Progress In Electromagnetics Research M, Vol. 14, 61–70, 2010

SIMPLE PROCEDURE FOR EVALUATING THE IMPEDANCE MATRIX OF FRACTAL AND FRACTILE ARRAYS

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Abstract—A fractal array is an antenna array which holds a property called "self-similarity". This means that parts of the whole structure are similar to the whole. A recursive procedure for evaluating the impedance matrix is allowed primarily by exploiting the self-similarity. However, numerous fractal arrays are extremely complicated in structure. Therefore, for these arrays, it is extremely elaborate to formulate explicitly a recursive relation. This paper proposes a simple procedure for evaluating, without formulating explicitly a recursive relation, the impedance matrix of fractal and fractile arrays; a fractile array is any array with a fractal boundary contour that tiles the plane without gaps or overlaps.

1. INTRODUCTION

The term "fractal", originally coined by Mandelbrot [1], means broken or irregular fragments. For fractals that have the property known as self-similarity, parts of their structure are similar to the whole in some way. The concept of fractal geometry was originated to describe complex shapes in nature that cannot be easily characterized using classical Euclidean geometry. Concepts based on fractal geometry have been finding an increasing number of applications in engineering and science [2, 3]; one of which is fractal array engineering.

A fractal array is an antenna array which holds a property called "self-similarity". This means that parts of the whole structure are similar to the whole. Recently a recursive procedure for evaluating the impedance matrix of linear and planar fractal arrays has been

Received 14 July 2010, Accepted 23 August 2010, Scheduled 25 August 2010
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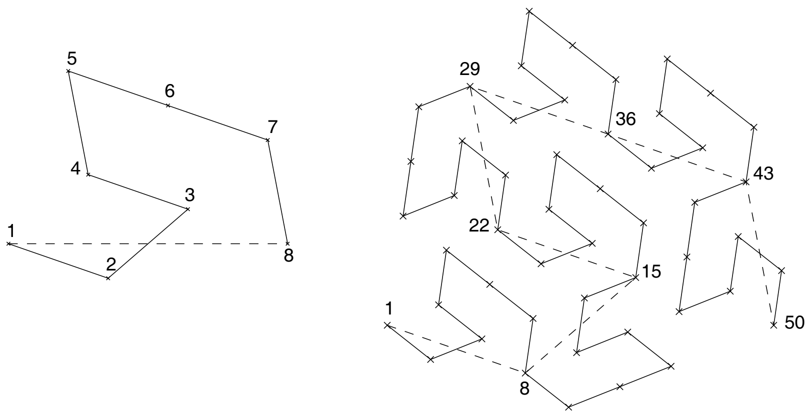
developed in [4]. A fractal array is an antenna array which holds a property called "self-similarity". This means that parts of the whole structure of the arrays are similar to the whole. However, only the procedure for the triadic Cantor linear array and Sierpinski carpet fractal array was investigated in depth. It is obvious that the recursive procedure for fractal arrays with complicated structures is extremely elaborate for implementation. The evidence is that Kuhirun [5] attempted to develop a recursive procedure for evaluating the impedance matrix of the Peano-Gosper fractal array but fail to fully formulate a recursive relation. Therefore, Kuhirun [6] developed a simple procedure for evaluating the impedance matrix of the Peano-Gosper fractal array. Extended from [6], this paper presents a simple procedure for evaluating the impedance matrix of fractal and fractile arrays, a further development of the recursive procedure for the impedance matrix investigated by Werner et al. [4] and Kuhirun [5]. The simple procedure enables us to evaluate the impedance matrix without formulating an explicit recursive relation for the impedance matrix of fractal and fractile arrays; fractile arrays are defined in [7] to be any array which has a fractal boundary contour that tiles the plane without gaps and overlaps. Tilings of the plane is extensively discussed in [8]. Examples of fractal and fractile arrays used for demonstration in this paper are the Peano-Gosper fractal array and the terdragon and 6-terdragon fractile arrays, respectively.

