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EFFECTS OF RANDOM ERRORS UPON EFFECTIVE PERMITTIVITY OF A COMPOSITE CONTAINING SHORT NEEDLES

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Abstract—A composite medium containing perfectly conducting short needles can have a range of frequency for which the real part of the effective permittivity of the composite is negative. Such a range of frequency can be taken as negative bandwidth. This negative bandwidth for a composite medium is dependent upon parameters like positioning, orientation, length and needle density of short needles. Effects of random errors in positioning and orientation of short needles upon the ensemble averaged effective permittivity are analyzed. It is studied theoretically that increasing error in positioning and orientation of short needles reduces negative bandwidth.

1. INTRODUCTION

A composite medium embedded with finite length conducting wires has many interesting electromagnetic properties [1–5]. One of the important property of such a composite is that it can occupy negative bandwidth. Due to finite lengths of needles or wires, its spectrum of effective permittivity is resonant contrary to the plasmonic [e.g., 6]. The effective permittivity of a composite containing finite length needles is nonlocal due to presence of spatial dispersion [e.g., 2]. The local effective permittivity can be assigned to such a composite provided that all needles are electrically short.

The propagation of electromagnetic waves through dielectric slab embedded with randomly distributed short thin wires has been studied by Deshpande and Deshpande [7]. They have analyzed the reflection and transmission coefficients numerically using finite element method.

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Experimental study of a composite containing periodic thin short needles is given by [8]. The study of negative bandwidth for a composite containing short needles has not been discussed too much in the literature. According to the authors, few references within this context are [9–11]. It was first shown by Koschny, et al. [9, 10] that a composite medium containing electrically short needles can have negative bandwidth. Later on, Fu and Zhao [11] found his results consistent with Koschny. Effects of random errors in positioning and orientation of short needles upon negative bandwidth is still unexplored. This is the study of a current paper. Random errors in positioning of a wire grid composed of infinitely long conducting wires has been discussed in [12].

In this paper, a three dimensional infinite lattice of electrically short thin needles is considered with background medium as a free space. The arrangement of needles along three orthogonal directions is periodic. Also, the needle density is taken to be high. The spacing along axial direction of two consecutive needles is kept very small compared to the remaining two orthogonal directions. This makes the adjacent needles to strongly couple at the both ends. Such a medium can be taken as a composite medium containing short needles. It can be shown that this composite medium can occupy negative bandwidth. It is studied in Section 3. Effects of random errors in positioning and orientation of short needles upon negative bandwidth are analyzed in Sections 4 and 5 respectively.

2. INDUCED POLARIZABILITY FOR A SHORT THIN PEC NEEDLE

At a first glance, consider a finite length needle with length l and radius a in free space. A needle is assumed to be perfectly electric conducting (PEC). Also a needle is taken to be thin, i.e., $a \ll \lambda_o$ where λ_o is wavelength in free space. An incident plane wave

$$\mathbf{E}^i(\mathbf{r}) = \hat{\mathbf{e}}_i E_o e^{i\mathbf{k}_i\cdot\mathbf{r}} = \mathbf{E}^i e^{i\mathbf{k}_i\cdot\mathbf{r}} \tag{1}$$

where

$$\mathbf{k}_i = k_o\hat{\mathbf{k}}_i = k_{ix}\hat{x} + k_{iy}\hat{y} + k_{iz}\hat{z}$$

propagating in the direction $\hat{\mathbf{k}}_i$ and polarized along $\hat{\mathbf{e}}_i$ impinges upon a finite length needle. It induces current in it. The time factor is assumed to be $e^{-i\omega t}$ and suppressed throughout. Free space propagation constant is $k_o = \omega\sqrt{\mu_o\epsilon_o}$ with μ_o and ϵ_o as permeability and permittivity of free space respectively. Once, the induced current density $\mathbf{J}^{ind}(\mathbf{r}')$ of a needle is known then its induced dipole moment

