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A NOVEL DUALITY BETWEEN PERMEABILITY AND PERMITTIVITY IN A CONCENTRIC SPHERE

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Abstract—Consider a TEM plane wave incident on a spherical multilayer structure, then the following theorem is valid. This theorem reveals a duality between permeability and permittivity of media in a spherical multilayer structure.

Theorem: Consider a sphere with arbitrary radius and parameters (ε_2, μ_2) surrounded by a homogenous medium with parameters (ε_1, μ_1). Then consider the case that each medium is filled by its dual medium according to the interchange $\varepsilon_i \to \mu_i$. Then, the forward and backward radar cross sections of the structure are the same for the two dual cases. However, in half planes $\varphi = \frac{(2k+1)\pi}{4}$; $(k = 0, 1, 2, 3)$, the interchange $\varepsilon_i \to \mu_i$ has no similar effect on the value of the radar cross section.

1. INTRODUCTION

The radar cross-section (RCS) of an object is a complicated function of observation angle, signal frequency and polarization, material and dimensions of the object. In this context planar structures have been mostly studied. However, in this paper we investigate multilayered spherical structures, which may be considered as basic and canonical shapes of practical objects. In general, electromagnetic wave scattering from multilayered spherical structures is studied by analytical, approximate and numerical methods [1–3]. Here we use

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the theoretical method of addition theorems, where the fields are expanded in terms of the spherical eigen functions in various media [1]. TEM plane wave incidence is considered. The addition theorem is employed to expand the fields as spherical wave functions inside the layers and outside in free space. Subsequently, we prove an interesting theorem for the radar cross section of spherical multilayered structures. This theorem is independent of frequency, number of layers, their thicknesses and the type of plane wave polarization. This theorem as described in the abstract, reveals some type of duality between media.

The results indicate that applying magnetic materials instead of electrical materials and vice versa has no effect in the radar cross section of such structures. This phenomenon shows one of the potential applications for this theorem. At first the theorem is proved for one layer structure and then generalized to multilayered structures. At last some examples and applications for this novel theorem are displayed.

2. NUMERICAL PROCEDURE

Consider a TEM plane wave incident on a sphere composed of a material by parameters (ε_2, μ_2) located in an isotropic space by parameters (ε_1, μ_1) as shown in Fig. 1.

Various methods have been devised for the analysis of wave incidence on multilayered structures, we follow the procedure developed in [1] for the analysis of multilayered spherical structures. The incident plane wave may be described in terms of the sum of spherical waves by using the addition theorem. The field components in spherical coordinates may be expressed in terms of the electric and magnetic Hertzian potentials (Π_1, Π_2) for the TM and TE modes,

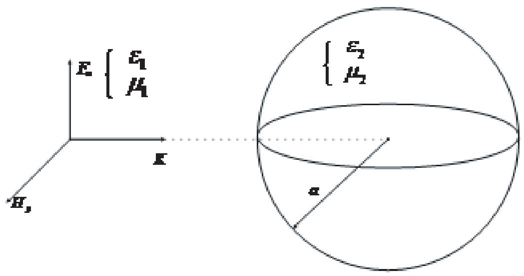

Figure 1. Geometry of the problem.

respectively, as:

$$\begin{cases} E_\varphi = \frac{1}{r\sin\theta}\ \frac{\partial^2}{\partial r\partial\varphi}(r\,\Pi_1) + j\omega\mu\frac{\partial}{\partial\theta}\Pi_2 \\ E_\theta = \frac{1}{r}\ \frac{\partial^2}{\partial r\partial\theta}(r\Pi_1) - j\omega\mu\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\Pi_2 \\ E_r = \frac{\partial^2}{\partial r^2}(r\Pi_1) + K^2 r\Pi_1 \\ \mathrm{H}_\varphi = -j\omega\varepsilon\frac{\partial}{\partial\theta}\Pi_1, +\frac{1}{r\sin\theta}\ \frac{\partial^2}{\partial r\partial\varphi}(r\Pi_2) \\ \mathrm{H}_\theta = j\omega\varepsilon\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\Pi_1 + \frac{1}{r}\ \frac{\partial^2}{\partial r\partial\theta}(r\Pi_2) \\ \mathrm{H}_r = \frac{\partial^2}{\partial r^2}(r\Pi_2) + K^2 r\Pi_2 \end{cases} \tag{1}$$