1.1. The Peano-Gosper Fractal Array

The Peano-Gosper fractal array is the first deterministic array which has no grating lobes even when the minimum spacing between elements is increased to at least a wavelength. It was first introduced in [1, 9]. Its elements are distributed uniformly along a curve known as "Peano-Gosper curve".

The Peano-Gosper curve for the stage of growth $P = 2$ can be generated from that for the stage of growth $P = 1$. The Peano-Gosper curve for the stage of growth $P = 2$ consists of 7 copies of the Peano-Gosper curve for the stage of growth $P = 1$. Figure 1 shows the construction of the Peano-Gosper fractal array for the stage of growth $P = 1$ and 2. The first to the last elements are distributed uniformly along the Peano-Gosper curve from the leftmost to the rightmost end, respectively; each of which is represented by "x". The dashed curve represents the Peano-Gosper curve at the previous stage.

Moreover, the Peano-Gosper curve at stage $P (P > 1)$ consists of 7 copies of the curve for the stage of growth $P - 1$. Hence, the Peano-Gosper curve consists of 7^P subsections. Figure 2 shows the stage 3 Peano-Gosper fractal array whose elements are distributed along the



(a) $P = 1$ (b) $P = 2$

Figure 1. The Peano-Gosper fractal array for the first two stages of growth the first to the last whose elements are distributed uniformly along the Peano-Gosper curve (darkened curve) from the leftmost to the rightmost ends, respectively; each of which is represented by "x". The dashed curve represents the Peano-Gosper curve represents the Peano-Gosper curve at the previous stage (From [1, 9]).

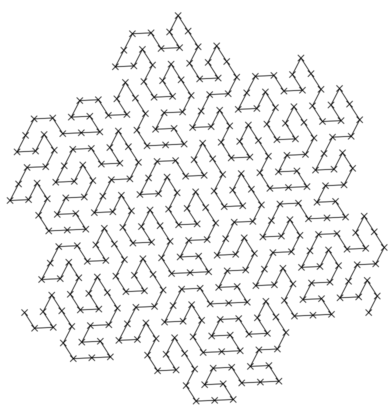
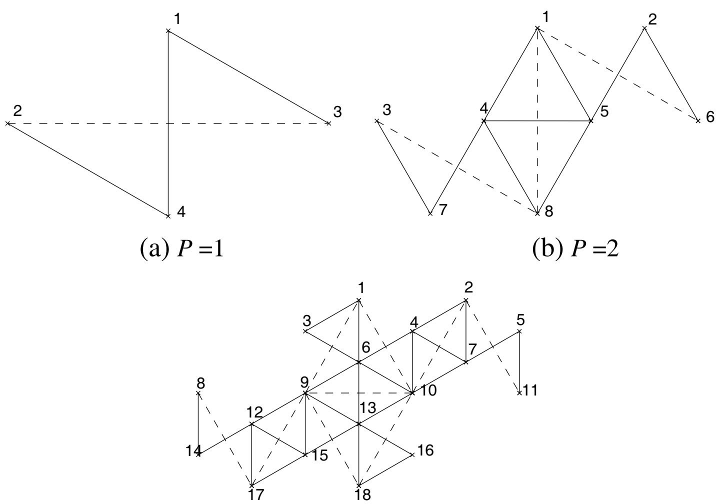


Figure 2. The Peano-Gosper fractal array for the stage of growth $P = 3$ whose the first to the last elements are distributed uniformly along the Peano-Gosper curve (darkened curve) from the leftmost to the rightmost ends, respectively; each of which is represented by "x". Note that the numbering scheme is not shown (From [1, 9]).

Peano-Gosper curve. The first to the last elements are numbered from the leftmost to the rightmost ends, respectively. It should be noted that the numbering scheme is not shown in Figure 2.

1.2. The Terdragon Fractile Array

Similar to the Peano-Gosper fractal array, the terdragon fractile array is a deterministic array which has no grating lobes even when the minimum spacing between elements is increased to at least a wavelength. It was first introduced in [1, 7]. Figure 3 shows numbering scheme and the construction of the terdragon array whose elements are distributed uniformly along a curve known as the terdragon curve for the stages of growth $P = 1, 2$ and 3. Elements are numbered from the farthest left to the farthest right and from the top to bottom; each of elements are represented by "x". The dashed curve represents the terdragon curve at the previous stage. Figure 4 shows that the stage 5 terdragon fractile array whose elements are distributed uniformly along the terdragon curve [10]. It should be noted that the numbering scheme is not shown in Figure 4.