$\mathbf{p}$ can be calculated using the well known relation [13],

$$\mathbf{p} = \frac{i}{\omega} \int_{V'} \mathbf{J}^{ind}(\mathbf{r}') dV' \tag{2}$$

where $\mathbf{r}'$ is a position vector for a needle scatterer and V' is a volume occupied by a needle. If the orientation of a jth needle is characterized by a unit vector $\hat{\mathbf{e}}_j$ and its position by $\mathbf{r}_j$, then the induced dipole moment $\mathbf{p}_j$ of a jth finite length needle can be written as [14],

$$\mathbf{p}_j = \bar{\mathbf{\Upsilon}}_j \cdot \mathbf{E}^i(\mathbf{r}_j) \tag{3}$$

where $\bar{\mathbf{\Upsilon}}_j$ is the electric polarizability tensor of a jth finite length needle and can be found by [14, Eq. (12)] with units of Fm^2. If the needle is taken to be electrically short, i.e., $l \ll \lambda_o$ then the quasistatic approximation can be used for the short needle scatterer. Under this assumption, the electric field distribution upon the needle can be taken as uniform, i.e., independent of position vector $\mathbf{r}_j$. In this case, the induced dipole moment for the jth short needle can be written as,

$$\mathbf{p}_j = \bar{\mathbf{\Upsilon}}_j \cdot \mathbf{E}^i \tag{4}$$

with

$$\bar{\mathbf{\Upsilon}}_j = \Lambda \bar{\mathbf{R}}_j \tag{5}$$

$$\Lambda = \frac{1}{3} \left(\frac{i}{\omega} \right) Y_{in}(\omega) l^2 \tag{6}$$

$$\bar{\mathbf{R}}_j = \begin{bmatrix} \sin^2\theta_j \cos^2\phi_j & \sin^2\theta_j \sin\phi_j \cos\phi_j & \sin\theta_j \cos\theta_j \cos\phi_j \\ \sin^2\theta_j \sin\phi_j \cos\phi_j & \sin^2\theta_j \sin^2\phi_j & \sin\theta_j \cos\theta_j \sin\phi_j \\ \sin\theta_j \cos\theta_j \cos\phi_j & \sin\theta_j \cos\theta_j \sin\phi_j & \cos^2\theta_j \end{bmatrix} \tag{7}$$

In deriving Eq. (7), the usual spherical coordinate representation has been used for the unit vector $\hat{\mathbf{e}}_j$ and is given by,

$$\hat{\mathbf{e}}_j = \cos\phi_j \sin\theta_j \hat{x} + \sin\phi_j \sin\theta_j \hat{y} + \cos\theta_j \hat{z} \tag{8}$$

The factor $Y_{in}(\omega)$ shows the input admittance of a short needle and is given by [15],

$$Y_{in}(\omega) = \left[20\pi^2 \left(\frac{l}{\lambda_o} \right)^2 + i \frac{120[\ln(l/a) - 1]}{\tan(k_o l)} \right]^{-1} \tag{9}$$

It can be seen from Eqs. (4)–(7) that the polarizability of a jth short needle is independent of the position of a needle. Likewise, the quasistatic polarizability tensor $\bar{\mathbf{\Upsilon}}_j$ given by Eqs. (5)–(7) is in agreement with [16].

3. EFFECTIVE PERMITTIVITY OF A COMPOSITE MEDIUM CONTAINING SHORT NEEDLES

Consider a composite medium which consists of an infinite three dimensional lattice of short needles with a background medium as free space. If the needles are taken to be far apart from each other then the multiple scattering effects can be ignored. This represents the dilute limit of a composite containing thin short needles. As it is well known that for such a limit, the effective permittivity of a composite can be found by Maxwell-Garnett mixing rule [e.g., 17]. This rule is dependent upon number density, i.e., number of inclusions per unit volume and polarizability of a single inclusion. Thus the obtained effective permittivity for a regular lattice of needles will be the same as that for a random distribution of needles provided that the needle density is same. Likewise, this theory becomes invalid for dense collection of needles. This is due to the reason that this theory does not properly account the contribution of dipole fields of its neighbouring needles.

Also, it is known that a short needle will act like a capacitor [15]. So, there exist no range of frequency for which the effective permittivity of the composite containing sparsely distributed short needles can be made negative. Because, it lacks of inductive effects. For this purpose, inductively loaded short needles can be used [16]. If needles are closely placed then the multiple scattering effects can not be ignored. As already mentioned in the introduction that adjacent needles couple strongly at the both ends. In this way, all the short needles along the axial direction can be taken as an infinite needle with periodic capacitive loading. This will make the spectrum of the effective permittivity of a composite containing short needles resonant. In this way, the negative bandwidth can be observed for a composite containing short needles.