We assume an incident plane electromagnetic wave and expand it and the scattered waves from the spherical structure and also the standing waves inside the layered media and the spherical core by the addition theorems:

$$\begin{aligned} &\begin{matrix} incident \\ wave: \end{matrix} \begin{cases} r\Pi_1^i = \frac{1}{K_1^2}\sum_{n=1}^{\infty}\frac{(-j)^{n-1}(2n+1)}{n(n+1)}\hat{J}_n(K_1 r)P_n^{(1)}(\cos\theta)\cos\varphi \\ r\Pi_2^i = \frac{1}{\eta_1 K_1^2}\sum_{n=1}^{\infty}\frac{(-j)^{n-1}(2n+1)}{n(n+1)}\hat{J}_n(K_1 r)P_n^{(1)}(\cos\theta)\sin\varphi \end{cases} \\ &\begin{matrix} scattered \\ wave: \end{matrix} \begin{cases} r\Pi_1^s = \frac{-1}{K_1^2}\sum_{n=1}^{\infty}\frac{(-j)^{n-1}(2n+1)}{n(n+1)}a_n\hat{H}_n^{(2)}(K_1 r)P_n^{(1)}(\cos\theta)\cos\varphi \\ r\Pi_2^s = \frac{-1}{\eta_1 K_1^2}\sum_{n=1}^{\infty}\frac{(-j)^{n-1}(2n+1)}{n(n+1)}b_n\hat{H}_n^{(2)}(K_1 r)P_n^{(1)}(\cos\theta)\sin\varphi \end{cases} \\ &\begin{matrix} \text{core's} \\ interior \\ wave: \end{matrix} \begin{cases} r\Pi_1 = \frac{1}{K_2^2}\sum_{n=1}^{\infty}\frac{(-j)^{n-1}(2n+1)}{n(n+1)}\left[c_n\hat{J}_n(K_2 r)\right]P_n^{(1)}(\cos\theta)\cos\varphi \\ r\Pi_2 = \frac{1}{\eta_2 K_2^2}\sum_{n=1}^{\infty}\frac{(-j)^{n-1}(2n+1)}{n(n+1)}\left[d_n\hat{J}_n(K_2 r)\right]P_n^{(1)}(\cos\theta)\sin\varphi \end{cases} \end{aligned} \tag{2}$$

where $k = \omega\sqrt{\mu\varepsilon}$ and $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ are the wave number and intrinsic impedance, respectively and $\hat{J}_n$ is the spherical Bessel functions of the first kind of degree n, and $\hat{H}_n^{(2)}$ is the spherical Hankle function of degree n. Notation ($'$) refers to the derivative with respect to the argument.

Amplitudes a_n, b_n, c_n and d_n are unknowns, which are determined by enforcing the boundary conditions, namely, the continuity of tangential E and H fields at the spherical boundaries. Now, we collect all the boundary conditions in a matrix equation as [A] [X] = [b] and solve them for the unknown amplitudes $a_n,\ b_n$.

$$a_n = \frac{\eta_2 \hat{J}_n\left(K_1 a\right) \hat{J}_n'\left(K_2 a\right) - \eta_1 \hat{J}_n\left(K_2 a\right) \hat{J}_n'\left(K_1 a\right)}{\eta_2 \hat{H}_n^{(2)}\left(K_1 a\right) \hat{J}_n'\left(K_2 a\right) - \eta_1 \hat{J}_n\left(K_2 a\right) \hat{H}_n^{(2)\prime}\left(K_1 a\right)}$$
$$b_n = \frac{\eta_1 \hat{J}_n\left(K_1 a\right) \hat{J}_n'\left(K_2 a\right) - \eta_2 \hat{J}_n\left(K_2 a\right) \hat{J}_n'\left(K_1 a\right)}{\eta_1 \hat{H}_n^{(2)}\left(K_1 a\right) \hat{J}_n'\left(K_2 a\right) - \eta_2 \hat{J}_n\left(K_2 a\right) \hat{H}_n^{(2)\prime}\left(K_1 a\right)} \tag{3}$$

By using the asymptotic formulas for spherical Hankle functions as the distance 'r' approaches infinity, the scattered field may be simplified as

$$\begin{cases} r\Pi_1^s = \frac{e^{-jk_1 r}}{K_1^2} \sum\limits_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} a_n P_n^{(1)}(\cos\theta)\cos\varphi \\ r\Pi_2^s = \frac{e^{-jk_1 r}}{\eta_1 K_1^2} \sum\limits_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} b_n P_n^{(1)}(\cos\theta)\sin\varphi \end{cases} \tag{4}$$

