders p and l. For $q = \sqrt{2}$, Eq. (1) denotes the CSD of a partially coherent standard LG beam; for $q = 1$, Eq. (1) denotes the CSD of a partially coherent elegant LG beam; also for $p = 0$ and $l = 0$, Eq. (1) degenerates to the CSD of a Gaussian Schell-model (GSM) beam [44–47].

By use of the following relation between an LG mode and an Hermite-Gaussian mode [48]

$$e^{il\varphi}\rho^l L_p^l\left(\rho^2\right)=\frac{(-1)^p}{2^{2p+l}p!}\sum_{m=0}^{p}\sum_{n=0}^{l}i^n\binom{p}{m}\binom{l}{n}H_{2m+l-n}\left(x\right)H_{2p-2m+n}\left(y\right), \quad (2)$$

with $H_m(x)$ being the Hermite polynomial of order m, $\binom{p}{m}$ and $\binom{l}{n}$ being binomial coefficients, Eq. (1) can be expressed in the following alternative form in Cartesian coordinates

$$W(x_1,y_1,x_2,y_2;0)$$
$$=\frac{1}{2^{4p+2l}(p!)^2}\sum_{m=0}^{p}\sum_{n=0}^{l}\sum_{h=0}^{p}\sum_{s=0}^{l}\left(i^n\right)^* i^s\binom{p}{m}\binom{l}{n}\binom{p}{h}\binom{l}{s}$$
$$\times H_{2m+l-n}\left(\frac{qx_1}{\omega_0}\right)H_{2h+l-s}\left(\frac{qx_2}{\omega_0}\right)H_{2p-2m+n}\left(\frac{qy_1}{\omega_0}\right)H_{2p-2h+s}\left(\frac{qy_2}{\omega_0}\right)$$
$$\times\exp\left(-\frac{x_1^2+y_1^2+x_2^2+y_2^2}{\omega_0^2}\right)\exp\left(-\frac{(x_1-x_2)^2+(y_1-y_2)^2}{2\sigma_g^2}\right) \quad (3)$$

Within the validity of the paraxial approximation, the propagation of a laser beam in turbulent atmosphere can be treated with the well-known extended Huygens-Fresnel integral formula, and the average intensity of a partially coherent laser beam in the receiver plane is given as follows [1–23]

$$\begin{aligned}\langle S(\xi,\eta;z)\rangle &= \left(\frac{1}{\lambda z}\right)^2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} W(x_1,y_1,x_2,y_2;0)\\ &\quad \exp\left[-\frac{ik}{2z}(x_1-\xi)^2-\frac{ik}{2z}(y_1-\eta)^2\right]\\ &\quad \times\exp\left[\frac{ik}{2z}(x_2-\xi)^2+\frac{ik}{2z}(y_2-\eta)^2\right.\\ &\quad \left.-\frac{1}{\rho_0^2}(x_1-x_2)^2-\frac{1}{\rho_0^2}(y_1-y_2)^2\right]dx_1dx_2dy_1dy_2, \quad (4)\end{aligned}$$

where $\rho_0 = (0.545C_n^2k^2z)^{-3/5}$ is the coherence length (induced by the atmospheric turbulence) of a spherical wave propagating in the turbulent medium, C_n^2 is the structure constant describing how strong the turbulence is [1], $k = 2\pi/\lambda$ is the wavenumber with λ being the wavelength of the light, z is the propagation axis. In the derivation of Eq. (4), we have employed Kolmogorov spectrum and quadratic approximation for Rytov's phase structure function [1–23]. The extended Huygens-Fresnel integral and the quadratic approximation are known to be valid for both weak and strong turbulence conditions [1, 9, 16].

Substituting from Eq. (3) into Eq. (4), and by applying following integral and expansion formulae [49, 50],

$$\int\limits_{-\infty}^{\infty} \exp\left[-(x-y)^2\right] H_n(ax)dx = \sqrt{\pi}(1-a^2)^{\frac{n}{2}} H_n\left(\frac{ay}{(1-a^2)^{1/2}}\right), \quad (5)$$

$$\int\limits_{-\infty}^{\infty} x^n \exp\left[-(x-\beta)^2\right]dx = (2i)^{-n}\sqrt{\pi}H_n(i\beta), \quad (6)$$

$$H_n(x+y) = \frac{1}{2^{n/2}}\sum_{k=0}^{n}\binom{n}{k} H_k(\sqrt{2}x)H_{n-k}(\sqrt{2}y), \quad (7)$$

$$H_n(x_1) = \sum_{m=0}^{[n/2]}(-1)^m\frac{n!}{m!(n-2m)!}(2x_1)^{n-2m}, \quad (8)$$

we obtain following expression for the average intensity of a partially coherent standard or elegant LG beam in the receiver plane

$$S(\varepsilon,\eta;z)=\left(\frac{1}{\lambda z}\right)\frac{1}{2^{4p+2l}(p!)^2}\frac{\pi^2}{M_1M_2}\left(\frac{1}{2M_2}-\frac{q^2}{2\omega_0^2M_1M_2}\right)^{(2p+l)/2}$$
$$\left(\frac{2q}{\omega_0}\right)^{2p+l}\exp\left(-\frac{k^2\xi^2}{4M_1z^2}\right)\times\exp\left[-\frac{k^2\xi^2}{4M_2z^2}\left(\frac{1}{2M_1L_c^2}-1\right)^2\right]$$
$$\exp\left(-\frac{k^2\eta^2}{4M_1z^2}\right)\exp\left[-\frac{k^2\eta^2}{4M_2z^2}\left(\frac{1}{2M_1L_c^2}-1\right)^2\right]$$
$$\times\sum_{m=0}^{p}\sum_{n=0}^{l}\sum_{h=0}^{p}\sum_{s=0}^{l}\sum_{d=0}^{2h+l-s}\sum_{c_1=0}^{[(2m+l-n)/2]}\sum_{c_2=0}^{[d/2]}\sum_{d_1=0}^{2p-2h+s}\sum_{e_1=0}^{[(2p-2m+n)/2]}\sum_{e_2=0}^{[d_1/2]}$$
$$(i^n)^*\,i^s\binom{p}{m}\binom{l}{n}\binom{p}{h}\binom{l}{s}\binom{2h+l-s}{d}\binom{2p-2h+s}{d_1}$$
$$\times(-1)^{e_1+e_2+c_1+c_2}\frac{(2m+l-n)!)}{c_1!(2m+l-n-2c_1)!}\frac{d!}{c_2!(d-2c_2)!}$$
$$\frac{(2p-2m+n)!}{e_1!(2p-2m+n-2e_1)!}\frac{d_1!}{e_2!(d_2-2e_2)!}$$
$$\times(2i)^{2c_1+2c_2+2e_1+2e_2-2p-l-d-d_1}\left(\frac{1}{\sqrt{M_2}}\right)^{d+d_1-2c_1-2c_2-2e_1-2e_2}$$
$$\left(\frac{2q}{\omega_0}\right)^{-2c_1-2e_1}\left(\frac{2q}{\sqrt{2}L_c^2\sqrt{M_1^2\omega_0^2-q^2M_1}}\right)^{d+d_1-2e_2-2c_2}$$
$$\times H_{2h+l-s-d}\left(-\frac{iqk\xi}{\sqrt{2}z\sqrt{\omega_0^2M_1^2-q^2M_1}}\right)$$
$$H_{2p-2h+s-d_1}\left(-\frac{iqk\eta}{\sqrt{2}z\sqrt{\omega_0^2M_1^2-q^2M_1}}\right)$$
$$\times H_{2m+l-n+d-2c_1-2c_2}\left[\frac{k\xi}{2z\sqrt{M_2}}\left(\frac{1}{2M_1L_c^2}-1\right)\right]$$
$$H_{2p-2m+n+d_1-2e_1-2e_2}\left[\frac{k\eta}{2z\sqrt{M_2}}\left(\frac{1}{2M_1L_c^2}\right)-1\right],\qquad(9)$$

