

MAGNETIC FIELD AND CURRENT ARE ZERO INSIDE IDEAL CONDUCTORS

M. C. N. Fiolhais

LIP-Coimbra, Department of Physics
University of Coimbra, Coimbra 3004-516, Portugal

H. Essén

Department of Mechanics, KTH
Stockholm SE-100 44, Sweden

C. Providência

Centro de Física Computacional, Department of Physics
University of Coimbra, Coimbra 3004-516, Portugal

A. B. Nordmark

Department of Mechanics, KTH
Stockholm SE-100 44, Sweden

Abstract—We prove a theorem on the magnetic energy minimum in a system of perfect, or ideal, conductors. It is analogous to Thomson's theorem on the equilibrium electric field and charge distribution in a system of conductors. We first prove Thomson's theorem using a variational principle. Our new theorem is then derived by similar methods. We find that magnetic energy is minimized when the current distribution is a surface current density with zero interior magnetic field; perfect conductors are perfectly diamagnetic. The results agree with currents in superconductors being confined near the surface. The theorem implies a generalized force that expels current and magnetic field from the interior of a conductor that loses its resistivity. Examples of solutions that obey the theorem are presented.

1. INTRODUCTION

Thomson's theorem states that electric charge density on a set of fixed conductors at static equilibrium is distributed on the surface of the conductors in such a way that the interior electric field is zero and the surface electric field is normal to the surface. Here, we will prove an analogous theorem on the magnetic field and current distribution in ideal conductors. We find that a stationary current density must distribute itself on the surface of the conductors in such a way that the interior magnetic field is zero while the surface magnetic field is perpendicular to both the current density and the surface normal.

When W. Thomson (Lord Kelvin) derived his theorem in 1848 a magnetic analog did not seem interesting since conductors with zero resistivity were unknown. Since the discovery of superconductivity in 1911 this has changed and stationary current distributions in type I superconductors, below the critical field, indeed obey our theorem. In spite of this the theorem is not stated, hardly even hinted at, in the literature. We wish to emphasize that even though the theorem applies to superconductors, phase transitions, statistical mechanics, or thermodynamics are irrelevant. Zero resistivity is simply assumed, not explained or derived. The theorem is purely a consequence of classical electromagnetism.

The outline of this article is as follows. We start by deriving Thomson's theorem using a variational principle. We then derive our minimum magnetic energy theorem in an analogous way and discuss previous work on the problem. After that an illuminating example is presented in which the energy reduction due the interior field expulsion can be calculated explicitly. Two Appendices give further motivation and explicit solutions illustrating the theorem for simple systems.

2. ENERGY MINIMUM THEOREMS

Electromagnetic energy can be written in a number of different ways. Since we will be interested in conductors and superconductors we will assume that microscopic dipoles play no essential role. The source of electric and magnetic fields from conductors and superconductors are known to be due to the distribution and motion of conduction electrons relative to the positive ion background[†]. Distinguishing between the $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{E}$, $\mathbf{B}$ fields is then unnecessary. That this is valid when treating the thermodynamics of the Meissner effect in type I superconductors is stressed by Carr [1]. Relevant energy expressions

[†] We return to the experimental evidence for this in Appendix B.

are then,

$$E_e + E_m = \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{B}^2) dV \quad (1)$$

$$= \frac{1}{2} \int \left(\varrho \phi + \frac{1}{c} \mathbf{j} \cdot \mathbf{A} \right) dV \quad (2)$$

$$= \frac{1}{2} \iint \left(\frac{\varrho(\mathbf{r})\varrho(\mathbf{r}') + \frac{1}{c^2} \mathbf{j}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) dV dV'. \quad (3)$$

Here, the first form is always valid while the two following assume quasi-statics, i.e., essentially negligible radiation.

According to Thomson's theorem, the electric charge on a set of conductors distributes itself on the conductor surfaces thereby minimizing the electrostatic energy. W. Thomson did not present a formal mathematical proof but such proofs may be found in most classic textbooks [2–5]. A recent derivation of the theorem in its differential form is by Bakhoun [6]. A derivation based on a variational principle can be found in the textbook by Kovetz [7]. A different approach also based on a variational principle, is presented below. Thomson's result is widely known and is useful in many applications. It has, e.g., been used to determine the induced surface charge density [8, 9], and in the tracing and the visualization of curvilinear squares field maps [6]. Other applications range from interesting teaching tools [10] to useful computational methods such as Monte Carlo energy minimization [11].

There are various similarities between electrostatics and magnetostatics, or quasi-statics, but for resistive media the magnetic field due to current dissipates[‡]. For *perfect*, or *ideal* conductors, however, there should be something corresponding to the magnetic version of Thomson's theorem. Indeed, below we will prove a theorem analogous to that of Thomson: *Magnetic energy is minimized by surface current distributions such that the magnetic field is zero inside while the surface field is normal to the current and the surface normal*. Energy conservation is assumed restricting the validity to perfect, or ideal, conductors. Previously somewhat similar results have appeared in the literature [12, 13] and we discuss those below.

[‡] Note that we are *not* concerned with magnetism due to microscopic dipole density.

2.1. Thomson's Theorem

In equilibrium, the electrostatic energy functional for a system of conductors surrounded by vacuum, may be written as[§],

$$E_e = \int_V \left[\varrho \phi - \frac{1}{8\pi} (\nabla \phi)^2 \right] dV, \quad (4)$$

by combining the electric parts of (1) and (2) and using $\mathbf{E} = -\nabla \phi$. We now split the integration region into the volume of the conductors, V_{in} , the exterior volume, V_{out} , and the boundary surfaces S ,

$$E_e = \int_{V_{\text{in}}} \left[\varrho \phi - \frac{1}{8\pi} (\nabla \phi)^2 \right] dV - \int_{V_{\text{out}}} \frac{1}{8\pi} (\nabla \phi)^2 dV + \int_S \sigma \phi dS, \quad (5)$$

where σ is the surface charge distribution.

We now use, $(\nabla \phi)^2 = \nabla \cdot (\phi \nabla \phi) - \phi \nabla^2 \phi$, and rewrite the divergencies using Gauss theorem. The energy functional then becomes,

$$\begin{aligned} E_e = & \int_{V_{\text{in}}} \left[\varrho \phi + \frac{1}{8\pi} \phi \nabla^2 \phi \right] dV - \int_{V_{\text{out}}} \frac{1}{8\pi} \phi \nabla^2 \phi dV \\ & + \int_S \left[\sigma \phi - \frac{1}{8\pi} \phi \hat{\mathbf{n}} \cdot (\nabla^+ \phi - \nabla^- \phi) \right] dS, \end{aligned} \quad (6)$$

where ∇^+ and ∇^- are the gradient operators at the surface in the outer and inner limits, respectively. The total charge in each conductor is constant and restricted to the conductor volume and surface. We handle this constraint by introducing a Lagrange multiplier λ . Infinitesimal variation of the energy then gives,

$$\begin{aligned} \delta E_e = & \int_{V_{\text{in}}} \left[\delta \phi \left(\varrho + \frac{1}{4\pi} \nabla^2 \phi \right) + \delta \varrho (\phi - \lambda) \right] dV + \int_{V_{\text{out}}} \delta \phi \frac{1}{4\pi} \nabla^2 \phi dV \\ & + \int_S \left\{ \delta \phi \left[\sigma + \frac{1}{4\pi} \hat{\mathbf{n}} \cdot (\nabla^+ \phi - \nabla^- \phi) \right] + \delta \sigma (\phi - \lambda) \right\} dS. \end{aligned} \quad (7)$$

From this energy minimization, the Euler-Lagrange equations become:

$$V_{\text{in}} : \quad \begin{cases} \nabla^2 \phi = -4\pi \varrho \\ \phi = \lambda \end{cases} \quad (8)$$

$$S : \quad \begin{cases} -\hat{\mathbf{n}} \cdot (\nabla^+ \phi - \nabla^- \phi) = 4\pi \sigma \\ \phi = \lambda \end{cases} \quad (9)$$

$$V_{\text{out}} : \quad \nabla^2 \phi = 0 \quad (10)$$

[§] The ultimate motivation for this specific form of the energy functional, and the corresponding one in the magnetic case, is that they lead to simple final equations.