In order to analyze the multiple scattering effects, it is assumed that the reference needle is placed at location $\mathbf{r}_q$. The local field $\mathbf{E}^{\rm loc}$ acting upon a reference needle is sum of an incident field and contribution of the scattered fields of all needles excluding the qth needle. Thus, assuming first order multiple scattering, the local field at qth needle can be written as,

$$\mathbf{E}^{\rm loc} = \mathbf{E}^i + \frac{1}{\epsilon_o} \sum_{j=-\infty,\, j\neq q}^{+\infty} k_o^2 \bar{\mathbf{G}}(\mathbf{r}_{jq}) \cdot \bar{\mathbf{\Upsilon}}_j \cdot \mathbf{E}^i \tag{10}$$

with

$$k_o^2 \bar{\mathbf{G}}(\mathbf{r}_{jq}) = \frac{1}{4\pi r_{jq}^3} [3\hat{\mathbf{r}}\hat{\mathbf{r}} - \bar{\mathbf{I}}] \tag{11}$$

where $\bar{\mathbf{I}}$ is a unit dyadic and $\mathbf{r}_{jq} = \mathbf{r}_j - \mathbf{r}_q$. The unit vector $\hat{\mathbf{r}} = \mathbf{r}_{jq}/|\mathbf{r}_{jq}| = \mathbf{r}_{jq}/r_{jq}$ is directed from $\mathbf{r}_j$ to $\mathbf{r}_q$.

The randomness in positioning of an ith short needle can be modeled by a random vector $\tilde{\mathbf{r}}_i = \mathbf{r}_i + \tilde{\mathbf{n}}_i$, where $\tilde{\mathbf{n}}_i$ is a random error vector with $i = j, q$. Likewise, randomness in orientation of an ith short needle can be described by $\tilde{\hat{\mathbf{e}}}_i = \cos\tilde{\phi}_i \sin\tilde{\theta}_i \hat{x} + \sin\tilde{\phi}_i \sin\tilde{\theta}_i \hat{y} + \cos\tilde{\theta}_i \hat{z}$, where $\tilde{\theta}_i$ and $\tilde{\phi}_i$ are two random variables. Random errors in positioning and orientation of a qth needle can be taken as independent of a jth needle without loss of generality. As random position of an ith short needle has no effect upon its random orientation, so they can be taken as independent. Thus the joint probability density function can be written as $p(\tilde{\mathbf{r}}_j, \tilde{\hat{\mathbf{e}}}_j, \tilde{\mathbf{r}}_q, \tilde{\hat{\mathbf{e}}}_q) = p(\tilde{\mathbf{r}}_j)p(\tilde{\hat{\mathbf{e}}}_j)p(\tilde{\mathbf{r}}_q)p(\tilde{\hat{\mathbf{e}}}_q)$. Likewise, it is assumed that random variables $\tilde{\theta}_i$ and $\tilde{\phi}_i$ are independent and identically distributed. Also it is further assumed that components of the random error vector $\tilde{n}_{xi}$, $\tilde{n}_{yi}$ and $\tilde{n}_{zi}$ along three orthogonal directions are independent and identically distributed. So, by taking ensemble average over random orientations $(\tilde{\theta}_i, \tilde{\phi}_i)$ and random positions $\tilde{\mathbf{r}}_i$ of an ith short needle with $i = j, q$, the ensemble averaged (EA) local electric field $\langle \tilde{\mathbf{E}}^{\mathrm{loc}} \rangle$ at qth needle can be written as,

$$\left\langle \tilde{\mathbf{E}}^{\mathrm{loc}} \right\rangle = \mathbf{E}^i + \frac{1}{\epsilon_o} \sum_{j=-\infty,\, j\neq q}^{+\infty} \left\langle \tilde{\bar{\mathbf{C}}}_{int} \right\rangle \cdot \left\langle \tilde{\bar{\boldsymbol{\Upsilon}}}_j \right\rangle \cdot \mathbf{E}^i \tag{12}$$

with

$$\begin{aligned} \left\langle \tilde{\bar{\mathbf{C}}}_{int} \right\rangle &= \left\langle k_o^2 \bar{\mathbf{G}} \left((\mathbf{r}_{jq} + \tilde{\mathbf{n}}_{jq}) \right) \right\rangle \\ &= \int_{\tilde{\mathbf{n}}_j} \int_{\tilde{\mathbf{n}}_q} k_o^2 \bar{\mathbf{G}} \left((\mathbf{r}_{jq} + \tilde{\mathbf{n}}_{jq}) \right) p(\tilde{\mathbf{n}}_j) p(\tilde{\mathbf{n}}_q) d\tilde{\mathbf{n}}_j d\tilde{\mathbf{n}}_q \end{aligned} \tag{13}$$