where $\frac{1}{L_c^2}=\frac{1}{\sigma_g^2}+\frac{2}{\rho_0^2}$, $M_1=\frac{1}{\omega_0^2}+\frac{1}{2L_c^2}-\frac{ik}{2z}$, $M_2=\frac{1}{\omega_0^2}+\frac{1}{2L_c^2}+\frac{ik}{2z}-\frac{1}{4M_1L_c^2}$. Eq. (9) can be used conveniently to study the properties of the average intensity of partially coherent standard and elegant LG beams in turbulent atmosphere.

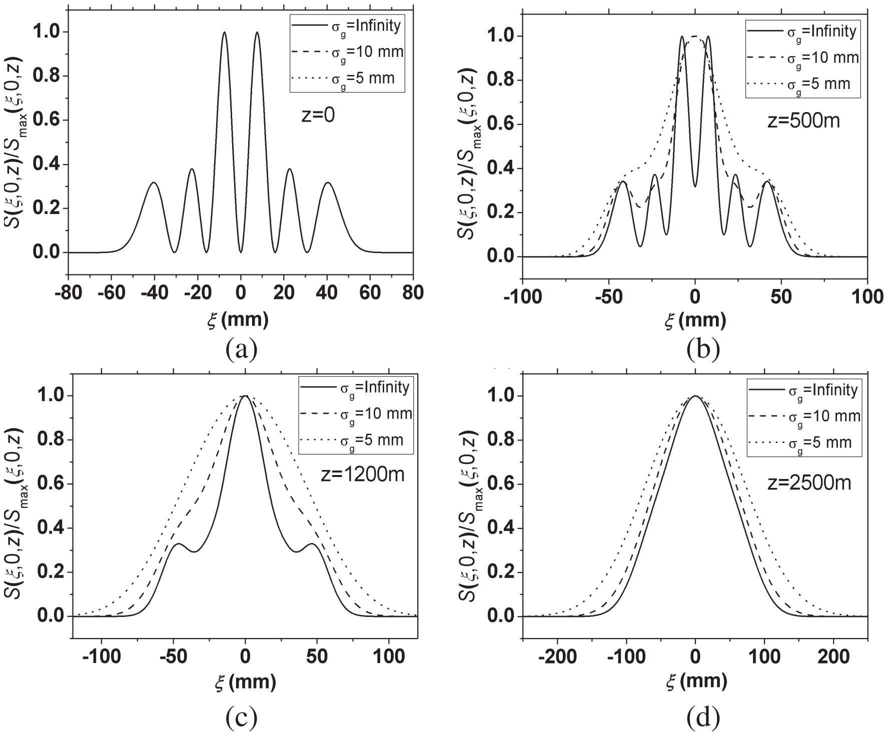

Figure 1. Normalized average intensity of a partially coherent standard LG beam with different initial transverse coherence length σ_g in turbulent atmosphere at several propagation distances.

Now we study the properties of the average intensity of partially coherent standard and elegant LG beams in turbulent atmosphere numerically using Eq. (9). Figure 1 shows the normalized average intensity of a partially coherent standard LG beam with different values of the initial transverse coherence length σ_g in turbulent atmosphere at several propagation distances with $q = \sqrt{2}$, $p = 2$, $l = 1$, $\omega_0 = 20\,\mathrm{mm}$, $\lambda = 632.8\,\mathrm{nm}$ and $C_n^2 = 10^{-14}\,\mathrm{m}^{-2/3}$. The average intensity at a propagation distance z, in Figure 1 and in the figures to follow, is normalized with respect to its maximum value at z. One finds from Figure 1 that the source beam profiles of coherent and partially coherent standard LG beams gradually disappear on propagation and eventually take Gaussian shapes in turbulent atmosphere. The evolution properties of the average intensity of a coherent standard LG beam in turbulent atmosphere are much different from its properties in free space, where its beam profile remains invariant on propagation although its beam spot spreads. The evolution properties of the average intensity of a coherent standard LG beam in turbulent atmosphere are similar to those of a partially coherent standard LG beam in free space [43]. This can be explained by the fact that the turbulence degrades the coherence of the standard LG beam [51]. It is clear from Figure 1, that the transition from a standard LG beam into a Gaussian beam becomes quicker as the initial coherence length decreases. Figure 2 shows the normalized average intensity of a partially coherent elegant LG beam with different values of the initial transverse coherence length σ_g in turbulent atmosphere at several propagation distances with $q = 1$, $p = 2$, $l = 1$, $\omega_0 = 20\,\mathrm{mm}$, $\lambda = 632.8\,\mathrm{nm}$ and $C_n^2 = 10^{-14}\,\mathrm{m}^{-2/3}$. As shown in Figure 2, similar to standard LG beams, the source beam profiles of coherent and partially