According to Equations (8) the potential is constant inside the conductor in the minimum energy state, and therefore $\varrho = 0$ there. Equations (9) mean the electric charge is distributed on the surface in such a way that the potential is constant there. The second of Equations (8) implies that $\nabla^-\phi = 0$ and the first of Equations (9) then implies that $\hat{\mathbf{n}} \cdot \mathbf{E}_+ = 4\pi\sigma$. This concludes the proof of Thomson's theorem.

2.2. Minimum Magnetic Energy Theorem

A similar procedure will now be applied to the magnetic field. We write the magnetic energy functional for a time independent magnetic field as,

$$E_m = \int_V \left[\frac{1}{c} \mathbf{j} \cdot \mathbf{A} - \frac{1}{8\pi} (\nabla \times \mathbf{A})^2 \right] dV, \quad (11)$$

i.e., as two times the form (2) of the magnetic energy minus the form (1), using $\mathbf{B} = \nabla \times \mathbf{A}$. As before we split the volume into the volume interior to conductors, the exterior vacuum, and the surface at the interfaces, and write,

$$\begin{aligned} E_m = & \int_{V_{\text{in}}} \left[\frac{1}{c} \mathbf{j} \cdot \mathbf{A} - \frac{1}{8\pi} (\nabla \times \mathbf{A})^2 \right] dV \\ & - \int_{V_{\text{out}}} \frac{1}{8\pi} (\nabla \times \mathbf{A})^2 dV + \frac{1}{c} \int_S \mathbf{k} \cdot \mathbf{A} dS, \end{aligned} \quad (12)$$

where $\mathbf{k}$ is the surface current density. We now use the identity,

$$(\nabla \times \mathbf{A})^2 = \nabla \cdot [\mathbf{A} \times (\nabla \times \mathbf{A})] + \mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{A})], \quad (13)$$

and then use Gauss theorem to rewrite the divergence terms. The energy functional then becomes:

$$\begin{aligned} E_m = & \int_{V_{\text{in}}} \left\{ \frac{1}{c} \mathbf{j} \cdot \mathbf{A} - \frac{1}{8\pi} \mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{A})] \right\} dV \\ & - \int_{V_{\text{out}}} \frac{1}{8\pi} \mathbf{A} \cdot [\nabla \times (\nabla \times \mathbf{A})] dV \\ & + \int_S \left\{ \frac{1}{c} \mathbf{k} \cdot \mathbf{A} - \frac{1}{8\pi} \mathbf{A} \cdot [\hat{\mathbf{n}} \times (\nabla^+ \times \mathbf{A} - \nabla^- \times \mathbf{A})] \right\} dS. \end{aligned} \quad (14)$$

As in the electric case, constraints must be imposed. Due to charge conservation, the electric current density must obey the continuity equation, for zero charge density, both inside and on the surface of the conductors [14]:

$$\nabla \cdot \mathbf{j} = 0 \quad (15)$$

$$\nabla_S \cdot \mathbf{k} = 0 \quad (16)$$

where ∇_S is the surface gradient operator. Notice that these constraints are local, not global. In other words, the relevant Lagrange multiplier is not constant but a scalar field $\lambda(\mathbf{r})$. Using this, infinitesimal variation of the magnetic energy gives,

$$\begin{aligned} c \delta E_m = & \int_{V_{\text{in}}} \left\{ \delta \mathbf{A} \cdot \left(\mathbf{j} - \frac{c}{4\pi} [\nabla \times (\nabla \times \mathbf{A})] \right) + \delta \mathbf{j} \cdot (\mathbf{A} - \nabla \lambda) \right\} dV \\ & - \int_{V_{\text{out}}} \delta \mathbf{A} \cdot \frac{c}{4\pi} [\nabla \times (\nabla \times \mathbf{A})] dV + \int_S \left\{ \delta \mathbf{A} \cdot \left[\mathbf{k} + \frac{c}{4\pi} \hat{\mathbf{n}} \cdot (\nabla^+ \times \mathbf{A} - \nabla^- \times \mathbf{A}) \right] \right. \\ & \left. + \delta \mathbf{k} \cdot (\mathbf{A} - \nabla_S \lambda) \right\} dS. \end{aligned} \quad (17)$$

Equating this to zero we find that,

$$V_{\text{in}} : \quad \begin{cases} \nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}, \\ \mathbf{A} = \nabla \lambda, \end{cases} \quad (18)$$

and,

$$S : \quad \begin{cases} \mathbf{k} = \frac{c}{4\pi} \hat{\mathbf{n}} \times (\nabla^+ \times \mathbf{A} - \nabla^- \times \mathbf{A}), \\ \mathbf{A} = \nabla_S \lambda, \end{cases} \quad (19)$$

and,

$$V_{\text{out}} : \quad \nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{B} = 0, \quad (20)$$

are the Euler-Lagrange equations for this energy functional.

According to Equation (18) $\mathbf{B} = \nabla \times \nabla \lambda = 0$, so the magnetic field must be zero in V_{in} . Consequently also the volume current density is zero, $\mathbf{j} = 0$, inside the conductor, in the minimum energy state.

2.2.1. Surface Currents

Now consider the results for the surface, Equation (19). Our results from V_{in} show that $\nabla^- \times \mathbf{A} = 0$, so the equation reads,

$$\mathbf{k} = \frac{c}{4\pi} \hat{\mathbf{n}} \times (\nabla^+ \times \nabla_S \lambda). \quad (21)$$

Let us introduce a local Cartesian coordinate system with origin on the surface, such that the surface is spanned by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ with unit normal $\hat{\mathbf{n}} = \hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$. Assuming that the surface is approximately flat we then have that,

$$\nabla_S = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y}, \quad \text{and,} \quad \nabla^+ = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{n}} \frac{\partial}{\partial z^+} = \nabla_S + \hat{\mathbf{n}} \frac{\partial}{\partial z^+}. \quad (22)$$

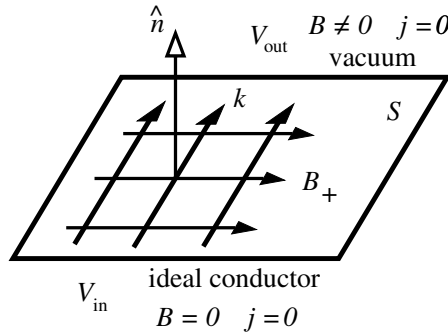


Figure 1. Some of the results of our theorem on the current density and magnetic field of an ideal conductor at minimum energy are illustrated here. Here $\mathbf{B}_+$ is the magnetic field on the outside of the surface S with surface unit normal $\hat{\mathbf{n}}$. The bulk current density $\mathbf{j}$ is zero, only the surface current density $\mathbf{k}$ is non-zero.