(a) $P = 1$ (b) $P = 2$

(c) $P = 3$

Figure 3. The terdragon fractile array for the first three stages of growth whose elements are distributed uniformly along the terdragon curve (darkened curve); each of which is represented by "x". Elements are numbered from the farthest left to the farthest right and from the top to bottom. The dashed curve represents the terdragon curve at the previous stage (From [1, 7]).

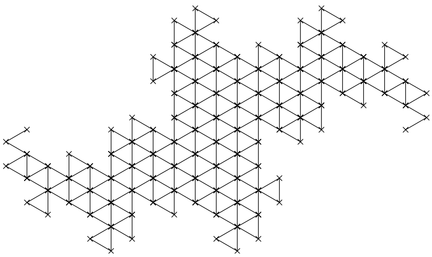


Figure 4. The terdragon fractile array for the stage of growth $P = 5$ whose elements are distributed uniformly along the terdragon curve; each of which is represented by "x". Note that the numbering scheme is not shown (From [1, 7]).

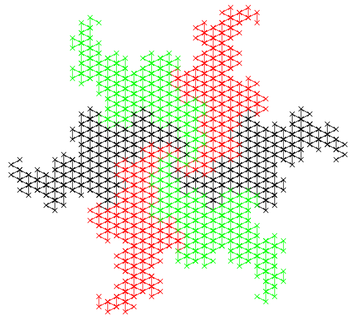


Figure 5. The 6-terdragon fractile array for the stage of growth $P = 5$ whose elements are distributed uniformly along the 6-terdragon; each of which is represented by "x". Note that the numbering scheme is not shown (From [1, 7]).

1.3. The 6-Terdragon Fractile Array

The 6-terdragon fractile array consists of 6 terdragon fractile arrays as shown in Figure 5. Similar to the numbering scheme shown in Figure 3, elements shown in Figure 5 are numbered from the farthest left to the farthest right and from the top to bottom. Note that the numbering scheme is not shown in Figure 5.

2. IMPEDANCE MATRIX AND SPACING MATRIX

For the stage of growth P, the impedance matrix $[Z]^P$ is an $N_P \times N_P$ matrix where N_P is the number of elements. Z_{ij}^P denotes the mutual impedance between the ith element and jth element for $i \neq j$ and denotes the self-impedance of the ith element for $i = j$.

In this paper, we consider only arrays consisting only of symmetric antenna elements with identical orientation, for example, circular patch antenna elements. More precisely, for the stage of growth P, mutual impedance Z_{ij}^P where $i \neq j$ depends only on the spacing between the ith element and jth element; the spacing is denoted by D_{ij}^P. Note that this is also held true for self-impedance Z_{ii}^P. As a result, if the associated spacing matrix $[D]^P$ holds symmetry, self-similarity properties, the impedance matrix $[Z]^P$ would hold the same properties. It should be noted that the assumption that mutual impedance Z_{ij}^P where $i \neq j$

depends only on the spacing between the ith element and jth element; the spacing is denoted by D_{ij}^P might not held true. For example, in the case of 3-D fractal arrays, if each individual element is an circular aperture, the mutual effect between each two individual elements does not depend only the interspacing between them but also depends on the presence of element in between them.

Figure 6 illustrates spacing matrix of the stage 3 Peano-Gosper fractal array. Each individual entry is represented by its associated picture element. It is found that a simple procedure for evaluating the impedance matrix is allowed by exploiting the symmetry and self-similarity properties. We can determine which entries in the impedance matrix $[Z]^P$ are required for evaluating and which entries are not by using the fact that the impedance Z_{ij}^P depends only on the spacing D_{ij}^P. It should be stated that this is not only applicable to the Peano-Gosper fractal array but also to any other fractal and fractile arrays, for example, the terdragon and 6-terdragon fractile arrays.