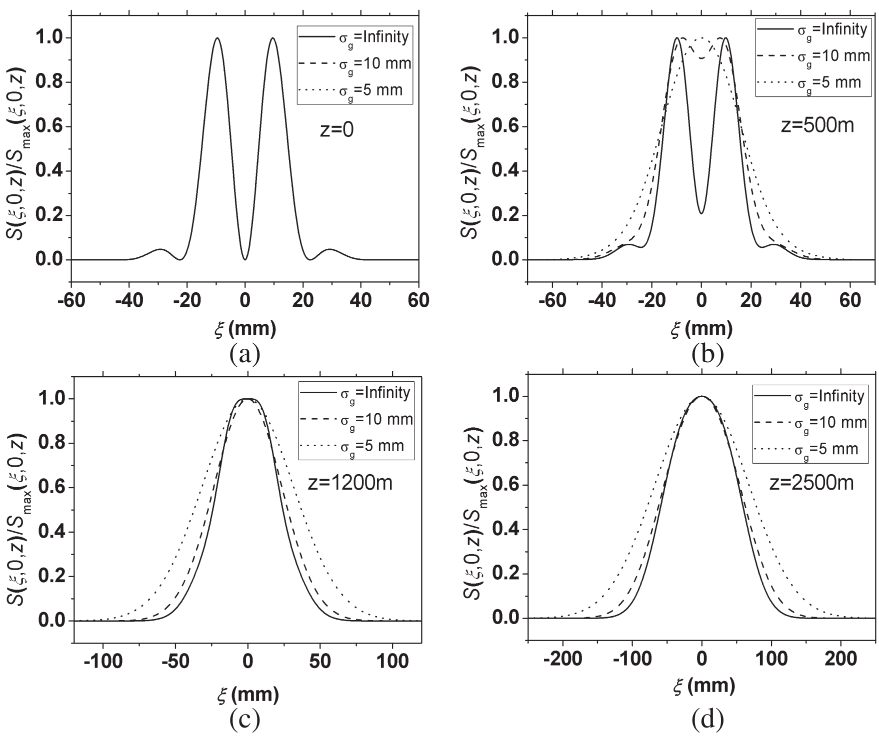

Figure 2. Normalized average intensity of a partially coherent elegant LG beam with different initial transverse coherence length σ_g in turbulent atmosphere at several propagation distances.

coherent elegant LG beams also gradually disappear on propagation and eventually take Gaussian shapes in turbulent atmosphere. At suitable distance, a flat-topped beam profile can be observed (see Figure 2(c)). The conversion from an elegant LG beam to a Gaussian beam becomes quicker for a smaller coherence length.

Figure 3 shows the normalized average intensity of partially coherent standard and elegant LG beams with different values of the beam orders in turbulent atmosphere at $z = 1500\,\mathrm{m}$ with $\omega_0 =$

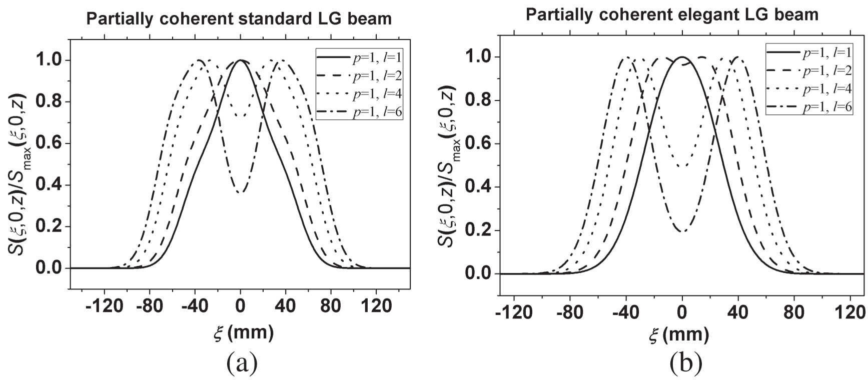

Figure 3. Normalized average intensity of partially coherent standard and elegant LG beams with different values of the beam orders in turbulent atmosphere at $z = 1500\,\mathrm{m}$.

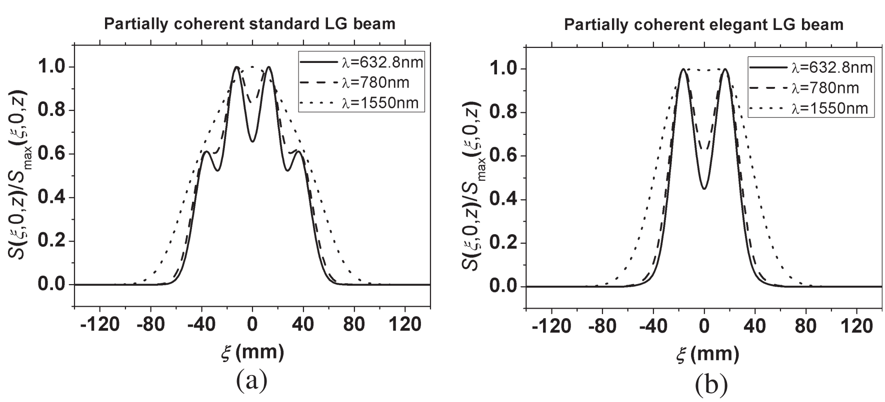

Figure 4. Normalized average intensity of partially coherent standard and elegant LG beams with different values of the wavelength λ in turbulent atmosphere at $z = 600\,\mathrm{m}$.

20 mm, $\lambda = 632.8\,\text{nm}$, $C_n^2 = 10^{-14}\,\text{m}^{-2/3}$ and $\sigma_g = 20\,\text{mm}$. Figure 4 shows the normalized average intensity of partially coherent standard and elegant LG beams with different values of the wavelength λ in turbulent atmosphere at $z = 600\,\text{m}$ with $p = 1$, $l = 2$, $\sigma_g = 10\,\text{mm}$, $\omega_0 = 20\,\text{mm}$ and $C_n^2 = 10^{-14}\,\text{m}^{-2/3}$. One finds from Figures 3 and 4 that the transition from a standard LG beam into a Gaussian beam becomes quicker for a longer wavelength and smaller beam orders. Furhermore, numerical results (omitted to save space) also shown this transition becomes quicker for a larger structure constant of turbulence.