Since, $\mathbf{A} = \nabla_S \lambda$, the vector potential is tangent to the conducting surface and we get,

$$\frac{4\pi}{c} \mathbf{k} = \hat{\mathbf{n}} \times \left[\left(\nabla_S + \hat{\mathbf{n}} \frac{\partial}{\partial z^+} \right) \times \nabla_S \lambda(x, y, z) \right] \quad (23)$$

$$= \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \frac{\partial}{\partial z^+} \times \left(\hat{\mathbf{x}} \frac{\partial \lambda}{\partial x} + \hat{\mathbf{y}} \frac{\partial \lambda}{\partial y} \right) \right] = \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \frac{\partial}{\partial z^+} \times \mathbf{A} \right), \quad (24)$$

for the surface current density. Rewriting the triple vector product we find,

$$\frac{4\pi}{c} \mathbf{k} = \frac{\partial}{\partial z^+} [(\hat{\mathbf{n}} \cdot \mathbf{A}) \hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) \mathbf{A}] = -\frac{\partial \mathbf{A}}{\partial z^+}, \quad (25)$$

so the surface current density is parallel to the outside normal derivative of the vector potential. We note that this agrees with the well known result [5],

$$\frac{4\pi}{c} \mathbf{k} = \hat{\mathbf{n}} \times (\mathbf{B}_+ - \mathbf{B}_-), \quad (26)$$

for the case of zero interior field ($\mathbf{B}_- = 0$). Our results are summarized in Fig. 1.

2.2.2. External Fields

Neither Thomson's theorem nor our minimum magnetic energy theorem are formally valid for conductors in constant external fields. In both cases, however, such a situation can be regarded as a limiting case.

In the case of Thomson's theorem one can include two large, distant, and oppositely charged parallel conducting plates. A small system of conductors between these can then approximately be regarded as in an electric field that approaches a constant external field at large distance. In a similar way the set of perfect conductors can be thought of as inside two large perfectly conducting Helmholtz coils (tori) which provide an approximately constant external magnetic field at large distance.

2.3. Previous Work

The fact that there is an energy minimum theorem for the magnetic energy of ideal, or perfect, conductors, analogous to Thomson's theorem, is not entirely new. In an interesting, but difficult and ignored, article by Karlsson [12] such a theorem is stated. Karlsson, however, restricts his theorem to conductors with holes in them. In the electrostatic case charge conservation prevents the energy minimum from being the trivial zero field solution. In our magnetic ideal conductor case the corresponding conservation law is the conservation of magnetic flux through a hole [15, 16]. As long as *one* conductor of the system has a hole with conserved flux there will be a non-trivial magnetic field. To require that all conductors of the system have holes, as Karlsson does, therefore seems unnecessarily restrictive. One of Karlsson's results is that the current distribution on a superconducting torus minimizes the magnetic energy.

A result by Badía-Majós [13] comes even closer to our own and we outline it here. The current density is assumed to be of the form,

$$\mathbf{j} = qn\mathbf{v}, \quad (27)$$

where q is the charge of the charge carriers and n is their number density. The time derivative is then given by,

$$\frac{d\mathbf{j}}{dt} = \frac{qn}{m} m \frac{d\mathbf{v}}{dt} = \frac{qn}{m} \left(q\mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{B} \right) = \frac{q^2 n}{m} \mathbf{E} + \frac{q}{mc} \mathbf{j} \times \mathbf{B}, \quad (28)$$

assuming that only the Lorentz force acts (ideal conductor). We now recall Poynting's theorem [17] for the time derivative of the field energy density of a system of charged particles,

$$\frac{d}{dt} \left(\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} \right) = -\mathbf{j} \cdot \mathbf{E} - \nabla \cdot \left(\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right). \quad (29)$$

The first term on the right hand side normally represents resistive energy loss. Here, we use the result for $\mathbf{E}$ from Equation (28),

$$\mathbf{E} = \frac{m}{q^2 n} \frac{d\mathbf{j}}{dt} - \frac{1}{qnc} \mathbf{j} \times \mathbf{B}, \quad (30)$$

and get,

$$\mathbf{j} \cdot \mathbf{E} = \frac{m}{q^2 n} \frac{d\mathbf{j}}{dt} \cdot \mathbf{j} = \frac{d}{dt} \left(\frac{m}{2q^2 n} \mathbf{j}^2 \right). \quad (31)$$

This is thus the natural form for this term for perfect conductors. We insert it into (29), neglect radiation, and assume that $\mathbf{E}^2 \ll \mathbf{B}^2$. This gives us,

$$\frac{d}{dt} \left(\frac{1}{8\pi} \mathbf{B}^2 + \frac{m}{2q^2 n} \mathbf{j}^2 \right) = 0. \quad (32)$$

Finally inserting, $\mathbf{j} = (c/4\pi)\nabla \times \mathbf{B}$, here, gives,

$$E_B = \frac{1}{8\pi} \left[\int_V \mathbf{B}^2 dV + \int_{V_{in}} \frac{mc^2}{4\pi q^2 n} (\nabla \times \mathbf{B})^2 dV \right], \quad (33)$$

for the conserved energy, after integration over space and time.

Badía-Majós [13] then notes that this energy functional implies flux expulsion from superconductors. Variation of the functional gives the London equation [18],

$$\mathbf{B} + \frac{1}{4\pi} \frac{mc^2}{q^2 n} \nabla \times (\nabla \times \mathbf{B}) = 0. \quad (34)$$

Badía-Majós, chooses not to point out that this classical derivation of flux expulsion is in conflict with frequent text book statements to the effect that no such classical result exists. Further work by Badía-Majós et al. [19] on variational principles for electromagnetism in conducting materials should be noted.

Finally, we should mention the pioneering work by Woltjer [20] on energy extremizing properties of, so called, force free magnetic fields. In plasma physics a lot of further work has been done in that tradition. It has, however, not been concerned with currents and fields inside or on the boundary of bounded domains separated by vacuum, as we are here.

3. IDEALLY CONDUCTING SPHERE IN EXTERNAL FIELD

We now know that magnetic energy minimum occurs when current flows only on the surface and the magnetic field is zero inside. Let us consider a perfectly conducting sphere surrounded by a fixed constant external magnetic field. Here, we will calculate the surface current needed to exclude the magnetic field from the interior and how much the total magnetic energy E_m is then reduced. In order to exclude a constant external field $\mathbf{B}_e$ from its interior the currents on the sphere must obviously produce an interior magnetic field $\mathbf{B}_i = -\mathbf{B}_e$, thereby

making the total field $\mathbf{B} = \mathbf{B}_e + \mathbf{B}_i$ zero in the interior. We will use that a constant field is produced inside a sphere by a current distribution due to rigid rotation of a constant surface charge density [21].