Figures 7 and 8 show the spacing matrix of the stage 5 terdragon and 6-terdragon fractile arrays, respectively. Each individual entry is represented by its associated picture element. Similar to that in Figure 6, the spacing matrices in Figures 7 and 8 are symmetric and repeated in pattern. Consequently, we can also exploit symmetry and self-similarity property in the same manner. The aforementioned simple procedure is explained in the next section.

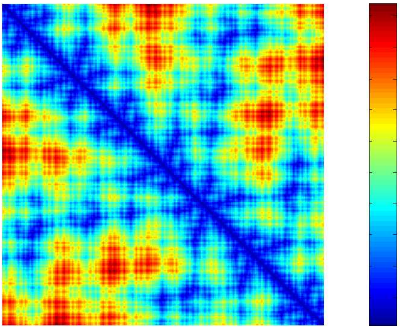


Figure 6. Spacing matrix D_P for the Peano-Gosper fractal array in terms of minimum spacing $d_{\min}$ for the stage of growth $P = 3$ (From [5]).

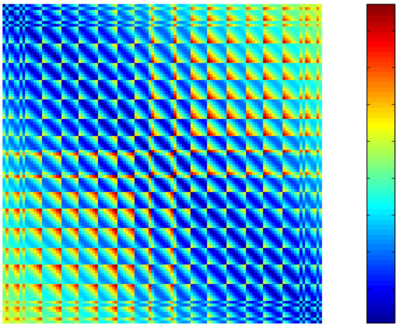


Figure 7. Spacing matrix D_P for the terdragon fractile array in terms of minimum spacing $d_{\min}$ for the stage of growth $P = 5$.

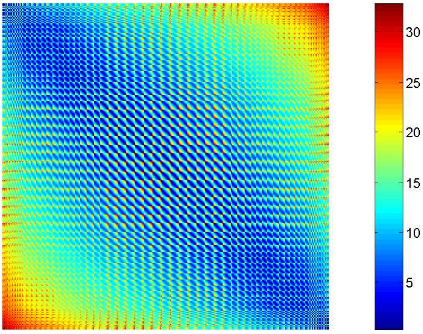


Figure 8. Spacing matrix D_P for the 6-terdragon fractile array in terms of minimum spacing $d_{\min}$ for the stage of growth $P = 5$.

3. PROCEDURE FOR EVALUATING THE IMPEDANCE MATRIX OF FRACTAL AND FRACTILE ARRAYS

First of all, determine which impedance matrix entries required for evaluating and determine which entries are the corresponding filled-in entries. This pre-procedure is shown below:

$d(1) = 0$ and $n(1) = 1$
$x(1, 1) = 1$ and $y(1, 1) = 1$
$n_S = 1$ and $check = 0$
$n(n_S) = 1$
for $i = 1$ to N_P **do**
  for $j = 1$ to N_P **do**
    evaluate $D^P(i, j)$
    if $i \neq 1$ or $j \neq 1$ **then**
      for $k = 1$ to n_S **do**
        if $d(k) = D^P(i, j)$ and $check = 0$ **then**
          $n(k) = n(k) + 1$, $x(n(k), k) = i$, $y(n(k), k) = j$ and $check = 1$
        end if
      end for
      if $check = 0$ **then**
        $n_S = n_S + 1$, $n(n_S) = 1$, $x(n(n_S), n_S) = i$, $y(n(n_S), n_S) = j$ and $d(n_S) = D^P(i, j)$
      end if
      $check = 0$
    end if
  end for
end for

The pre-procedure shows how to determine the indices $x(i,j)$ and $y(i,j)$ of the filled-in impedance matrix entry $Z^P(x(i,j), y(i,j))$ corresponding to the evaluated impedance $z(j)$ for $j = 1, 2, \ldots, n_S$.

A simple procedure is listed as follows:

(i) Determine the impedances required for evaluating $z(1), z(2), \ldots, z(n_S)$.