3. MEAN-SQUARED BEAM WIDTH AND ANGULAR SPREAD OF PARTIALLY COHERENT STANDARD AND ELEGANT LG BEAMS IN TURBULENT ATMOSPHERE

In the rectangular coordinates, the mean-squared beam width of a laser beam is defined as [52, 53]

$$w(z) = \sqrt{\frac{4\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\xi^2 S(\xi,\eta;z)d\xi d\eta}{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}S(\xi,\eta;z)d\xi d\eta}}, \tag{10}$$

Substituting from Eqs. (3) and (4) into Eq. (10) and making use of the integral transform technique, after some integral calculations and operations (see Appendix A), we obtain the following expressions for the mean-squared beam width of partially coherent standard and elegant LG beams in the receiver plane in turbulent atmosphere

$$w_{SLG}(z) = 2\sqrt{\begin{array}{l}\frac{(2p+l+1)\omega_0^2}{4}+\frac{z^2}{k^2}\left(\frac{(2p+l+1)}{\omega_0^2}+\frac{1}{\sigma_g^2}\right)\\ +2\left(0.545C_n^2\right)^{6/5}k^{2/5}z^{16/5}\end{array}}. \tag{11}$$

$$w_{ELG}(z) = 2\sqrt{\begin{array}{l}\frac{A_3\omega_0^2}{4}+\frac{z^2}{k^2}\left(\frac{(2p+l+1)}{\omega_0^2}+\frac{1}{\sigma_g^2}\right)\\ +2(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}\end{array}}, \tag{12}$$

where

$$A_3 = \begin{cases} 1, \ p=0 \text{ and } l=0 \\ \left(2p+l^2+l\right)/(2p+l) \ p\neq 0 \text{ or } l\neq 0 \end{cases} \tag{13}$$

The first three terms inside the square root sign in Eq. (11) or Eq. (12) represent the effect of diffractive spreading of a partially coherent standard or elegant LG beam in free space, while the fourth term stands for the spreading due to turbulence. From Eqs. (11) and (12), it is clear that the mean-squared beam width of partially coherent standard and elegant LG beams are determined by the beam parameters (beam orders p, l, beam waist width ω_0, wavelength λ, initial transverse coherence length σ_g) and the structure constant of turbulence (C_n^2). The value of mean-squared beam width increases as the beam orders and the structure constant increases or as the initial coherence length decreases.

Now we analyze three special cases of Eqs. (11) and (12):

(a) For $p = 0$ and $l = 0$, Eqs. (11) and (12) reduce to

$$w(z) = 2\sqrt{\frac{\omega_0^2}{4} + \frac{z^2}{k^2}\left(\frac{1}{\omega_0^2} + \frac{1}{\sigma_g^2}\right) + 2(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}} \tag{14}$$

Eq. (14) gives the expression for the mean-squared beam width of a GSM beam in turbulent atmosphere [16]. Under the condition of $\sigma_g = \infty$, Eq. (14) reduces to the expression for the mean-squared beam width of a fully coherent Gaussian beam in turbulent atmosphere.

(b) For $p = 0$, Eqs. (11) and (12) reduce to

$$w(z) = 2\sqrt{\frac{(l+1)\omega_0^2}{4} + \frac{z^2}{k^2}\left(\frac{(l+1)}{\omega_0^2} + \frac{1}{\sigma_g^2}\right) + 2(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}}. \tag{15}$$

In this case, partially coherent standard and elegant LG beams have the same mean-squared beam width when the beam parameters are the same. This is caused by the fact that $L_0^l(x) = 1$, the expressions of the CSD of partially coherent standard and elegant LG beams with $p = 0$ (see Eq. (1)) are almost the same except for a constant coefficient q^{2l}.

(c) For $l = 0$, $p \neq 0$, Eqs. (11) and (12) reduce to

$$w_{PSLG}(z) = 2\sqrt{\begin{array}{c}\frac{(2p+1)\omega_0^2}{4} + \frac{z^2}{k^2}\left(\frac{(2p+1)}{\omega_0^2} + \frac{1}{\sigma_g^2}\right) \\ +2(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}\end{array}}, \tag{16}$$

$$w_{PELG}(z) = 2\sqrt{\frac{\omega_0^2}{4} + \frac{z^2}{k^2}\left(\frac{(2p+1)}{\omega_0^2} + \frac{1}{\sigma_g^2}\right) + 2(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}}. \tag{17}$$

In this case, the mean-squared beam width of a partially coherent elegant LG beam in the source plane ($z = 0$) is independent of the beam order p, while its value depends on p upon propagation. The mean-squared beam width of a partially coherent standard LG beam depends on p everywhere.

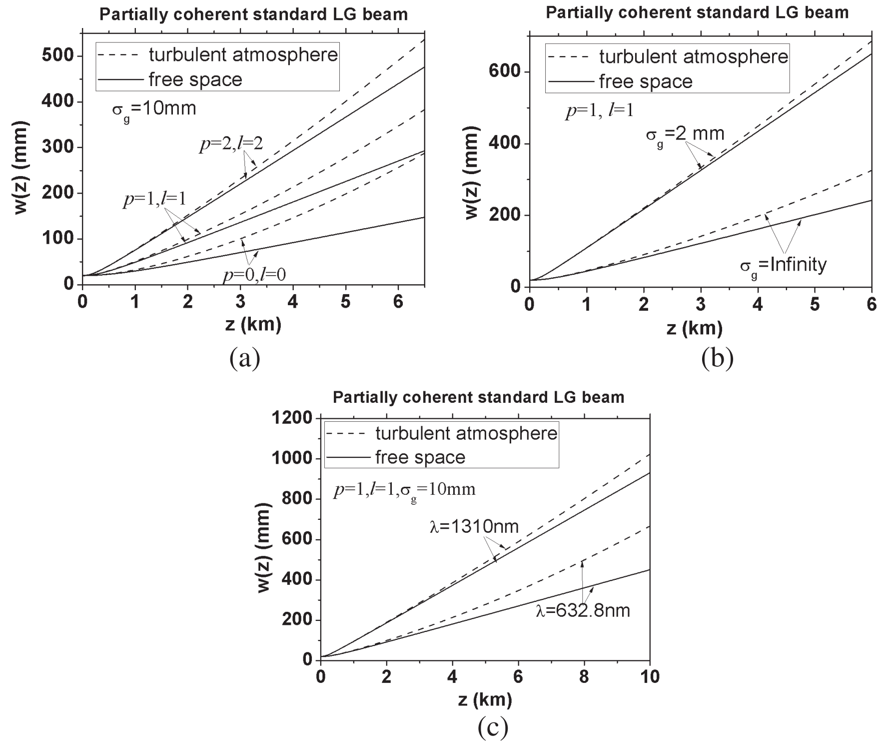

Figure 5. Mean-squared beam width of a partially coherent standard LG beam versus the propagation distance z in free space and in turbulent atmosphere for different values of beam parameters.