3.1. Energy of the External Field

For the total magnetic field to have a finite energy E_m , we can not assume that the constant external field extends to infinity. Instead of using Helmholtz coils to produce it, we simplify the mathematics and produce our external field by a spherical shell of current that is equivalent to a rigidly rotating current distribution on the surface. This can be done in practice by having a set of rings representing closely spaced longitudes on a globe with the right amount of current maintained in each of them. Such a spherical shell of rigidly rotating charge produces a magnetic field that is constant inside the sphere and a pure dipole field outside the sphere (see Fig. 2):

$$\mathbf{B}_e(\mathbf{r}) = \begin{cases} \frac{2\mathbf{m}}{R^3} & \text{for } r \leq R \\ \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} & \text{for } r > R \end{cases} \quad (35)$$

Here $r = |\mathbf{r}|$ and the center of the sphere is at the origin. If Q is the total rotating surface charge and $\boldsymbol{\omega}$ its angular velocity,

$$\mathbf{m} = \frac{QR^2}{3c} \boldsymbol{\omega}, \quad (36)$$

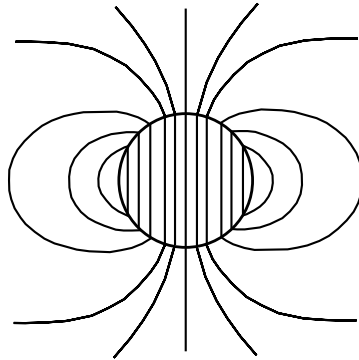


Figure 2. The field lines of the field of Equation (35) for a nonmagnetic sphere with a rigidly rotating homogeneous surface charge density.

see Equation (A6) below. It is now easy to calculate the magnetic energy of this field. One finds^{||},

$$E_m^0 = \frac{1}{8\pi} \left(\int_{r < R} \mathbf{B}_e^2 dV + \int_{r > R} \mathbf{B}_e^2 dV \right) = \left(\frac{2}{3} + \frac{1}{3} \right) \frac{\mathbf{m}^2}{R^3} = \frac{m^2}{R^3}. \quad (37)$$

Inside this sphere, which is assumed to maintain a constant current density on its surface, we now place a smaller perfectly conducting sphere.

3.2. Magnetic Energy of the Two Sphere System

We assume that the small sphere in the middle of the big one has radius $a < R$ and that it also produces a magnetic field by a rigidly rotating charged shell on its surface. We denote its dipole moment by $\mathbf{m}_i$ so that its total energy would be,

$$E_m^i = \frac{\mathbf{m}_i^2}{a^3} = \frac{m_i^2}{a^3}, \quad (38)$$

if it was far from all other fields, according to our previous result (37). We now place the small sphere inside the large one and assume that $\mathbf{m}_i$ makes an angle α with $\mathbf{m} = m \hat{\mathbf{m}}$,

$$\mathbf{m} \cdot \mathbf{m}_i = m m_i \cos \alpha. \quad (39)$$

The total energy of the system is now,

$$E_m = \frac{1}{8\pi} \int (\mathbf{B}_e + \mathbf{B}_i)^2 dV = E_m^0 + E_m^i + E_c, \quad (40)$$

where the coupling (interaction) energy is,

$$E_c = \frac{1}{4\pi} \int \mathbf{B}_e \cdot \mathbf{B}_i dV. \quad (41)$$

This integral must be split into the three radial regions: $0 \leq r < a$, $a \leq r < R$, and $R \leq r$. The calculations are elementary using spherical coordinates. The contribution from the inner region is,

$$E_{c1} = \frac{1}{4\pi} \int_{r < a} \mathbf{B}_e \cdot \mathbf{B}_i dV = \frac{4}{3} \frac{m m_i}{R^3} \cos \alpha. \quad (42)$$

The middle region, where there is a superposition of a dipole field from the small sphere and a constant field from the big one, contributes zero: $E_{c2} = 0$. The outer region gives $E_{c3} = (2/3) m m_i \cos \alpha / R^3$. Summing up one finds,

$$E_c = 2 \frac{m m_i}{R^3} \cos \alpha, \quad (43)$$

for the magnetic interaction energy of the two spheres.

^{||} If (36) is inserted for $\mathbf{m}$ here we get $E_m^0 = (\omega/c)^2 R Q^2 / 9$ which is equal to E_m of Equation (A8) below, for $\xi = 1$, corresponding to surface current only, as it should.

3.3. Minimizing the Total Magnetic Energy

The total magnetic energy of the system discussed above is thus,

$$E_m(m_i, \alpha) = E_m^0 + E_m^i + E_c = \frac{m^2}{R^3} + \frac{m_i^2}{a^3} + 2 \frac{mm_i}{R^3} \cos \alpha. \quad (44)$$

We assume that m and m_i are positive quantities. This means that as a function of α this quantity is guarantied to have its minimum when $\cos \alpha = -1$, i.e., for $\alpha = \pi$. Thus, at minimum, the dipole of the inner sphere has the opposite direction to that of the constant external field, $\mathbf{m}_i = -m_i \hat{\mathbf{m}}$.

Now, assuming $\alpha = \pi$, we can look for the minimum as a function of m_i . Elementary algebra shows that this minimum is attained for,

$$m_i = \left(\frac{a}{R}\right)^3 m \equiv m_{i \min}. \quad (45)$$

The magnetic field in the interior of the inner sphere ($r < a$) is then,

$$\mathbf{B}_e + \mathbf{B}_i = \frac{2\mathbf{m}}{R^3} + \frac{2\mathbf{m}_{i \min}}{a^3} = \left(\frac{2m}{R^3} - \frac{2m_{i \min}}{a^3}\right) \hat{\mathbf{m}} = 0, \quad (46)$$

so it has been *expelled*. The minimized energy (44) of the system is found to be,

$$E_{m \min} = E_m(m_{\min}, \pi) = \frac{m^2}{R^3} \left[1 - \left(\frac{a}{R}\right)^3\right]. \quad (47)$$

The relative energy reduction is thus given by the volume ratio of the two spheres.

Using E_m^0 from (37), the energy lowering is now found to be:

$$E_m^0 - E_{m \min} = \frac{m^2}{R^3} \frac{a^3}{R^3} = \frac{\mathbf{B}_e^2}{4} a^3 = 3 \left(\frac{\mathbf{B}_e^2}{8\pi}\right) \left(\frac{4\pi a^3}{3}\right). \quad (48)$$

This result is independent of the radius R of the big sphere introduced to produce the constant external field. It shows that the energy lowering corresponds to three times the external magnetic energy in the volume $4\pi a^3/3$ of the perfectly conducting interior sphere.

4. THE MECHANISM OF FLUX EXPULSION

In 1933 Meissner and Ochsenfeld [22], discussing their experimental discovery, stated that it is understandable that an external magnetic field does not penetrate a superconductor[¶] but that the expulsion

[¶] Eddy currents induced in accordance with Lenz law do not dissipate because of zero resistivity.

of a pre-existing field at the phase transition cannot be understood by classical physics. This statement has since been repeated many times. We are not aware, however, of any deeper investigations of what classical electromagnetism predicts regarding the behavior of perfect conductors in this respect, at least not prior to the work of Karlsson [12] and Badía-Majós [13]. Our theorem strengthens the conclusion of Badía-Majós that a magnetic field is expelled according to classical electromagnetism. Hirsch [23] has pointed out that the Meissner effect is not explained by BCS theory. Here we briefly speculate on the microscopic physical mechanism of this expulsion.

Assume that a resistive metal sphere is penetrated by a constant magnetic field. Lower the temperature until the resistance vanishes. How does the metal sphere expel the magnetic field, or equivalently, how does it produce surface currents that screen the external field? According to Forrest [24] this can not be understood from the point of view of classical electrodynamics since in a perfectly conducting medium the field lines must be frozen-in. This claim is motivated thus: When the resistivity is zero there can be no electric field according to Ohm's law, since this law then predicts infinite current. But if the electric field is zero the Maxwell equation, $\nabla \times \mathbf{E} - \partial \mathbf{B} / \partial t = 0$, requires that the time derivative of the magnetic field is zero. Hence it must be constant.