Consequently, we can write

$$\begin{cases} E_\theta^s = f_\theta\left(\theta,\varphi\right) \frac{e^{-jK_1 r}}{r} \\ E_\varphi^s = f_\varphi\left(\theta,\varphi\right) \frac{e^{-jK_1 r}}{r} \end{cases} \tag{5}$$

where

$$\begin{cases} f_\theta = -\frac{j\cos\varphi S_2(\theta)}{K_1}, \\ S_2(\theta) = \sum\limits_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \Omega_n(\cos\theta) + b_n \Lambda_n(\cos\theta)\right] \\ f_\varphi = \frac{j\sin\varphi S_1(\theta)}{K_1}, \\ S_1(\theta) = \sum\limits_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[a_n \Lambda_n(\cos\theta) + b_n \Omega_n(\cos\theta)\right] \end{cases} \tag{6}$$

and

$$\Lambda_n(\cos\theta) = \frac{P_n^{(1)}(\cos\theta)}{\sin\theta},\ \ \Omega_n(\cos\theta) = \frac{d}{d\theta} P_n^{(1)}(\cos\theta) \tag{7}$$

At last the amplitude of total scattered field may be written as

$$\left|E^s\right|^2 = \left|E_\theta^s\right|^2 + \left|E_\varphi^s\right|^2 = \frac{\left|f_\theta\right|^2 + \left|f_\varphi\right|^2}{r^2} \tag{8}$$

Therefore, the radar cross section may be obtained as

$$RCS = \sigma(\theta,\varphi) = \lim 4\pi r^2 \frac{\left|E^s\right|^2}{\left|E_{inc}\right|^2} = 4\pi(\left|f_\theta\right|^2 + \left|f_\varphi\right|^2) \tag{9}$$

3. PROOF OF THE THEOREM

3.1. Part 1:

For forward scattering cross section, according to the Eq. (7) we have

$$\theta = 0,\ \Lambda_n(\cos 0) = \Omega_n(\cos 0) = \frac{n(n+1)}{2} \tag{10}$$

And for back scattering cross section

$$\theta = \pi,\ \Lambda_n(\cos \pi) = -\Omega_n(\cos \pi) = -(-1)^n \frac{n(n+1)}{2} \tag{11}$$

Consequently, we have

$$\begin{aligned} &\theta = 0,\ S_1(0) = S_2(0) = \sum_{n=1}^{\infty} \frac{2n+1}{2}(a_n + b_n) \\ &\theta = \pi,\ S_1(\pi) = -S_2(\pi) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{2}(a_n - b_n) \end{aligned} \tag{12}$$

Now, we consider the media interchanges $\varepsilon_i \leftrightarrow \mu_i$ where all materials are replaced by their dual materials, then

$$(\eta_i)_{\text{new}} = (\eta_i)_{\text{old}}^{-1}, \quad (k_i)_{\text{new}} = (k_i)_{\text{old}} \tag{13}$$

From Eqs. (3) & (13) after some mathematical manipulations, we obtain

$$\begin{aligned} (a_n)_{\text{new}} &= (b_n)_{\text{old}} \\ (b_n)_{\text{new}} &= (a_n)_{\text{old}} \end{aligned} \tag{14}$$

Then from Eq. (12)

$$\begin{cases} (S_1)_{\text{new}} = (S_1)_{\text{old}} \\ (S_2)_{\text{new}} = -(S_2)_{\text{old}} \end{cases} \tag{15}$$

Then the relations in Eq. (6), discover

$$|f_\theta|^2_{\text{new}} = |f_\theta|^2_{\text{old}}, \quad |f_\varphi|^2_{\text{new}} = |f_\theta|^2_{\text{old}} \tag{16}$$

Finally, according to Eq. (9)

$$\text{RCS}_{\text{new}} = \text{RCS}_{\text{old}} \tag{17}$$

So the proof of the first part of the theorem is complete.