Figure 5 shows the mean-squared beam width of a partially coherent standard LG beam versus the propagation distance z in free space and in turbulent atmosphere for different values of beam parameters p, l, σ_g and λ with $C_n^2 = 10^{-14}\,\mathrm{m}^{-2/3}$. For the convenience of comparison, we have set $\omega_0 = 10\,\mathrm{mm}$ for the partially coherent standard LG beam with $p = 1$, $l = 1$, and $\omega_0 = 7.56\,\mathrm{mm}$ for the corresponding beam with $p = 2$, $l = 2$, so that they have the same mean-squared beam width in the source plane ($z = 0$). Figure 6 shows the mean-squared beam width of a partially coherent elegant LG beam versus the propagation distance z in free space and in turbulent atmosphere for different values of beam parameters p, l, σ_g and λ with $C_n^2 = 10^{-14}\,\mathrm{m}^{-2/3}$. Similarly we have set $\omega_0 = 17.32\,\mathrm{mm}$ for the partially coherent elegant LG beam with $p = 1$, $l = 1$, and $\omega_0 = 15.49\,\mathrm{mm}$ for $p = 2$, $l = 2$ for the corresponding beam with $p = 2$,

$l = 2$. One finds from Figures 5 and 6 that the difference between solid and dashed curves decreases as the initial coherence length σ_g decreases or as the beam orders p, l and the wavelength λ increase, which means that partially coherent standard and elegant LG beams with larger beam orders, longer wavelength and smaller initial coherence length are less affected by the turbulence. Furthermore, we calculate in Figure 7 the mean-squared beam width of partially coherent standard and elegant LG beams versus the propagation distance z in turbulent atmosphere for different values of the beam parameters and structure constant for comparison. We find from Figure 7 that a partially coherent elegant LG beam is less affected by the turbulence than a partially coherent standard LG beam, and this advantage is more apparent for smaller structure constant C_n^2, larger beam orders p, l, larger initial coherence length σ_g, and longer wavelength λ.

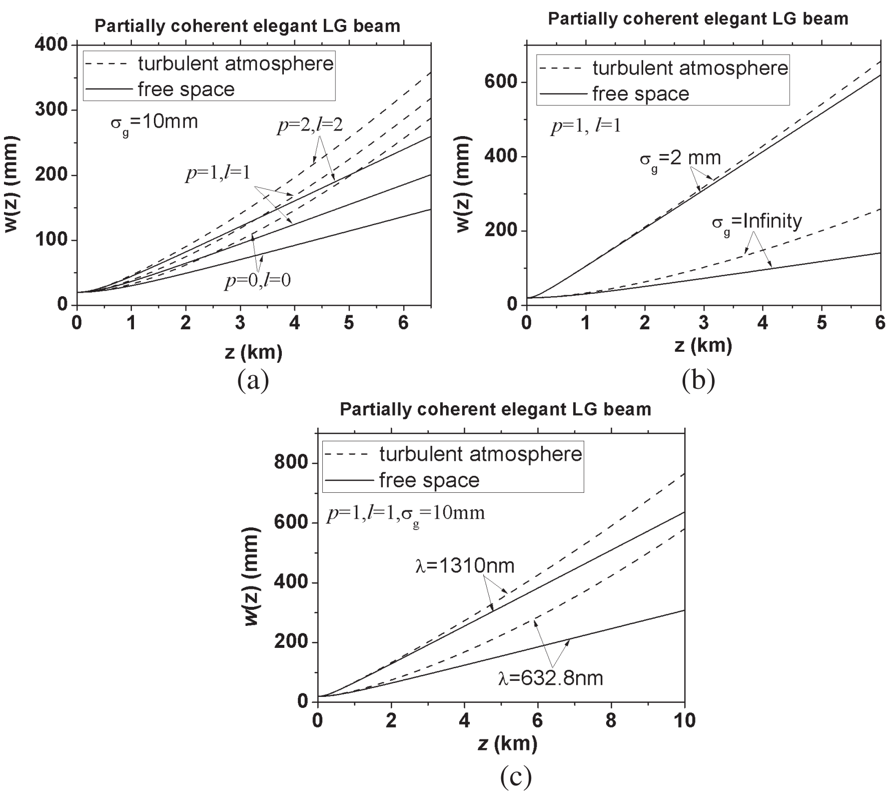

Figure 6. Mean-squared beam width of a partially coherent elegant LG beam versus the propagation distance z in free space and in turbulent atmosphere for different values of beam parameters.

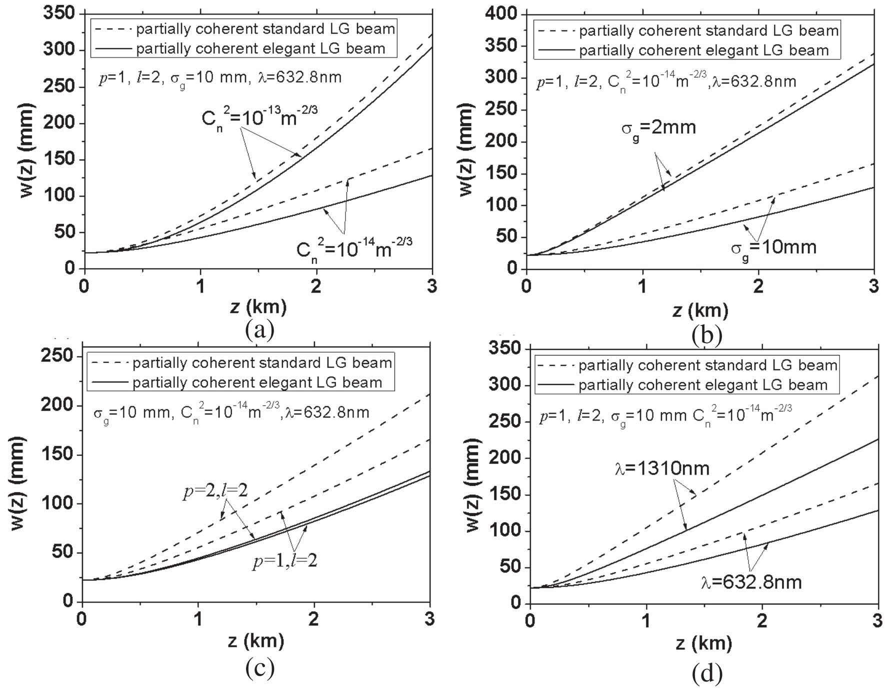

Figure 7. Mean-squared beam width of partially coherent standard and elegant LG beams versus the propagation distance z in turbulent atmosphere for different values of the beam parameters and structure constant.

