INVERSE JOUKOWSKI MAPPING

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Abstract—This paper discusses the inverse Joukowski mapping, $w = z + \sqrt{z^2 - c^2}$ $(c > 0)$, which can be classified into active and passive inverse transformation. By using the active inverse Joukowski mapping, the generalized image problems that the line charge ρ_l is located outside the elliptical conducting cylinder, or the finite conducting plate can be solved. By using the passive logarithmic inverse Joukowski mapping, the capacitance C of a finite conducting plate placed vertically above the infinite conducting plate can be solved. Thus the conformal mapping method can replace the image method and electrical axis method and become the uniform method to solve the electrostatic problems.

1. INTRODUCTION

The Joukowski mapping is a famous complex conformal transformation, whose function is

$$w = \frac{1}{2} \left(z + \frac{c^2}{z} \right) \quad (c > 0) \tag{1}$$

and it is called Joukowski function. The Joukowski mapping has two well-know applications. One application is simulation that the airfoil flow can be substituted by flow around the cylinder. Then the analytic solution was first obtained and used to explain the lifting of crafts in low speed aerodynamics. The other is transforming the circle cluster into corresponding confocal elliptical cluster. By this method, the analytic solution of elliptical coaxial line capacitance C and characteristic impedance Z_0 can be deduced. Thus, analytic

solution of capacitance is an important research topic and needs to be extensively discussed. Although some researchers analyzed the capacitance of a disc [1] or arbitrarily shaped conducting plates [2], the capacitance of other kinds still needs to be studied in detail.

It is well known that image method, electric axis method and complex conformal mapping are three typical methods to solve the electrostatic problem [3–6]. If the conducting bodies have boundaries of a simple geometry, the method of images has great advantage. When the system consists of two-wire transmission line, the equipotential surface of the two wires can be considered to have been generated by a pair of line charges. The location of the pair of line charges is called the electric axis of the two-wire transmission, so the method is named by method of electric axis. However, the two methods are restrained by the ways of electric charges and the location of conducting bodies. Furthermore, in many practical problems, it is difficult to find the potential by applying the two methods directly. Thus a more general and uniform method is needed to solve the complex electrostatic problem. Here we provide a useful extension of the conformal method which will allow us to solve complex electrostatic problem which was previously done by a much more laborious approach [2–6]. It was proved that the methods of images can be replaced by active conformal mapping method [7]. Then the application of conformal mapping method is discussed extensively.

This paper discusses the electrostatic problem by using conformal mapping method. The inverse Joukowski mapping discussed in this paper can be classified into active and passive conformal transformations. Specifically, the generalized image problem that the line charge ρ_l outside the elliptical conducting cylinder or outside the finite conducting plate can be solved by active inverse Joukowski mapping. When the finite conducting plate is placed vertically above the infinite conducting plate, the compatible capacitance solution of the whole system can be solved by the passive logarithmic inverse Joukowski mapping. It can be seen that the conformal mapping method can become the uniform method to solve electrostatic problems.

2. INVERSE JOUKOWSKI MAPPING

The standard form of inverse Joukowski mapping is

$$w = z + \sqrt{z^2 - c^2} \quad (c > 0) \tag{2}$$

and the normalized form is
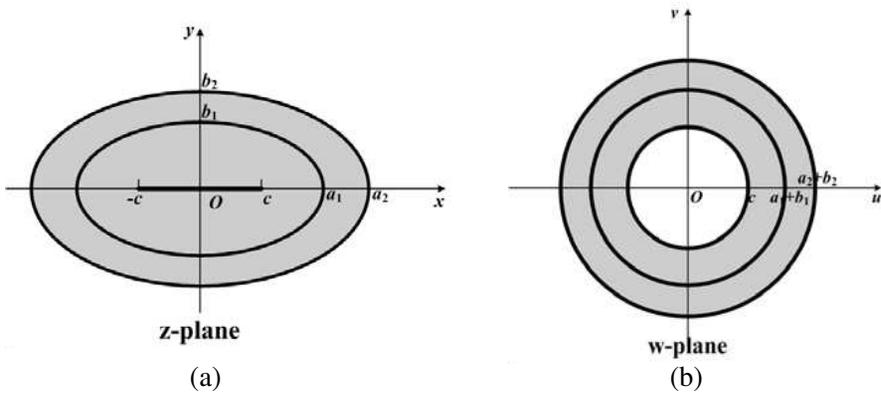
$$w = z + \sqrt{z^2 - 1} \tag{3}$$