3.2. Part 2

According to Eqs. (6) & (13) and interchange $\varepsilon_i \leftrightarrow \mu_i$, we have

$$\begin{cases} (S_1)_{\text{new}} = (S_2)_{\text{old}} \\ (S_2)_{\text{new}} = (S_1)_{\text{old}} \end{cases} \tag{18}$$

From Eqs. (6) & (13), after simplifying these relations we obtain

$$\begin{aligned} |f_\theta|^2_{\text{new}} + |f_\varphi|^2_{\text{new}} &= \frac{\cos^2\varphi\,(S_2)^2_{\text{new}} + \sin^2\varphi\,(S_1)^2_{\text{new}}}{k^2_{1\text{new}}} \\ &= \frac{\cos^2\varphi\,(S_1)^2_{\text{old}} + \sin^2\varphi\,(S_2)^2_{\text{old}}}{k^2_{1\text{old}}} \end{aligned} \tag{19}$$

Again from Eq. (6), it is obvious that

$$|f_\theta|^2_{\text{old}} + |f_\varphi|^2_{\text{old}} = \frac{\cos^2\varphi\,(S_2)^2_{\text{old}} + \sin^2\varphi\,(S_1)^2_{\text{old}}}{k^2_{1\text{old}}} \tag{20}$$

Observe that in the half planes $\varphi = \frac{(2k+1)\pi}{4}$, the equality $|\sin\varphi| = |\cos\varphi|$ holds. Then according to Eqs. (9), (19) & (20), again Eq. (17) is deduced. This theorem is also valid for multilayer metamaterial structures [4–7].

4. NUMERICAL EXAMPLES

Two examples are presented here for the lossless and lossy media, which provide appropriate basis for the verification of the proposed theorem and understanding the behavior of dual material media.

4.1. Example 1: Electrical Sphere and Its Dual Magnetic Sphere

Consider a sphere of radius $a = 7\,\text{cm}$ composed of a magnetic material with parameters $\begin{cases} \varepsilon_r = 1 \\ \mu_r = 2.5 \end{cases}$ located in free space with normally incident TEM plane wave. The permittivity and permeability of metamaterials are assumed independent of frequency [8–11]. According to our theorem we can replace this sphere with a non-magnetic sphere by dual parameters $\begin{cases} \varepsilon_r = 2.5 \\ \mu_r = 1 \end{cases}$. The forward and backward scattering cross sections of these two dual spheres versus frequency are calculated and drawn in Fig. 2. It is observed that our theorem is valid.

Now for the verification of the second part of the theorem, we calculate and draw the bistatic radar cross section of these two dual spheres at half planes $\varphi = \frac{\pi}{4}, \pi$ and at frequency $f = 3\,\text{GHz}$ versus the angle of observation ($\Delta\theta = [0, \pi]$) in Fig. 3. It is seen that the RCS of dual spheres at $\varphi = \frac{\pi}{4}$ are equal which confirms our theorem. This example may be considered as a potential application of this theorem.

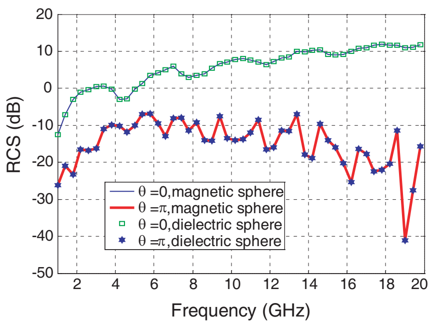

Figure 2. The forward and backward scattering cross section of a sphere located in free space versus frequency for two dual cases.

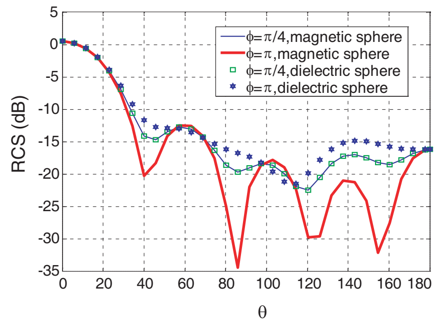

Figure 3. The bistatic radar cross section of a sphere located in free space versus angle of receiver antenna (θ) for two dual cases.

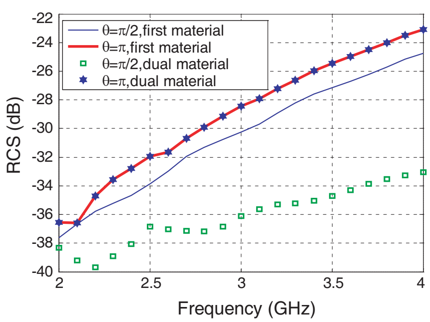

Figure 4. The radar cross section of a lossy sphere coated with a lossy material, located in free space, at half plane $\varphi = \frac{\pi}{2}$ and for two directions $\theta = \frac{\pi}{2}, \pi$ versus frequency, for two dual cases.