This argument is flawed since Ohm's law is not applicable. Inertia, inductive or due to rest mass, prevents infinite acceleration. Instead the system of charged particles undergoes thermal fluctuations and these produce electric and magnetic fields. These fields accelerate charges according to the Lorentz force law. In the normal situation the corresponding currents and fields remain microscopic. When there is an external magnetic field present the overall energy is lowered if these microscopic currents correlate and grow to exclude the external field. According to standard statistical mechanics the system will then eventually relax to the energy minimum state consistent with constraints. We note that Alfvén and Fälthammar [25] state that "in low density plasmas the concept of frozen-in lines of force is questionable".

Another argument by Forrest [24] is that the magnetic flux through a perfectly conducting current loop is conserved. Since one can imagine arbitrary current loops in the metal that just lost its resistivity with a magnetic field inside, the field must remain fixed, it seems. It is indeed correct that the flux through an ideal current loop is conserved, but the actual physical current loop will not remain intact unless constrained by non-electromagnetic forces. There will be forces on a loop of current that encloses a magnetic flux that expands

it [5]. This is the well know mechanism behind the rail gun, see, e.g., Essén [26]. All the little current loops in the metal will thus expand until they come to the surface where the expansion stops. In this way the interior field is thinned out and current concentrates near the surface. So, the flux is conserved through the loops, but the loops expand.

It was noted early in the history of superconductivity that the Meissner flux expulsion is necessary if superconductors obey normal thermodynamics. Gorter and Casimir [27] observed that the final thermal equilibrium state of a superconductor must be independent of whether the external magnetic field existed inside the body prior to the phase transition or if it was added after the transition already had occurred. Our theorem indicates that it is perfectly natural that an interior magnetic field is expelled in the approach to thermodynamic equilibrium.

5. CONCLUSIONS

It is amazing that the theorem derived here has not been stated before. Zero resistivity conductors have been known since 1911 and zero resistivity is also considered a good approximation in many plasmas, so the ideal, or perfect conductor, is a well known concept and its magnetic energy minimum state ought to be of great importance. The well known symmetry between source free electric and magnetic fields, as revealed by Maxwell's equations, indicates that a magnetic analog of Thomson's theorem might exist. In a relativistic context the scalar Lagrangian density $\sim A^\mu j_\mu$ also implies a symmetry between electricity and magnetism but the fact that space is three-dimensional and time one-dimensional implies a vector source for the magnetic field, as compared to the scalar electric case. This explains why our derivation for the magnetic theorem is somewhat more complicated.

It is clear from our results above, and further motivated by the examples in the Appendices below, that type I superconductors below their critical field obey the theorem, and the reason that these only have surface current and zero interior field is simply minimization of magnetic energy. Naturally there is some other energy involved that is responsible for the phase transition to a zero resistivity state. But apart from being implicitly assumed constant in our variations its nature and origin is irrelevant to the current investigation.

ACKNOWLEDGMENT

Essén and Nordmark are grateful for research support from the Department of Mechanics at KTH.

APPENDIX A. MAGNETIC ENERGY MINIMIZATION IN SIMPLE ONE DEGREE OF FREEDOM MODEL SYSTEMS

We investigate two simple one degree of freedom model systems and use them to illustrate how the minimum magnetic energy theorem works. We take systems in which the magnetic energy E_m can be calculated exactly so that energy minimization amounts to minimizing a function of a single variable. The systems are both related to a system used by Brito and Fiolhais [10] to study electric energy.

A.0.1. Magnetic Energy of Coaxial Cable

The minimum magnetic energy theorem can be illustrated in such a simple system as a coaxial cable. The cable can be modeled by an outer cylindrical conducting shell with radius b , carrying an electric current I , and a concentric solid cylindrical conductor with radius $a < b$, carrying the same electric current in the opposite direction. We now assume that the total current on the inner cylinder is the sum of surface current I_s and bulk interior current I_v . See Fig. (A1).

Here, we use cylindrical coordinates, ρ , φ , z and put $I_s = (1 - \eta)I$ at $\rho = a$ for the surface current, and $I_v = \eta I$ for the bulk current in

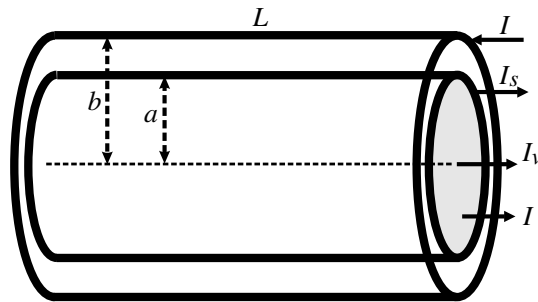


Figure A1. Notation for the coaxial cable. Magnetic energy is minimized when the current on the inner conductor is pure surface current $I_s = I$, and $I_v = 0$.

$0 \leq \rho < a$. Then $I = I_s + I_v$, and we get the magnetic field,

$$\mathbf{B}(\rho, \eta) = \frac{2I}{c} \cdot \begin{cases} \frac{\eta\rho}{a^2} \hat{\varphi} & 0 \leq \rho < a \\ \frac{1}{\rho} \hat{\varphi} & a \leq \rho \leq b \\ 0 & b < \rho \end{cases} \quad (\text{A1})$$

using Ampère's law. Thus, the magnetic field energy for a length L of the cable is given by:

$$\begin{aligned} E_m(\eta) &= \frac{1}{8\pi} \int_V \mathbf{B}^2 dV \\ &= \frac{1}{8\pi} \left(\frac{2I}{c} \right)^2 \left[\int_0^a \left(\frac{\eta\rho}{a^2} \right)^2 L 2\pi\rho d\rho + \int_a^b \left(\frac{1}{\rho} \right)^2 L 2\pi\rho d\rho \right] \\ &= \frac{LI^2}{c^2} \left[\frac{\eta^2}{4} + \ln\left(\frac{b}{a}\right) \right]. \end{aligned} \quad (\text{A2})$$

This magnetic energy reaches its minimum for zero bulk current, $\eta = 0$, corresponding to surface current only, and zero field for $0 \leq \rho < a$.

A.0.2. Current in Sphere Due to Rigidly Rotating Charge

Consider an ideally conducting sphere of radius R . Assume that there is a circulating current in the sphere which can be seen as the rigid rotation of a charge Q evenly distributed in the thick spherical shell between $r = a < R$ and $r = R$. The charge density,

$$\varrho(\mathbf{r}) = \begin{cases} 0 & \text{for } 0 \leq r < a \\ \frac{3Q}{4\pi(R^3 - a^3)} & \text{for } a \leq r \leq R \\ 0 & \text{for } R < r \end{cases} \quad (\text{A3})$$

is assumed to rotate with angular velocity $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$ relative to an identical charge density of opposite sign at rest. The current density is then,

$$\mathbf{j}(\mathbf{r}) = \varrho(\mathbf{r}) \boldsymbol{\omega} \times \mathbf{r}, \quad (\text{A4})$$

and the current, $I = \frac{Q}{2\pi}\omega$, passes through a half plane with the z -axis as edge.