Figure 1. Inverse Joukowski mapping (a) the confocal elliptical cluster. (b) Circle cluster with a cut circle of radius c.

Formulas (2) and (3) are both inverse Joukowski mapping in this paper. It is worth pointing out that another solution is existed and that is

$$w' = z - \sqrt{z^2 - c^2} \quad (c > 0) \tag{4}$$

However,

$$w' = \frac{c^2}{w} \tag{5}$$

It is apparent that the two solutions are mutual inversions. So this situation will not be discussed.

The typical inverse Joukowski transformation maps a family of confocal elliptical in the z-plane with the same focal length onto a family of circles in the w-plane which have a cut circle of radius c. As shown in Figure 1, the focal length is $c = \sqrt{a_1^2 - b_1^2} = \sqrt{a_2^2 - b_2^2}$, the ellipse of major semi-axis a_1 and minor semi-axis b_1 are corresponding to circle of radius $a_1 + b_1$, and the ellipse of major semi-axis a_2 and minor semi-axis b_2 are corresponding to circle of radius $a_2 + b_2$. From the view of electromagnetic field, the focal strip with the range $-c \leq x \leq c$ of the x axis is mapped onto the conducting circle with radius c in w-plane.

3. ACTIVE INVERSE JOUKOWSKI MAPPING

This paper proposes the conformal mapping that can be classified into two types: active conformal mapping with singular line charge ρ_l and passive conformal mapping without singular line charge ρ_l.

The former can be used to solve complex potential problems in two-dimensional generalized image method. The latter can be used to solve the capacitance between the two conducting plates. Therefore, the complex conformal mapping method which can be utilized to replace methods of image and electrical axis will become the uniform method for electrostatic problems.

The typical application of the inverse Joukowski mapping is a line charge ρ_l located at a distance h ($h > b$) from the axis of a parallel, conducting, elliptical cylinder, as shown in Figure 2(a). Figure 2(b) plots the corresponding inverse mapping outside the circle. Substitute the position of the line charge ρ_l, $z_0 = jh$ into (2). Then the formula can be given as

$$w_0 = j \left(\sqrt{h^2 + c^2} + h \right) \tag{6}$$

Under this consideration, the circle of radius c in w-plane is equipotential, which is not plotted.

In w-plane, the corresponding inverse point of the image charge is

$$w_0' = j \frac{(a + b)^2}{\left(\sqrt{h^2 + c^2} + h \right)} = j \left(\frac{a + b}{a - b} \right) \left(\sqrt{h^2 + c^2} - h \right) \tag{7}$$
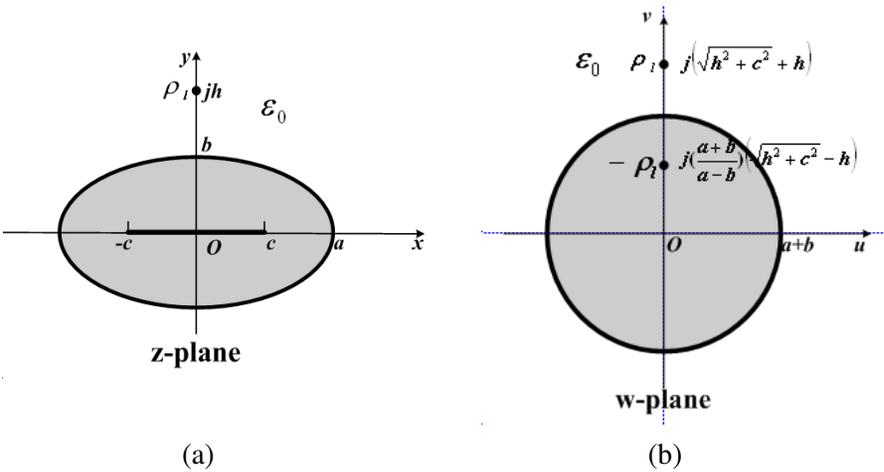
According to the principle of conformal mapping, the potential