$$\left\langle \tilde{\bar{\boldsymbol{\Upsilon}}}_j \right\rangle = \int_{\tilde{\theta}_j} \int_{\tilde{\phi}_j} \tilde{\bar{\boldsymbol{\Upsilon}}}_j p(\tilde{\theta}_j) p(\tilde{\phi}_j) d\tilde{\theta}_j d\tilde{\phi}_j \tag{14}$$

with $\tilde{\mathbf{n}}_{jq} = \tilde{\mathbf{n}}_j - \tilde{\mathbf{n}}_q$ and $\tilde{\bar{\boldsymbol{\Upsilon}}}_j$ is given by Eqs. (5) to (8) with replacement of $\theta_j = \tilde{\theta}_j$ and $\phi_j = \tilde{\phi}_j$ respectively. Once the EA local electric field is known then the EA induced dipole moment $\langle \tilde{\mathbf{p}}_q \rangle$ at qth needle can be found. It can be expressed in terms of an effective EA polarizability tensor $\langle \tilde{\bar{\boldsymbol{\Upsilon}}}_q^e \rangle$ as,

$$\langle \tilde{\mathbf{p}}_q \rangle = \left\langle \tilde{\bar{\boldsymbol{\Upsilon}}}_q^e \right\rangle \cdot \mathbf{E}^i \tag{15}$$

where

$$\left\langle \tilde{\bar{\bar{\mathbf{\Upsilon}}}}_q^e \right\rangle = \left\langle \tilde{\bar{\bar{\mathbf{\Upsilon}}}}_q \right\rangle + \frac{1}{\epsilon_o} \left\langle \tilde{\bar{\bar{\mathbf{\Upsilon}}}}_q \right\rangle \cdot \sum_{j=-\infty,\, j \neq q}^{+\infty} \left\langle \tilde{\bar{\bar{\mathbf{C}}}}_{\mathbf{int}} \right\rangle \cdot \left\langle \tilde{\bar{\bar{\mathbf{\Upsilon}}}}_j \right\rangle \tag{16}$$

Thus the effective EA polarizability tensor is a sum of two tensors. The first tensor can be taken as self EA tensor whereas the second tensor models effects of EA interaction effects for the qth needle. The factor $\langle \tilde{\bar{\bar{\mathbf{C}}}}_{int} \rangle$ completely models effects of positioning errors in short needles. It can be taken as EA quasistatic interaction constant which describes interaction effects in the infinite lattice with positioning errors. Likewise, random errors in orientations of needles are completely described by the EA polarizability tensors $\langle \tilde{\bar{\bar{\mathbf{\Upsilon}}}}_j \rangle$ and $\langle \tilde{\bar{\bar{\mathbf{\Upsilon}}}}_q \rangle$.

If there are N short needles distributed per unit volume V with number density $n = N/V$ then the EA polarization $\langle \tilde{\mathbf{P}} \rangle$ can be written as,

$$\langle \tilde{\mathbf{P}} \rangle = \frac{1}{V} \sum_{q=1}^{N} \langle \tilde{\mathbf{p}}_q \rangle \tag{17}$$

Thus using Eqs. (15)–(17) and the constitutive relation for the electric displacement vector $\mathbf{D}$, i.e.,

$$\mathbf{D} = \epsilon_o \mathbf{E}^i + \langle \tilde{\mathbf{P}} \rangle = \epsilon_o \langle \bar{\epsilon}_{eff}(\omega) \rangle \cdot \mathbf{E}^i \tag{18}$$

the EA effective permittivity tensor $\langle \bar{\epsilon}_{eff}(\omega) \rangle$ of the composite medium containing short needles can be written as,

$$\langle \bar{\epsilon}_{eff}(\omega) \rangle = \bar{\mathbf{I}} + \frac{1}{V} \sum_{q=1}^{N} \frac{\left\langle \tilde{\bar{\bar{\mathbf{\Upsilon}}}}_q^e \right\rangle}{\epsilon_o} \tag{19}$$

The EA effective permittivity given by Eq. (19) is general and can be applied to all types of random errors in orientation and positioning of short needles. For this one only needs to know probability density functions associated with positioning and orientational errors. Once they are known, the EA effective permittivity can be found using Eq. (19) for any real physical situation.

As needles are electrically short so the EA effective permittivity shows frequency dispersion only. If needles are taken to be finite in length, i.e., not electrically short then the electric field distribution is not uniform over a finite length needle scatterer. This makes the effective permittivity spatially dispersive. Thus, the EA effective permittivity of a composite containing finite length needles becomes non local, i.e., it becomes a function of frequency ω and wave vector $\mathbf{k}_i$.