The vector potential produced by this current density can be found using the methods of Essén [28], see also [29–31]. If we introduce $\xi = a/R$, we find,

$$\mathbf{A}(\mathbf{r}) = \frac{Q}{c} (\boldsymbol{\omega} \times \mathbf{r}) \cdot \begin{cases} \frac{(1 - \xi^2)}{2(1 - \xi^3)R} & \text{for } 0 \leq r < a \\ \frac{R^2}{10(1 - \xi^3)} \left(\frac{5}{R^3} - 3 \frac{r^2}{R^5} - 2 \frac{\xi^5}{r^3} \right) & \text{for } a \leq r \leq R \\ \frac{(1 - \xi^5)}{5(1 - \xi^3)} \frac{R^2}{r^3} & \text{for } R < r \end{cases} \quad (\text{A5})$$

The parameter $\xi = a/R$ is zero, $\xi = 0$, for a homogeneous ball of rotating charge, while $\xi = 1$ corresponds to a rotating shell of surface charge, see Fig. A2. Comparing with the vector potential for a constant field, $\mathbf{A} = \frac{1}{2} \mathbf{B}_0 \times \mathbf{r}$ we see that,

$$\mathbf{B}_0 = \frac{Q\omega}{cR} \frac{(1 - \xi^2)}{(1 - \xi^3)} = \frac{Q\omega}{cR} \frac{(1 + \xi)}{(1 + \xi + \xi^2)} \quad (\text{A6})$$

is the field in the central current free region $0 \leq r < a$.

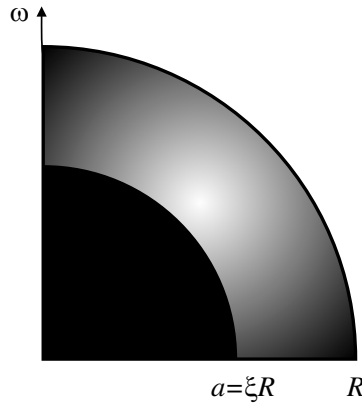


Figure A2. Some notation for the system considered here. Current density flows in a thick spherical shell as a rigid rotation of constant charge density between $r = a = \xi R$ and $r = R$.

A.0.3. Magnetic Energy of Rotating Spherical Shell Current

We now calculate the magnetic energy of this system using the formula,

$$E_m = \frac{1}{2c} \int \mathbf{j} \cdot \mathbf{A} \, dV. \quad (\text{A7})$$

Performing the integration using spherical coordinates gives,

$$E_m(\omega, \xi) = \left(\frac{R\omega}{c} \right)^2 \frac{Q^2}{R} f(\xi), \quad (\text{A8})$$

where,

$$f(\xi) = \frac{2 + 4\xi + 6\xi^2 + 8\xi^3 + 10\xi^4 + 5\xi^5}{35(1 + \xi + \xi^2)^2}, \quad (\text{A9})$$

is a function of the dimensionless parameter ξ . Note that $f(0) = 2/35$, that $f(1) = 1/9$, and that $f(\xi)$ is monotonically increasing, by a factor of almost 2 in the interval 0 to 1.

This expression for the energy is the Lagrangian form of a kinetic energy which depends on the generalized velocity $\omega = \dot{\varphi}$,

$$L_m(\dot{\varphi}, \xi) = \frac{R^2}{c^2} \frac{Q^2}{R} f(\xi) \dot{\varphi}^2. \quad (\text{A10})$$

Since the generalized coordinate φ does not appear in the Lagrangian L_m the corresponding generalized momentum (the angular momentum),

$$p_\varphi = \frac{\partial L_m}{\partial \dot{\varphi}} = 2 \frac{R^2}{c^2} \frac{Q^2}{R} f(\xi) \dot{\varphi}, \quad (\text{A11})$$

is a conserved quantity. The corresponding Hamiltonian, and relevant, expression for the magnetic energy is then $H_m = p_\varphi \dot{\varphi} - L_m$, expressed in terms of p_φ ,

$$E_m(\xi) = H_m(p_\varphi, \xi) = \frac{c^2}{4} \frac{p_\varphi^2}{Q^2 f(\xi) R}. \quad (\text{A12})$$

The function $1/f(\xi)$ is plotted in Fig. A3. We now consider the two energy expressions (A10) and (A12) separately.

Case of constant current: We first consider the case that the *current*, $I = Q\omega/2\pi$, is *constant*. Changing ξ then means changing the conductor geometry while keeping a constant total current, or, equivalently, angular velocity $\omega = \dot{\varphi}$. One might regard the total current as flowing in a continuum of circular wires. Changing ξ from zero to one means changing the distribution of these circular wires from a bulk distribution in the sphere to a pure surface distribution, while maintaining constant current. According to a result by Greiner [32] a

system will tend to *maximize* its magnetic energy when the conductor geometry changes while currents are kept constant. This has also been discussed in Essén [26]. In conclusion, if currents are kept constant the magnetic energy (A10) will tend (thermodynamically) to a stable equilibrium with at a *maximum* value and we note that this corresponds to a pure surface current $\xi = 1$.

Case of constant angular momentum: Assume now that we pass to the Hamiltonian (canonical) formalism. Thermodynamically this type of system should tend to minimize its phase space energy (A12) in accordance with ordinary Maxwell-Boltzmann statistical mechanics. As a function of ξ this Hamiltonian form of the energy $E_m(\xi)$ clearly has a minimum at $\xi = 1$, see Fig. A3, corresponding to pure surface current. In this case, therefore, there will be current density only on the *surface* in the energy minimizing state. This is in accordance with our minimum magnetic energy theorem. It is notable that *both* the assumption of constant current and the assumption of constant angular momentum lead to a pure *surface current density* as the stable equilibrium.

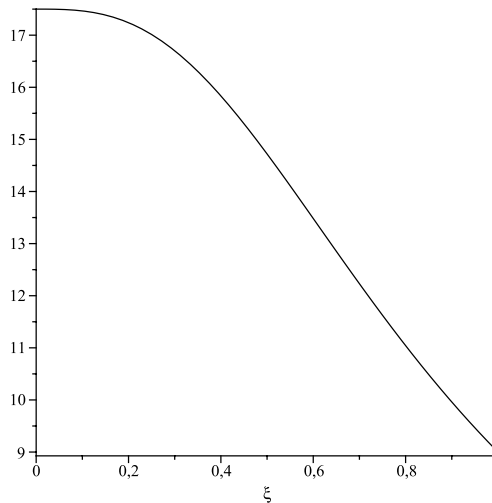


Figure A3. A graph of the function $1/f(\xi)$ which is proportional to the Hamiltonian form of the magnetic energy (A12) of our model system. Note that $\xi = 0$ corresponds to volume (bulk) current and $\xi = 1$ to pure surface current.

APPENDIX B. EXPLICIT SOLUTIONS WITH MINIMUM MAGNETIC ENERGY

To further illustrate our theorem we present here three explicit solutions for current distributions and magnetic fields that minimize the magnetic energy. We do not repeat the solution for a torus since it is a bit lengthy and has been published several times already, probably first by Fock [33], but, independently, several times since then, see, e.g., [34–39]. Karlsson [12], however, was probably the first to notice that the solution minimizes magnetic energy for constant flux. Dolecek and de Launay [15] verified experimentally that a type I superconducting torus behaves exactly as the corresponding classical perfectly diamagnetic system for field strength below the critical field. Here we treat three cases all involving a constant external field. For a cylinder, perpendicular to the field, and for a sphere, analytical solutions are found. Finally, for a cube with a space diagonal parallel to the field, we present a numerical solution.

B.0.4. Cylinder in External Perpendicular Magnetic Field

Consider an infinite cylindrical ideal conductor with radius R in a external constant perpendicular magnetic field. To get the vector potential one must solve the following differential equation,

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = 0. \quad (\text{B1})$$

To solve this one should look for the symmetries of the system. We assume that the external constant magnetic field points in the y -direction and that the cylinder axis coincides with the z -axis. There will then be no dependence on the z -coordinate so the magnetic field is,

$$\mathbf{B} = \frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} \hat{\rho} - \frac{\partial A_z}{\partial \rho} \hat{\varphi} + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi} \right) \hat{z}. \quad (\text{B2})$$

Moreover, due to the symmetry of the system, the z -component of the magnetic field must be zero,

$$\frac{\partial}{\partial \rho} (\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi} = 0. \quad (\text{B3})$$

These assumptions and constraints transform Equation (B1) into,

$$\nabla \times \mathbf{B} = - \left(\frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial \varphi^2} + \frac{\partial^2 A_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A_z}{\partial \rho} \right) \hat{z} = 0 \quad (\text{B4})$$

which simply is Laplace equation in cylindrical coordinates (ρ, φ, z) .