Figure 2. (a) Line charge ρ_l located at h ($h > b$) which is outside the elliptical cylinder of major semi-axis a and minor semi-axis b. (b) Inverse Joukowski mapping of line charge outside the circular cylinder.

function φ is

$$\varphi = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{w-w_0'}{w-w_0}\right| = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{\left(z+\sqrt{z^2-c^2}\right)-j\left(\frac{a+b}{a-b}\right)\left(\sqrt{h^2+c^2}-h\right)}{\left(z+\sqrt{z^2-c^2}\right)-j\left(\sqrt{h^2+c^2}+h\right)}\right| \tag{8}$$

Let $c = 0$, $R = a = b$, the elliptical cylinder will be turned into cylinder. Formula (8) is simplified as

$$\varphi = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{z-j\frac{R^2}{h}}{z-jh}\right| \tag{9}$$

Then, it is well known that the result of this solution is familiar and correct.

The second example is shown in Figure 3, where line charge ρ_l is placed at the vertical center line of the finite conducting strip with the range $-c \le x \le c$ of the x axis, and the distribution of the potential function φ is requested.

Suppose that the line charge is placed on the confocal ellipse of major semi-axis a and minor semi-axis b, we have

$$\begin{cases} c^2 = a^2 - b^2 \\ b = h \end{cases} \tag{10}$$

then

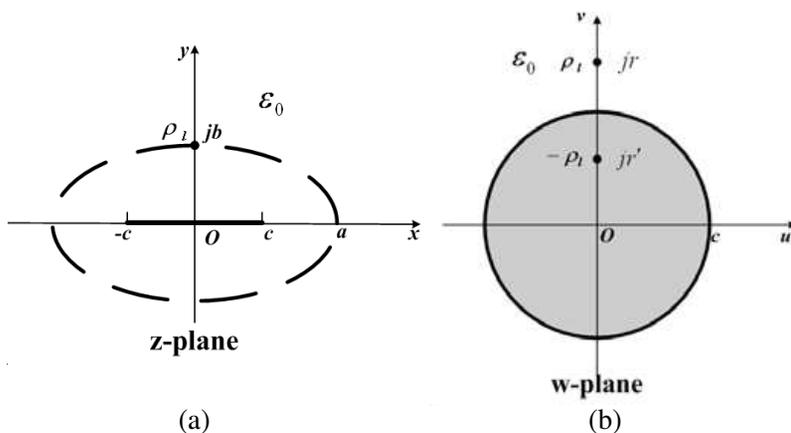$$a = \sqrt{h^2 + c^2} \tag{11}$$

Figure 3. (a) Line charge ρ_l is placed at the vertical center line of the finite conducting plate with width of $2c$, (b) Corresponding inverse Joukowski mapping.

Similarly, taking the inverse Joukowski mapping into account, the conducting strip is mapped onto the conducting circle of radius c, and the position of the line charge is jr, where

$$r = \sqrt{h^2 + c^2} + h \tag{12}$$

Applying the cylinder image method, we can get

$$r' = \sqrt{h^2 + c^2} - h \tag{13}$$

It is satisfied that

$$rr' = c^2 \tag{14}$$

Finally, the distribution of the potential function is

$$\varphi = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{w - jr'}{w - jr}\right| = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{\left(z + \sqrt{z^2 - c^2}\right) - j\left(\sqrt{h^2 + c^2} - h\right)}{\left(z + \sqrt{z^2 - c^2}\right) - j\left(\sqrt{h^2 + c^2} + h\right)}\right| \tag{15}$$

Especially, let $c \to \infty$, the problem becomes the line charge ρ_l placed above the infinite conducting plate. We can get

$$\lim_{c\to\infty} \varphi = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{z + jh}{z - jh}\right| \tag{16}$$

Formula (16) is the known result.