Now we discuss the directionality of partially coherent standard and elegant LG beams in turbulent atmosphere. From Eqs. (11) and (12), the angular spread of partially coherent standard and elegant LG beams in turbulent atmosphere turn out to be

$$\theta_{PSLG}(z) = \lim_{z\to\infty} \frac{w_{SLG}(z)}{z} = \theta_{PELG}(z) = \lim_{z\to\infty} \frac{w_{ELG}(z)}{z}$$
$$= 2\sqrt{\frac{1}{k^2}\left(\frac{2p+l+1}{\omega_0^2} + \frac{1}{\sigma_g^2}\right) + 2(0.545C_n^2)^{6/5}k^{2/5}z^{6/5}}, \quad (18)$$

Under the condition of $C_n^2 = 0$, Eq. (18) reduces to the expressions for the angular spread of partially coherent standard and elegant LG beams in free space. One finds from Eq. (18) that partially coherent standard and elegant LG beams with the same beam parameters have the same directionality both in free space and in turbulent atmosphere.

Under the condition of $p = 0, l = 0$, Eq. (18) reduces to

$$\theta_{GSM}(z) = 2\sqrt{\frac{1}{k^2\omega_0^2} + \frac{1}{k^2\sigma_g^2} + 2(0.545C_n^2)^{6/5}k^{2/5}z^{6/5}}. \qquad (19)$$

Eq. (19) stands for the angular spread of a GSM beam in turbulent atmosphere, which is somewhat different from Eq. (14) in [54] because of a quadratic approximation of Rytov's phase structure function is used; i.e., $z^{6/5}$ appears in the second term of Eq. (19) here, while z occurs in [54]. Eq. (19) agrees well with Eq. (31) in [16].

Under the condition of $\sigma_g = \infty$, Eq. (18) reduces to

$$\theta_{SLG}(z) = \theta_{ELG}(z) = 2\sqrt{\frac{1}{k^2}\left(\frac{2p+l+1}{\omega_0^2}\right) + 2(0.545C_n^2)^{6/5}k^{2/5}z^{6/5}}, \qquad (20)$$

Under the condition of $p = 0,\ l = 0$ and $\sigma_g = \infty$, Eq. (18) reduces to

$$\theta_{GS}(z) = 2\sqrt{\frac{1}{k^2\omega_0^2} + 2(0.545C_n^2)^{6/5}k^{2/5}z^{6/5}}. \qquad (21)$$

Eq. (21) is the angular spread of a fully coherent Gaussian beam, which agrees well with Eq. (35) in [16]. From a comparison of Eqs. (18)–(21), one comes to the conclusion that partially coherent standard and elegant LG beams, fully coherent standard and elegant LG beams, GSM beams and Gaussian beams will generate the same angular spread both in free space and in turbulence if following condition is satisfied

$$\begin{aligned}\left(\frac{2p_1+l_1+1}{\omega_{01}^2}+\frac{1}{\sigma_{g1}^2}\right)_{PSLG} &= \left(\frac{2p_2+l_2+1}{\omega_{02}^2}+\frac{1}{\sigma_{g2}^2}\right)_{PELG} \\ &= \left(\frac{2p_3+l_3+1}{\omega_{03}^2}\right)_{SLG} = \left(\frac{2p_4+l_4+1}{\omega_{04}^2}\right)_{ELG} \\ &= \left(\frac{1}{\omega_{05}^2}+\frac{1}{\sigma_{g5}^2}\right)_{GSM} = \left(\frac{1}{\omega_{06}^2}\right)_{GS}. \qquad (22)\end{aligned}$$

Above result is valid no matter how strong the turbulence is. Such beams are called equivalent partially coherent standard and elegant LG beams, equivalent fully coherent standard and elegant LG beams, equivalent GSM beams. Figure 8 shows the mean-squared beam width of the equivalent partially coherent standard and elegant LG beams, the equivalent fully coherent standard and elegant LG beams, the equivalent GSM beam and the corresponding fully coherent Gaussian beam propagating both in free space and in turbulent atmosphere.

The calculation parameters are listed in Table 1. As expected, the equivalent partially coherent standard and elegant LG beams, the equivalent fully coherent standard and elegant LG beams, and the equivalent GSM beam exhibit the same directionality as the corresponding fully coherent Gaussian beam both in free space and in turbulent atmosphere.

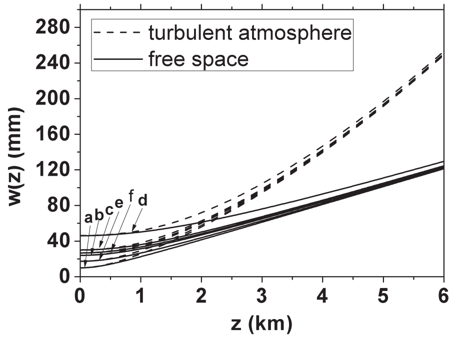

Figure 8. Mean-squared beam width of the equivalent partially coherent standard and elegant LG beams, the equivalent fully coherent standard and elegant LG beams, the equivalent GSM beam and the corresponding fully coherent Gaussian beam propagating both in free space ($C_n^2 = 0$) and in turbulent atmosphere ($C_n^2 = 10^{-14}\,\mathrm{m}^{-2/3}$) with $\lambda = 632.8\,\mathrm{nm}$. The other calculation parameters are listed in Table 1.

Table 1. Parameters of the equivalent partially coherent standard and elegant LG beams, the equivalent fully coherent standard and elegant LG beams, the equivalent GSM beam and the corresponding fully coherent Gaussian beam.