Before writing down the general solution, let us consider the boundary conditions. As $\rho \rightarrow \infty$, the magnetic field must approach the external one: $\mathbf{B}_0 = B_0 \hat{\mathbf{y}} = B_0(\hat{\boldsymbol{\varphi}} \cos \varphi + \hat{\boldsymbol{\rho}} \sin \varphi)$. Furthermore, since the magnetic field is zero inside the perfect conductor, one concludes from Equation (B2) that the vector potential vector must be constant inside the cylinder. Therefore, the solution is,

$$A_z = \text{const.} + B_0 \left(\frac{R^2}{\rho} - \rho \right) \cos \varphi, \quad (\text{B5})$$

for $\rho > R$. The magnetic field outside the cylinder becomes,

$$\mathbf{B} = \hat{\boldsymbol{\rho}} B_0 \left(1 - \frac{R^2}{\rho^2} \right) \sin \varphi + \hat{\boldsymbol{\varphi}} B_0 \left(1 + \frac{R^2}{\rho^2} \right) \cos \varphi, \quad (\text{B6})$$

which implies that,

$$\mathbf{B}(\rho = R) = 2B_0 \cos \varphi \hat{\boldsymbol{\varphi}} \quad (\text{B7})$$

The magnetic field on the cylinder's surface determines the surface current according to Equation (26), so we get,

$$\mathbf{k} = \frac{c}{2\pi} B_0 \cos \varphi \hat{\mathbf{z}}. \quad (\text{B8})$$

The total current obtained through integration of the surface current is zero as expected, otherwise the energy would diverge. A more detailed analysis on this problem can be found in [40].

B.0.5. Superconducting Sphere in Constant Magnetic Field

Similar calculations can be performed for a superconducting sphere with radius R in a constant external magnetic field pointing in the direction of the z -axis. As for the cylinder case, Equation (B1) is considerably simplified using the symmetries of the system. Since the external constant magnetic field points along the z -axis, there won't be any dependence on the φ coordinate and the magnetic field along this coordinate must be zero. Therefore, the magnetic field simplifies to,

$$\mathbf{B} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\varphi \sin \theta) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial r} (A_\varphi r) \hat{\boldsymbol{\theta}}, \quad (\text{B9})$$

where we use spherical coordinates r, θ, φ . Again, using these assumptions and constraints Equation (B1) becomes,

$$\nabla \times \mathbf{B} = -\frac{1}{r} \left[\frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} (r A_\varphi) \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (A_\varphi \sin \theta) \right) \right] \hat{\boldsymbol{\varphi}} = 0 \quad (\text{B10})$$

Since the magnetic field is zero inside the sphere, Equation (B9) implies that the vector potential has the form,

$$A_\varphi(r < R) = \frac{C}{r \sin \theta}, \quad (\text{B11})$$

where C is a constant. To prevent the vector potential from diverging at $r = 0$ and $\theta = 0$, the constant C must be zero. Furthermore, as $r \rightarrow \infty$, the magnetic field must go to the external field, $\mathbf{B}_0 = B_0 \hat{\mathbf{z}} = B_0(\hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta)$. Therefore, the solution of Equation (B10) for this case is,

$$A_\varphi(r > R) = \frac{B_0}{2} \left(r - \frac{R^3}{r^2} \right) \sin \theta, \quad (\text{B12})$$

which leads to the following magnetic field outside the sphere,

$$\mathbf{B} = \hat{\mathbf{r}} B_0 \left(1 - \frac{R^3}{r^3} \right) \cos \theta - \hat{\boldsymbol{\theta}} B_0 \left(1 + \frac{1}{2} \frac{R^3}{r^3} \right) \sin \theta. \quad (\text{B13})$$

The magnetic field at the sphere surface is thus,

$$\mathbf{B} = -\frac{3}{2} B_0 \sin \theta \hat{\boldsymbol{\theta}}. \quad (\text{B14})$$

One notes that this is the same field as that of Section 3.2 at the surface of the inner sphere when energy is minimized.

Using Equation (26), the surface current density becomes,

$$\mathbf{k} = -\frac{3c}{8\pi} B_0 \sin \theta \hat{\boldsymbol{\varphi}}. \quad (\text{B15})$$

Unlike the infinite cylinder in a perpendicular external field, the sphere must have a total non-zero electric current, $I = \frac{3c}{4\pi} R B_0$, to keep the magnetic field from entering. A similar approach to this problem can be found in [41].

B.0.6. Perfectly Conducting Cube in Constant Magnetic Field

Starting from the magnetic energy functional we have made finite element calculations of the current and magnetic field produced when a perfectly conducting cube is placed in a (previously) constant field. The constant external field is produced by given currents on the surface of a sphere that encloses the cube, as in Section 3.1. The cube is placed inside this sphere with one of its space diagonals parallel to the external field. The calculations verify the results of the variational principle: current density and magnetic field become zero inside the cube and the surface current density adjusts to achieve this. In Figures B1 and B2, we show results for the current distribution of the surface of the

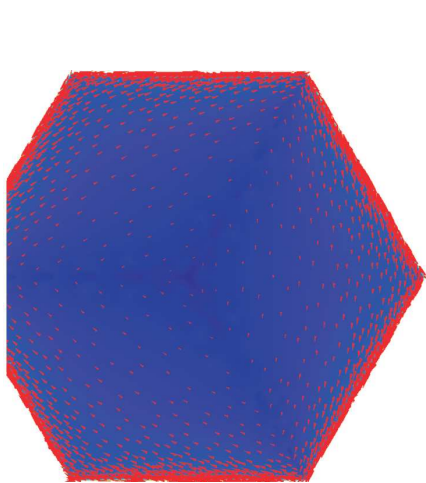


Figure B1. Result of finite element calculation of the current distribution on an ideally conducting cube in a constant magnetic field. The external field is along a space diagonal of the cube and points up from the figure. The current is on the surface and is indicated by arrowheads.

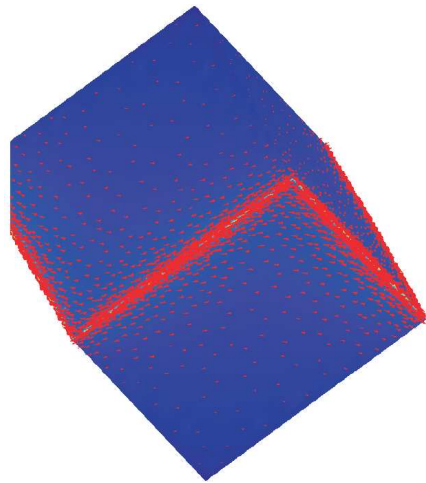


Figure B2. Same as Figure B1 but with a sideways view of the cube. The current is seen to concentrate along the edges that form a closed path round the cube. The induced surface current expels the external field from the interior of the cube.

cube as seen first from the top (the direction of the external field) and then from the side. One observes that the current density concentrates along the edges that form a closed path round the cube.