The third example is the system that the line charge ρ_l is placed at a distance d from the left of the origin at the x axis, and the range $0 \le x \le 2c$ of the x axis is placed with the finite conducting strip, as described in Figure 4(a). Taking the translation inverse Joukowski
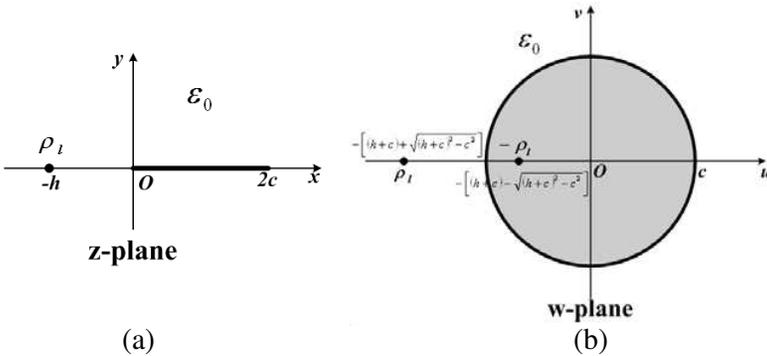

Figure 4. (a) The line charge ρ_l placed at left of the conducting strip with width of $2c$. (b) Corresponding inverse mapping.

mapping into account, the positions of the line charge and its image line charge are

$$w_0 = -\left[(h+c) + \sqrt{(h+c)^2 - c^2}\right] \tag{17}$$

$$w_0' = -\left[(h+c) - \sqrt{(h+c)^2 - c^2}\right] \tag{18}$$

Then the potential function φ is

$$
\begin{aligned}
\varphi &= \frac{\rho_l}{2\pi\varepsilon_0} \ln\left|\frac{w - w_0'}{w - w_0}\right| \\
&= \frac{\rho_l}{2\pi\varepsilon_0} \ln\left|\frac{\left[(z-c) + \sqrt{(z-c)^2 - c^2}\right] + \left[(h+c) - \sqrt{(h+c)^2 - c^2}\right]}{\left[(z-c) + \sqrt{(z-c)^2 - c^2}\right] + \left[(h+c) + \sqrt{(h+c)^2 - c^2}\right]}\right|
\end{aligned}
\tag{19}
$$

Especially let $c \to \infty$, the problem becomes the system that the line charge ρ_l is located at a distance d from the left of the origin at the x axis and is parallel to the conducting half-plate. We can get

$$\lim_{c \to \infty} \varphi = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left|\frac{\sqrt{z} + j\sqrt{h}}{\sqrt{z} - j\sqrt{h}}\right| \tag{20}$$

This is also a known result.

At last, when the line charge ρ_l is located at (x_0, y_0) above the conducting plate which is placed within the range $-c \le x \le c$ of the x axis, as shown in Figure 5, the generalized potential distribution function can be deduced as follows.

Without loss of generality, suppose that (x_0, y_0) locates at the first quadrant. The line charge is placed on the confocal ellipse of major semi-axis a and minor semi-axis b, it is constrained that