(ii) Fill in $Z^P(x(i,j), y(i,j))$ with the corresponding evaluated impedance $z(j)$ for $i = 1, 2, \ldots, n(j)$ and $j = 1, 2, \ldots, n_S$.

4. ANALYSIS OF SIMPLE PROCEDURE FOR EVALUATING IMPEDANCE MATRIX

Assume that the time required for evaluating self impedance equals that for evaluating mutual impedance. Let n_P be the number of times required for evaluating impedance of fractal and fractile arrays for the stage of growth P. n_P can be determined numerically. Hence, the time required for evaluating impedance matrix is:

$$t_s = n_P t_0 \tag{1}$$

where t_0 is the time required for evaluating each individual entry of the impedance matrix. In comparison, the time required for evaluating the impedance matrix directly t_d is:

$$t_d = N_P^2 t_0 \tag{2}$$

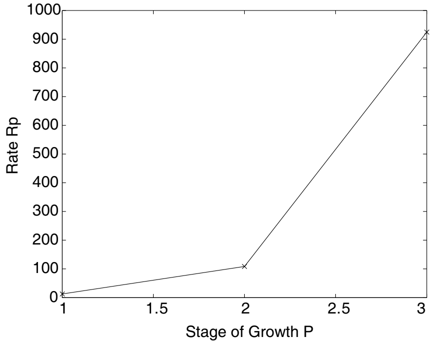


Figure 9. Plot of R_P versus Stage of Growth P for the first three stages in the Peano-Gosper fractal array (from [5]).

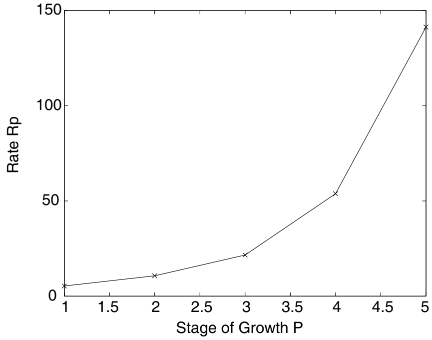


Figure 10. Plot of R_P versus Stage of Growth P for the first five stages in the Terdragon fractile array.

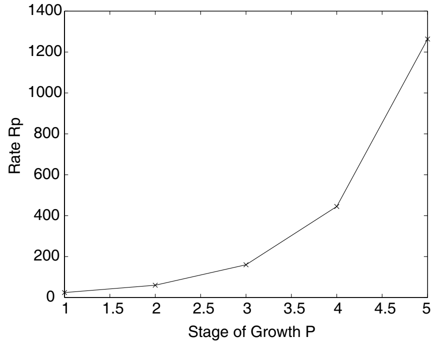


Figure 11. Plot of R_P versus Stage of Growth P for the first five stages in the 6-terdragon fractile array.

The ratio of the time for evaluating the impedance matrix directly and that for evaluating the impedance matrix by a simple procedure R_P is:

$$R_P = \frac{t_d}{t_s} = \frac{N_P^2}{n_P} \tag{3}$$

The plots of R_P versus P for the various stages of growth for the Peano-Gosper fractal array, the terdragon and 6-terdragon fractile arrays are illustrated in Figures 9, 10 and 11, respectively.

5. CONCLUSION

This paper presents a simple procedure for evaluating the impedance matrix of fractal and fractile arrays. The procedure can be implemented primarily by exploiting the symmetry and self-similarity property. The procedure can be achieved without explicitly formulating the recursive relation. The most striking benefit is due to the fact that fully formulating the recursive relation for complicated-structured arrays is not easily obtainable. That is, the simple procedure is better than the recursive procedure in the sense that the simple procedure can fully exploit self-similarity property in an easy manner whereas the recursive procedure can not do that easily. The evidence is that, by comparing Figure 9 with Figure 4 in [5], the plot of R_P versus stage of growth P of the simple procedure for the Peano-Gosper Fractal Array is better to that of the recursive procedure shown in Figure 4 in [5].

ACKNOWLEDGMENT

The author would like to thank Kasetsart University Research and Development Institute (KURDI) for financial support.

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