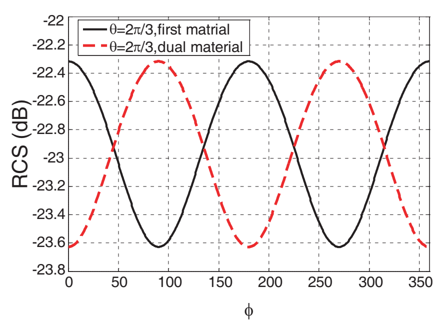

Figure 5. The radar cross section of a lossy sphere coated with a lossy material, located in free space, at direction $\theta = \frac{2\pi}{3}$ and $f = 3\,\text{GHz}$ versus angle φ, for two dual cases.

4.2. Example 2: Lossy Multilayered Spherical Structures and Their Dual Structures

Consider a sphere of radius a =10 cm composed of a lossy material with parameters $\left\{\begin{array}{l}\varepsilon=5-j0.1\\ \mu=3-j0.2\end{array}\right.$ covered by one layer of lossy material coating with thickness 2 cm and parameters $\left\{\begin{array}{l}\varepsilon=1-j0.3\\ \mu=7-j0.05\end{array}\right.$ located in free space. Now, we replace this sphere and its coating with their dual media according to the interchange $\varepsilon_i \leftrightarrow \mu_i$. The radar cross section of these two dual structures at half plane $\varphi=\frac{\pi}{2}$ and for two directions $\theta=\frac{\pi}{2},\pi$ versus frequency are calculated and drawn in Fig. 4. It is observed that the RCS of the two dual structures become equal at the direction $\theta=\pi$, (but not at $\theta=\frac{\pi}{2}$). Therefore, our theorem is also valid for lossy and multilayered structures.

We also calculate the radar cross section of these two dual structures at the zenith angle $\theta=\frac{2\pi}{3}$ at frequency $f=6\,\text{GHz}$ and draw it versus the azimuth angle of observation $(0\le\varphi\le 2\pi)$ in Fig. 5. It is observed that the RCS of dual structures at half planes $\varphi=\frac{(2k+1)\pi}{4}$ $(k=0,1,2,3)$ become equal for two dual cases, verifying the second part of our theorem.

As a suggestion, it worth mentioning that these theorems may be extended to broader forms, such as anisotropic materials. In this case, one can also calculate that the total RCS should be the same with replacements $\varepsilon_i \leftrightarrow \mu_i$. One can prove this from a more generalized material, which is radially anisotropic [12]. Based on the extended Mie theory [12] for such anisotropic spheres, we can prove that these theorems are valid for an isotropic sphere, because an isotropic sphere is just a special case of the general anisotropic sphere by making all elements in diagonal tensor of $\overline{\overline{\varepsilon}}$, $\overline{\overline{\mu}}$ equal.

In the Ref. [13] based on the extended Mie theory (using fractional-order Ricatti-Bessel function), similar relations to what we achieved in Eq. (3) has been concluded, considering which, it can be seen that the same results as of the Eq. (14) could be proven.

Furthermore, as the RCSs are a function of (a_n+b_n), $(a_n-b_n)^2$, $(|a_n|^2+|b_n|^2)$, and based on relations given in Ref. [13], also the Eq. (14), the theorems of this article can be extended to all RCSs (extinction, scattering, absorption, radar backscattering) as well.

Sometimes we should use special magnetic scatterers for a particular application (e.g., achieving desired patterns). Despite this, it occurs that some magnetic materials are less available or even not realizable. Instead, considering these theorems, one may use dual electrical materials. This is in fact one of the applications of our theorems. Some researchers even discussed the isorefractive sphere in the literatures [14], which can have a lot of interesting applications.

Nevertheless, these theorems can be examined as an initiative to be extended to other structures, forms and kinds of materials by the researchers and to find physical applications for theses duality theorems.

5. CONCLUSION

A theorem has been proved for the TEM plane wave incidence on a spherical structure. The interchange of materials $\varepsilon_i \leftrightarrow \mu_i$, leads to identical forward ($\theta = 0$) and backward ($\theta = \pi$) radar cross sections for the two dual structures. Therefore, a duality is established for the radar cross section between two dual structures with replacements $\varepsilon_i \leftrightarrow \mu_i$. One consequence of these theorems is that what is achievable for some specifications (such as zero reflection conditions) by some type of materials is also realizable by their dual materials. Two examples are provided and appropriate conclusions are derived. In the absence of a specific material, its dual can be used and it may be considered as a potential application of this theorem.

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