$$c^2 = a^2 - b^2 \tag{21}$$

Further, suppose that

$$\begin{cases} x_0 = a\cos t \\ y_0 = b\sin t \end{cases} \tag{22}$$

where t is the parameter, and x_0, y_0 are known. We can easily get

$$\frac{x_0^2}{\cos^2 t} - \frac{y_0^2}{\sin^2 t} = c^2 \tag{23}$$

and

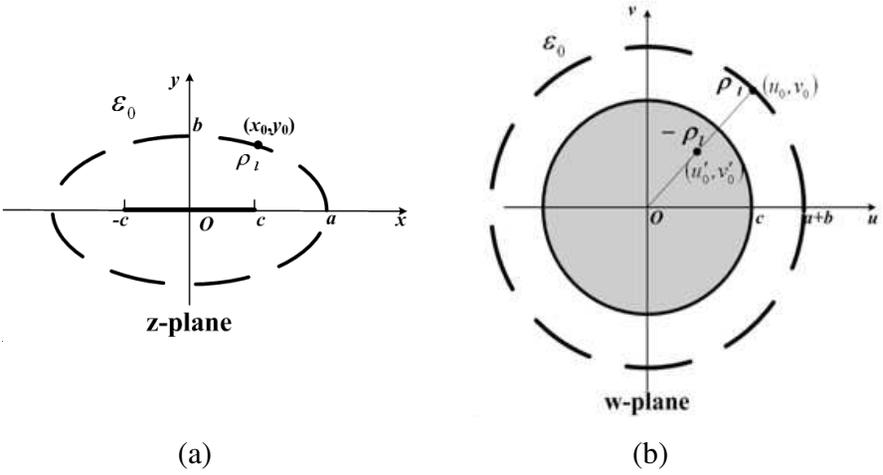$$c^4 \sin^4 t + \left(x_0^2 + y_0^2 - c^2\right)\sin^2 t - y_0^2 = 0 \tag{24}$$

Figure 5. (a) the line charge ρ_l placed above the conducting plate with width of $2c$, (b) generalized inverse mapping.

Then, it is deduced that

$$a = \frac{x_0}{\sqrt{\frac{1}{2c^2}\left[-\sqrt{\left(x_0^2 + y_0^2 - c^2\right) + 4c^2 y_0^2} + \left(x_0^2 + y_0^2 + c^2\right)\right]}} \tag{25}$$

$$b = \frac{y_0}{\sqrt{\frac{1}{2c^2}\left[\sqrt{\left(x_0^2 + y_0^2 - c^2\right) + 4c^2 y_0^2} - \left(x_0^2 + y_0^2 - c^2\right)\right]}} \tag{26}$$

It is easy to conclude from Figure 5 that

$$w_0 = u_0 + jv_0 = (a + b)\cos\varphi + j(a + b)\sin\varphi \tag{27}$$

Further, the Joukowski mapping is

$$z = \frac{1}{2}\left(w + \frac{c^2}{w}\right) \tag{28}$$

Thus, we get

$$x_0 + jy_0 = \frac{1}{2}\left[(a + b) + \frac{c^2}{a + b}\right]\cos\varphi + j\frac{1}{2}\left[(a + b) - \frac{c^2}{a + b}\right]\sin\varphi \tag{29}$$

where $c^2 = a^2 - b^2$, it is

$$\begin{cases} x_0 = a\cos\varphi \\ y_0 = b\sin\varphi \end{cases} \tag{30}$$

Surprising discovery is that

$$\varphi = t \tag{31}$$

Then, we can get

$$u_0 = (a+b)\sqrt{\frac{1}{2c^2}\left[-\sqrt{\left(x_0^2+y_0^2-c^2\right)^2+4c^2y_0^2}+\left(x_0^2+y_0^2+c^2\right)\right]} \tag{32}$$

$$v_0 = (a+b)\sqrt{\frac{1}{2c^2}\left[\sqrt{\left(x_0^2+y_0^2-c^2\right)^2+4c^2y_0^2}-\left(x_0^2+y_0^2-c^2\right)\right]} \tag{33}$$

and

$$w_0' = u_0' + jv_0' = \frac{c^2}{(a+b)^2}(u_0+jv_0) = \left(\frac{a-b}{a+b}\right)(u_0+jv_0) \tag{34}$$

Finally, the potential function φ is

$$\varphi = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{w-w_0'}{w-w_0}\right| = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left|\frac{\left(z+\sqrt{z^2-c^2}\right) - \left(\frac{a-b}{a+b}\right)(u_0+jv_0)}{\left(z+\sqrt{z^2-c^2}\right) - (u_0+jv_0)}\right| \tag{35}$$

Formula (35) is the generalized result of the case where the line charge ρ_l is placed at (x_0, y_0) above the conducting strip with width of $2c$.