Beam parameter	p	l	ω_0 (mm)	σ_g (mm)
a. Gaussian beam	0	0	10	∞
b. Partially coherent GSM	0	0	24.004	11
c. Fully coherent SLGB	1	0	17.32	∞
d. Partially coherent SLGB	1	1	23.094	20
e. Fully coherent ELGB	1	0	17.32	∞
f. Partially coherent ELGB	1	1	23.094	20

4. CONCLUSION

The average intensity, spreading and directionality of partially coherent standard and elegant LG beams propagating in turbulent atmosphere have been studied in detail both theoretically and numerically. Analytical expressions for the average intensity, mean-squared beam width and angular spread of such beams in turbulent atmosphere have been derived. It is found that the properties of the partially coherent standard and elegant LG beams in turbulent atmosphere are closely related to its beam parameters and the structure constant of turbulence. In general, the smaller the coherence length is, the larger beam orders are and the longer wavelength is, the less partially coherent standard and elegant LG beams are affected by the turbulence although the beams with smaller coherence length, larger beam orders and longer wavelength have greater spreading in free space. A partially coherent elegant LG beam has advantage over a partially coherent standard LG beam for overcoming the destructive effect of turbulence, and this advantage is more apparent for smaller structure constant C_n^2, larger beam orders p, l, larger initial coherence length σ_g, and longer wavelength λ. Furthermore, we have found that there exist equivalent partially coherent standard and elegant LG beams, equivalent fully coherent standard and elegant LG beams, Gaussian Schell-model beams that may have the same directionality as a fully coherent Gaussian beam both in free space and in turbulence atmosphere under the condition of Eq. (22). Our formulae are valid for both weak and strong turbulence conditions. Our results will be useful in long distance free-space optical communications, remote sensing and laser radar.

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APPENDIX A. DERIVATION OF EQS. (11) AND (12)

Equation (10) can be rewritten as:

$$w(z) = \sqrt{\frac{4F_1}{F_2}}, \tag{A1}$$

with

$$F_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi^2 S(\xi,\eta,z)d\xi d\eta, \quad \text{(A2)}$$

$$F_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\xi,\eta,z)d\xi d\eta. \quad \text{(A3)}$$

First we derive the expression (Eq. (11)) for the mean-squared beam width of a partially coherent standard LG beam ($q = \sqrt{2}$) propagating in turbulent atmosphere. Substituting from Eqs. (3) and (4) into Eq. (A2) and after integrating over y_1, y_2 and η, we obtain

$$\begin{aligned} F_1 = & \frac{\omega_0\sqrt{\pi}}{\sqrt{2}\lambda z}\frac{1}{2^{4p+2l}(p!)^2}\sum_{m=0}^{p}\sum_{n=0}^{l}\sum_{h=0}^{p}\sum_{s=0}^{l}(i^n)^* i^s \binom{p}{m}\binom{l}{n}\binom{p}{h}\binom{l}{s} \\ & 2^{2p-2m+n}(2p-2m+n)!\delta_{2p-2m+n,2p-2h+s} \\ & \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi^2 H_{2m+l-n}\left(\frac{\sqrt{2}x_1}{\omega_0}\right)H_{2h+l-s}\left(\frac{\sqrt{2}x_2}{\omega_0}\right)\exp\left(-\frac{x_1^2+x_2^2}{\omega_0^2}\right) \\ & \exp\left(-\frac{(x_1-x_2)^2}{2\sigma_g^2}\right)\exp\left(-\frac{(x_1-x_2)^2}{\rho_0^2}\right) \\ & \times\exp\left[-\frac{ik}{2z}(x_1^2-x_2^2)+\frac{ik\xi}{z}(x_1-x_2)\right]dx_1dx_2d\xi, \end{aligned} \quad \text{(A4)}$$

where $\delta_{2p-2m+n,2p-2h+s} = 1$ for $2p-2m+n = 2p-2h+s$ and $\delta_{2p-2m+n,2p-2h+s} = 0$ for $2p-2m+n \neq 2p-2h+s$. In the derivation of Eq. (A4), we have applied following integral and expansion formulae [49, 50]

$$\int_{-\infty}^{\infty}\exp(i\xi x)d\xi = 2\pi\delta(x), \quad \text{(A5)}$$

$$\int_{-\infty}^{\infty}H_m(x)H_n(x)\exp(-x^2)dx = \sqrt{\pi}2^m m!\delta_{m,n}, \quad \text{(A6)}$$

where $\delta(x)$ stands for the Dirac Delta function. By use of the following variable transformation

$$u = \frac{x_1+x_2}{2}, \quad v = x_2 - x_1 \quad \text{(A7)}$$

Eq. (A4) becomes

$$F_1 = \frac{\omega_0\sqrt{\pi}}{\sqrt{2}\lambda z}\frac{1}{2^{4p+2l}(p!)^2}\sum_{m=0}^{p}\sum_{n=0}^{l}\sum_{h=0}^{p}\sum_{s=0}^{l}(i^n)^*\, i^s \binom{p}{m}\binom{l}{n}\binom{p}{h}\binom{l}{s}$$
$$2^{2p-2m+n}\times(2p-2m+n)!\delta_{2p-2m+n,2p-2h+s}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\xi^2 H_{2m+l-n}$$
$$\left[\frac{\sqrt{2}}{\omega_0}\left(u-\frac{v}{2}\right)\right] H_{2h+l-s}\left[\frac{\sqrt{2}}{\omega_0}\left(u+\frac{v}{2}\right)\right]\times\exp\left(-\frac{2u^2}{\omega_0^2}-\frac{v^2}{2\omega_0^2}\right)$$
$$\exp\left(-\frac{v^2}{L_c^2}\right)\exp\left(\frac{ikuv}{z}-\frac{ik\xi v}{z}\right)dudvd\xi. \quad \text{(A8)}$$

Applying the following integral formulae [49, 50]

$$\int\limits_{-\infty}^{\infty} x^2\exp\left(\frac{-i2\pi xs}{\lambda z}\right)dx = -\frac{\lambda^3 z^3}{(2\pi)^2}\delta''(s), \quad \text{(A9)}$$

$$\int\limits_{-\infty}^{\infty} f(x)\delta''(x)dx = f''(0), \quad \text{(A10)}$$

$$\int\limits_{-\infty}^{\infty}\exp(-x^2)H_m(x+y)H_m(x+z)dx = 2^m\sqrt{\pi}L_m(-2yz), \quad \text{(A11)}$$

where $f(x)$ is an arbitrary function and $f''(x)$ is its second derivative. After integration over u, v, and ξ, Eq. (A8) becomes

$$F_1 = \frac{2\pi\omega_0^2}{2^{2p+l}p!}\sum_{m=0}^{p}\sum_{n=0}^{l}\sum_{h=0}^{p}\sum_{s=0}^{l}(i^n)^*\, i^s\binom{p}{m}\binom{l}{n}\binom{p}{h}\binom{l}{s}$$
$$(l+2m-n)!(2p-2m+n)!\delta_{2p-2m+n,2p-2h+s}$$
$$\left[(l+2m-n+1/2)\frac{\omega_0^2}{4}+\frac{z^2}{k^2}\left(\frac{(l+2m-n+1/2)}{\omega_0^2}+\frac{1}{2\sigma_g^2}\right)\right.$$
$$\left.+(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}\right]. \quad \text{(A12)}$$