We also calculated the magnetic flux through the circular surface enclosed by the equator of the enclosing sphere, which we take to have radius $R = 1$ and to produce the constant magnetic field $B_e = 1$ in the interior, when empty. For this case the flux becomes,

$$\Phi_0 = B_e \pi R^2 = \pi \approx 3.1416 \quad (\text{B16})$$

When an ideally conducting sphere of volume $V = 1$, i.e., radius $a = \sqrt[3]{\frac{3}{4\pi}}$, is placed inside (as in Section 3) the flux is reduced to,

$$\Phi_{\text{sp}} = \pi \left(1 - \frac{3}{4\pi} \right) \approx 2.3916 \quad (\text{B17})$$

When the sphere is replaced by a cube of volume $V = 1$ and a space diagonal parallel to the external fields we find the magnetic flux,

$$\Phi_{\text{cu}} \approx 2.2733 \quad (\text{B18})$$

numerically. One notes that such a cube excludes more flux than a sphere of the same volume.

REFERENCES

1. Carr, Jr., W. J., "Macroscopic theory of superconductors," *Phys. Rev. B*, Vol. 23, 3208–3212, 1981.
2. Jackson, J. D., *Classical Electrodynamics*, 3rd edition, John Wiley & Sons, New York, 1999.
3. Coulson, C. A., *Electricity*, 3rd edition, Oliver and Boyd, Edinburgh, 1953.
4. Panofsky, W. K. H. and M. Phillips, *Classical Electricity and Magnetism*, 2nd edition, Dover, New York, 2005.
5. Landau, L. D. and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd edition, Butterworth-Heinemann, Oxford, 1984.
6. Bakhoun, E. G., "Proof of Thomson's theorem of electrostatics," *J. Electrostatics*, Vol. 66, 561–563, 2008.
7. Kovetz, A., *Electromagnetic Theory*, Oxford University Press, Oxford, 2000.
8. Sancho, M., J. L. Sebastián, and V. Giner, "Distribution of charges on conductors and Thomson's theorem," *Eng. Sci. Educ. J.*, Vol. 10, 26–30, 2001.
9. Donolato, C., "An application of Thomson's theorem to the determination of induced charge density," *Eur. J. Phys.*, Vol. 24, L1–L4, 2003.
10. Brito, L. and M. Fiolhais, "Energetics of charge distributions," *Eur. J. Phys.*, Vol. 23, 427–431, 2002.
11. Sancho, M., J. L. Sebastián, S. Muñoz, and J. M. Miranda, "Computational method in electrostatics based on monte carlo energy minimization," *IEE Proc., Sci. Meas. Technol.*, Vol. 148, 121–124, 2009.
12. Karlsson, P. W., "Inductance inequalities for ideal conductors," *Arch. f. Elektrotech.*, Vol. 67, 29–33, 1984.
13. Badía-Majós, A., "Understanding stable levitation of superconductors from intermediate electromagnetics," *Am. J. Phys.*, Vol. 74, 1136–1142, 2006.
14. McAllister, I. W., "Surface current density K: An introduction," *IEEE Trans. Elect. Insul.*, Vol. 26, 416–417, 1991.
15. Dolecek, R. L. and J. de Launay, "Conservation of flux by a superconducting torus," *Phys. Rev.*, Vol. 78, 58–60, 1950.
16. Hehl, F. W. and Y. N. Obukhov, "Dimensions and units in

- electrodynamics,” *Gen. Relativ. Gravit.*, Vol. 37, 733–749, USA, 2005.
17. Landau, L. D. and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edition, Pergamon, Oxford, 1975.
 18. London, F. and H. London, “The electromagnetic equations of the supraconductor,” *Proc. Roy. Soc. A*, Vol. 149, 71–88, 1935.
 19. Badía-Majós, A., J. F. Cariñena, and C. López, “Geometric treatment of electromagnetic phenomena in conducting materials: variational principles,” *J. Phys. A: Math. Gen.*, Vol. 39, 14699–14726, 2006.
 20. Woltjer, L., “A theorem on force-free magnetic fields,” *Proc. Nat. Acad. Sci.*, Vol. 44, 489–491, 1958.
 21. Griffiths, D. J., *Introduction to Electrodynamics*, 3rd edition, Prentice Hall, New Jersey, 1999.
 22. Meissner, W. and R. Ochsenfeld, “Ein neuer Effekt bei eintritt der Supraleitfähigkeit,” *Naturwiss.*, Vol. 21, 787–788, 1933.
 23. Hirsch, J. E., “Charge expulsion, spin Meissner effect, and charge inhomogeneity in superconductors,” *J. Supercond. Nov. Magn.*, Vol. 22, 131–139, 2009.
 24. Forrest, A. M., “Meissner and Ochsenfeld revisited,” *Eur. J. Phys.*, Vol. 4, 117–120, 1983. Comments on and translation into English of Meissner and Ochsenfeld.
 25. Alfvén, H. and C.-G. Fälthammar, *Cosmical Electrodynamics*, 2nd edition, Oxford University Press, Oxford, 1963.
 26. Essén, H., “From least action in electrodynamics to magnetomechanical energy,” *Eur. J. Phys.*, Vol. 30, 515–539, 2009.
 27. Gorter, C. J. and H. Casimir, “On supraconductivity I,” *Physica*, Vol. 1, 306–320, 1934.
 28. Essén, H., “Magnetic fields, rotating atoms, and the origin of diamagnetism,” *Phys. Scr.*, Vol. 40, 761–767, 1989.
 29. Essén, H., “Darwin magnetic interaction energy and its macroscopic consequences,” *Phys. Rev. E*, Vol. 53, 5228–5239, 1996.
 30. Essén, H., “Magnetic dynamics of simple collective modes in a two-sphere plasma model,” *Phys. of Plasmas*, Vol. 12, 122101–1–7, 2005.
 31. Essén, H., “Electrodynamic model connecting superconductor response to magnetic field and to rotation,” *Eur. J. Phys.*, Vol. 26, 279–285, 2005.
 32. Greiner, W., *Classical Electrodynamics*, Springer, New York, 1998.

33. Fock, V., "Skineffekt in einem Ringe," *Phys. Z. Sowjetunion*, Vol. 1, 215–236, 1932.
34. De Launay, J., "Electrodynamics of a superconducting torus," Technical Report NRL-3441, Naval Research Lab, Washington DC, 1949.
35. Carter, G. W., S. C. Loh, and C. Y. K. Po, "The magnetic field of systems of currents circulating in a conducting ring," *Quart. Journ. Mech. and Applied Math.*, Vol. 18, 87–106, 1965.
36. Bhadra, D., "Field due to current in toroidal geometry," *Rev. Sci. Instrum.*, Vol. 39, 1536–1546, 1968.
37. Haas, H., "Das Magnetfeld eines gleichstromdurchflossenen Torus," *Arch. f. Elektrotech.*, Vol. 58, 197–209, 1976.
38. Belevitch, V. and J. Boersma, "Some electrical problems for a torus," *Philips J. Res.*, Vol. 38, 79–137, 1983.
39. Ivaska, V., V. Jonkus, and V. Palenskis, "Magnetic field distribution around a superconducting torus," *Physica C*, Vol. 319, 79–86, 1999.
40. Zhilichev, Y. N., "Superconducting cylinder in a static transverse magnetic field," *IEEE Trans. Appl. Supercond.*, Vol. 7, 3874–3879, 1997.
41. Matute, E. A., "On the superconducting sphere in an external magnetic field," *Am. J. Phys.*, Vol. 67, 786–788, 1999.