4. PASSIVE LOGARITHMIC INVERSE JOUKOWSKI MAPPING

The passive logarithmic inverse Joukowski mapping is the inverse hyperbolic cosine mapping, that is

$$w = \text{ch}^{-1}(z) = \ln\left(z + \sqrt{z^2 - 1}\right) \tag{36}$$

It actually contains several transformations, such as arccosine transformation

$$w = \cos^{-1}(z) = -j\text{ch}^{-1}(z) \tag{37}$$

and arcsine transformation

$$w = \sin^{-1}(z) = -j\text{ch}^{-1}(z) + \frac{\pi}{2} \tag{38}$$

The transformations mentioned above are all normalized logarithmic inverse Joukowski mapping. Thus, all these functions can be considered as one kind. Then, taking arccosine transformation as the example, we get

$$w = \cos^{-1}\left(\frac{z}{A}\right) \tag{39}$$

Formula (39) is equivalent to

$$x + jy = A\cos(u + jv) \tag{40}$$

It is easy to get that

$$\frac{x^2}{A^2\text{ch}^2 v} + \frac{y^2}{A^2\text{sh}^2 v} = 1 \tag{41}$$

$$\frac{x^2}{A^2\cos^2 u} - \frac{y^2}{A^2\sin^2 u} = 1 \tag{42}$$

Formula (41) shows that $v = $ constant is the family of confocal elliptical curves with the center at origin. Major semi-axis and minor semi-axis respectively are

$$\left\{ \begin{array}{l} a_1 = A\text{ch}v \\ b_1 = A\text{sh}v \end{array} \right. \tag{43}$$

Then the distance between the two focuses is

$$2\sqrt{a_1^2 - b_1^2} = 2A \tag{44}$$

Formula (42) shows that $u = $ constant is the family of confocal hyperbola curves. We get

$$\left\{ \begin{array}{l} a_2 = A\cos u \\ b_2 = A\sin u \end{array} \right. \tag{45}$$

The distance between the two focuses is

$$2\sqrt{a_2^2 + b_2^2} = 2A \tag{46}$$

It is obvious that the confocal elliptical cluster and confocal hyperbola cluster are conjugate and orthogonal, as shown in Figure 6. By using this inverse mapping, many difficult electromagnetic problems of capacitance C can be tackled.

The capacitance C of the vertical plate AB placed above the infinite conducting plate can be solved by the passive conformal mapping. Figure 7 shows that the distance between O and A is denoted by A, and the distance between O and B is denoted by B. The length of the plate is

$$L = B - A \tag{47}$$

As an approximate solution, the problem in Figure 7 can be considered as the finite plate $(B - A)$ at $u = 0$. Under this consideration, power lines are in semi-elliptical shape. For the purpose of convenience, suppose that the potential of the finite plate satisfies $u' = u_0$ and that the potential of the infinite conducting plane satisfies $u' = 0$. It gets that

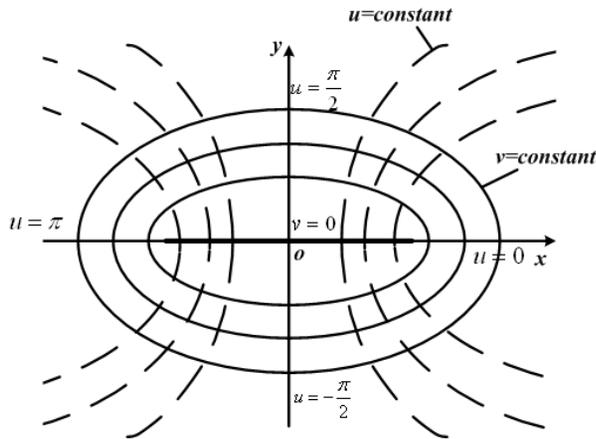$$w' = K + H\cos^{-1}\left(\frac{z}{A}\right) = u' + jv' \tag{48}$$