In a similar way, we obtain following expression for F_2

$$F_2 = \frac{\pi\omega_0^2}{2}\frac{(p+l)!}{p!}. \quad \text{(A13)}$$

Substituting from Eqs. (A12) and (A13) into Eq. (A1), after some operation, we obtain following expression for the mean-squared beam width of a partially coherent standard LG beam propagating in turbulent atmosphere

$$w_{SLG}(z)=2\sqrt{\frac{(2p+l+1)\omega_0^2}{4}+\frac{z^2}{k^2}\left(\frac{2p+l+1}{\omega_0^2}+\frac{1}{\sigma_g^2}\right)+2(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}}. \tag{A14}$$

Eq. (A14) is the same as Eq. (11) in the text.

Now we derive the expression (Eq. (12)) for the mean-squared beam width of a partially coherent standard LG beam ($q = 1$) propagating in turbulent atmosphere. Substituting from Eqs. (4) and (5) into Eq. (A2) and after integrating over y_1, y_2 and η, we obtain

$$F_1 = \frac{\omega_0}{\lambda z}\frac{1}{2^{4p+2l}(p!)^2}\sum_{m=0}^{p}\sum_{n=0}^{l}\sum_{h=0}^{p}\sum_{s=0}^{l}(i^n)^* i^s \binom{p}{m}\binom{l}{n}\binom{p}{h}\binom{l}{s}$$
$$f_1(2p-2m+n, 2p-2h+s)\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi^2 H_{2m+l-n}\left(\frac{x_1}{\omega_0}\right)H_{2h+l-s}\left(\frac{x_2}{\omega_0}\right)$$
$$\exp\left(-\frac{x_1^2+x_2^2}{\omega_0^2}\right)\exp\left(-\frac{(x_1-x_2)^2}{2\sigma_g^2}\right)\times\exp\left(-\frac{(x_1-x_2)^2}{\rho_0^2}\right)$$
$$\exp\left[-\frac{ik}{2z}(x_1^2-x_2^2)+\frac{ik\xi}{z}(x_1-x_2)\right]dx_1dx_2d\xi, \tag{A15}$$

with

$$f_1(a,b)=\begin{cases}0, & a+b \text{ is an odd number}\\ (-1)^{(a+3b)/2}2^{(a+b-1)/2}\Gamma\left[(a+b+1)/2\right], & a+b \text{ is an even number}\end{cases} \tag{A16}$$

where $\Gamma(x)$ denotes the Gamma function. In the derivation Eq. (A15), we have applied integral formula:

$$\int_{-\infty}^{\infty}\exp(-2x^2)H_m(x)H_n(x)dx$$
$$=\begin{cases}0, & m+n \text{ is an odd number}\\ (-1)^{(m+3n)/2}2^{(m+n-1)/2}\Gamma((m+n+1)/2), & m+n \text{ is an even number}\end{cases} \tag{A17}$$

By use of the variable transformation (see Eq. (A7)) and after integration over u, v, and ξ, Eq. (15) becomes

$$F_1 = \frac{\pi\sqrt{\pi}\omega_0^2}{\sqrt{2}}\frac{1}{2^{4p+2l}(p!)^2}\sum_{m=0}^{p}\sum_{n=0}^{l}\sum_{h=0}^{p}\sum_{s=0}^{l}(i^n)^* i^s$$
$$\left(\begin{array}{c} p \\ m \end{array}\right)\left(\begin{array}{c} l \\ n \end{array}\right)\left(\begin{array}{c} p \\ h \end{array}\right)\left(\begin{array}{c} l \\ s \end{array}\right)\sum_{m_1=0}^{2m+l-n}\sum_{l_1=0}^{2h+l-s}\frac{1}{2^{(2l+2m+2h-n-s)/2}}$$
$$\times\left(\begin{array}{c} 2m+l-n \\ m_1 \end{array}\right)\left(\begin{array}{c} 2h+l-s \\ l_1 \end{array}\right)2^{l_1+m_1}2^{2m+l-n-m_1}$$
$$(2m+l-n-m_1)!\delta_{2m+l-n-m_1,2h+l-s-l_1}$$
$$\times f_1(2p-2m+n, 2p-2h+s)\left[(b_1+b_2+b_3+b_4)\omega_0^2/8\right.$$
$$\left.+(b_1+b_3-b_2-b_4)\frac{z^2}{2k^2\omega_0^2}+\frac{b_1 z^2}{2\sigma_g^2 k^2}+b_1\left(0.545C_n^2\right)^{6/5}k^{2/5}z^{16/5}\right], \tag{A18}$$

where

$$\begin{aligned} b_1 &= \frac{2}{\Gamma(1/2-l_1/2)\,\Gamma(1/2-m_1/2)}, \\ b_2 &= \frac{l_1(l_1-1)}{\Gamma(3/2-l_1/2)\,\Gamma(1/2-m_1/2)}, \\ b_3 &= \frac{2m_1 l_1}{\Gamma(1-l_1/2)\,\Gamma(1-m_1/2)}, \\ b_4 &= \frac{m_1(m_1-1)}{\Gamma(1/2-l_1/2)\,\Gamma(3/2-m_1/2)}. \end{aligned} \tag{A19}$$

In the derivation of Eq. (A18), we have used Eq. (7). Similarly, we obtain the expression for F_2

$$F_2 = \omega_0^2\pi\frac{(2p+l)!}{(p!)^2 2^{2p+l+1}}. \tag{A20}$$

Substituting from Eqs. (A18) and (A20) into Eq. (A1), after some operation, we obtain following expression for the mean-squared beam width of a partially coherent elegant LG beam propagating in turbulent atmosphere

$$w_{ELG}(z) = 2\sqrt{\frac{A_3\omega_0^2}{4}+\frac{z^2}{k^2}\left(\frac{(2p+l+1)}{\omega_0^2}+\frac{1}{k^2\sigma_g^2}\right)+2(0.545C_n^2)^{6/5}k^{2/5}z^{16/5}}, \tag{A21}$$

where A_3 is given by Eq. (13). Eq. (A21) is the same as Eq. (12) in the text.

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