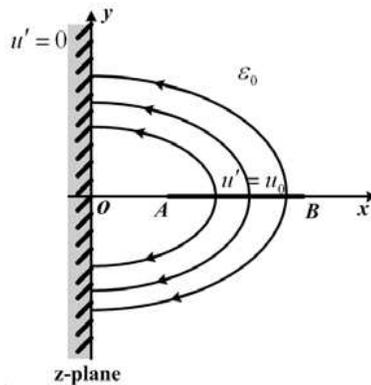**Figure 6.** Complex potential distribution of $z = A \cos w$.

Figure 7. Capacitance C of the vertical plate placed above the infinite conducting plate can be solved by passive conformal mapping.

where K and H are constants to be determined; u' represents the potential function; v' represents the force line function. Figure 6 is compared with Figure 7. The first boundary condition is

$$\begin{cases} u = \frac{u'-K}{H} = 0 \\ u' = u_0 \end{cases} \tag{49}$$

It is easy to get that

$$K = u_0 \tag{50}$$

Then the second boundary condition is

$$\begin{cases} u = \frac{u'-K}{H} = \frac{\pi}{2} \\ u' = 0 \end{cases} \tag{51}$$

and

$$H = -\frac{2}{\pi}u_0 \tag{52}$$

Finally, we get

$$z = x + jy = A\cos\left[\left(1 - \frac{w'}{u_0}\right)\frac{\pi}{2}\right] \tag{53}$$

Figure 7 shows the vertical plate AB

$$x = A\mathrm{ch}\left(\frac{\pi v'}{2u_0}\right) \tag{54}$$

Then the force line function is

$$v' = \frac{2u_0}{\pi}\mathrm{ch}^{-1}\left(\frac{x}{A}\right) \tag{55}$$

According to the definition of charging density σ, we have

$$\sigma = -\varepsilon_0\frac{\partial\varphi}{\partial y} = -\varepsilon_0\frac{\partial u'}{\partial y} = \varepsilon_0\frac{\partial v'}{\partial x} \tag{56}$$

The charge of the vertical plate AB is

$$Q = \int_A^B \sigma dx = \int_A^B \varepsilon_0\frac{\partial v'}{\partial x}dx = \varepsilon_0 v'\big|_A^B \tag{57}$$

Substituting formula (55) into formula (57), we obtain

$$Q = \frac{2\varepsilon_0 u_0}{\pi}\mathrm{ch}^{-1}\left(\frac{B}{A}\right) \tag{58}$$

According to the definition of the capacitance, we have

$$C = \frac{Q}{u_0} = \frac{2\varepsilon_0}{\pi}\mathrm{ch}^{-1}\left(\frac{B}{A}\right) = \frac{2\varepsilon_0}{\pi}\ln\left[\left(\frac{B}{A}\right) + \sqrt{\left(\frac{B}{A}\right)^2 - 1}\right] \tag{59}$$

Let $B \gg A$. The expression is

$$C \approx \frac{2\varepsilon_0}{\pi}\ln\left(\frac{2B}{A}\right) \tag{60}$$

Suppose that $A = 1$, $B = 10$. From (59), we get $C = 1.68487 \times 10^{-11}$ F.

5. CONCLUSION

Joukowski mapping is a famous complex conformal transformation. The inverse Joukowski mapping $w = z + \sqrt{z^2 - c^2}$ $(c > 0)$ is discussed in this paper. The conformal mapping application can be classified into two types. Active inverse Joukowski mapping is used to solve the generalized image problem where the line charge ρ_l is placed outside the elliptical conducting cylinder or outside the finite conducting plate. The passive logarithmic inverse Joukowski mapping is used to solve the capacitance approximate solution of the finite conducting plate which is placed above and vertical to the infinite conducting plate. The conformal mapping method which replaces the image method and electrical axis method becomes the uniform method to solve electrostatic problems.

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