The expression of EA effective permittivity given by Eq. (19) is general. But to simplify the analysis, it is desired to make some assumptions so that EA effective permittivity can be expressed in simplified form. Firstly, qth reference needle is placed at origin and its orientation is described by random variables $\tilde{\theta}$ and $\tilde{\phi}$. As it is of interest to know that how maximum error in positioning and orientation of a single short needle can be tolerated. That is why, secondly, it is assumed that $\tilde{\mathbf{n}}_j = \tilde{\mathbf{n}}$ and $\tilde{\theta}_j = \tilde{\theta}$ and $\tilde{\phi}_j = \tilde{\phi}$. Using these assumptions, the EA effective permittivity tensor $\langle\bar{\epsilon}_{\mathit{eff}}(\omega)\rangle$ of a composite containing short needles can be written in simplified form as,

$$\langle\bar{\epsilon}_{\mathit{eff}}(\omega)\rangle = \bar{\mathbf{I}} + n\left[\bar{\mathbf{I}} - \frac{1}{\epsilon_o}\left\langle\tilde{\bar{\boldsymbol{\Upsilon}}}\right\rangle \cdot \left\langle\tilde{\bar{\mathbf{C}}}_{int}\right\rangle\right]^{-1} \cdot \frac{1}{\epsilon_o}\langle\tilde{\bar{\boldsymbol{\Upsilon}}}\rangle \tag{20}$$

with

$$\left\langle\tilde{\bar{\mathbf{C}}}_{int}\right\rangle = \sum_{j=-\infty,\, j\neq 0}^{+\infty} \int_{\tilde{\mathbf{n}}} k_o^2 \bar{\mathbf{G}}(\tilde{\mathbf{r}}_j) p(\tilde{\mathbf{n}}) d\tilde{\mathbf{n}} \tag{21}$$

$$\left\langle\tilde{\bar{\boldsymbol{\Upsilon}}}\right\rangle = \int_{\tilde{\theta}}\int_{\tilde{\phi}} \tilde{\bar{\boldsymbol{\Upsilon}}} p(\tilde{\theta}) p(\tilde{\phi}) d\tilde{\theta} d\tilde{\phi} \tag{22}$$

where $\tilde{\mathbf{r}}_j = \mathbf{r}_j + \tilde{\mathbf{n}}$ and the expression of EA effective permittivity given by Eq. (20) is consistent with Silveirinha [18, Eq. (40)] under quasistatic limit.

4. EFFECTS OF POSITIONING ERRORS

In order to analyze effects of positioning errors upon EA effective permittivity, it is assumed that there exist no errors in orientations of short needles. Also the incident wave polarization is taken to be aligned along z-axis, i.e., $\hat{\mathbf{e}}_i = \hat{\mathbf{z}}$. For no errors in orientation of needles, we can have $\tilde{\theta} = 0$ and $\tilde{\phi} = \pi/2$. Thus, for the z-component of the EA local field, it is desired to take $\tilde{\mathbf{r}}_j/|\tilde{\mathbf{r}}_j| = \hat{\tilde{\mathbf{r}}}_j = \hat{\mathbf{z}}$, where $\hat{\tilde{\mathbf{r}}}_j$ is the unit vector directed along $\tilde{\mathbf{r}}_j$. Also the mean vector $\mathbf{r}_j$ is periodic in x, y and z directions with periods d_x, d_y and d_z respectively. Thus, the EA effective permittivity becomes,

$$\langle\bar{\epsilon}_{\mathit{eff}}(\omega)\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \langle\epsilon_z(\omega)\rangle \end{bmatrix} \tag{23}$$

with

$$\langle \epsilon_z(\omega) \rangle = 1 + \frac{n}{\epsilon_o \mathrm{Re}\left(\frac{1}{\Lambda}\right) - i\left(\frac{k_o^3}{6\pi}\right) - n\left\langle \tilde{C}_{int} \right\rangle} \tag{24}$$

$$\left\langle \tilde{C}_{int} \right\rangle = \frac{d_x d_y d_z}{4\pi} \left\langle \sum_{j=-\infty,\, j\neq 0}^{+\infty} \left\{ \frac{3(\hat{\mathbf{z}}\cdot\tilde{\mathbf{r}}_j)^2}{\tilde{r}_j^5} - \frac{1}{\tilde{r}_j^3} \right\} \right\rangle \tag{25}$$

As a composite medium consists of perfectly conducting needles, so it seems that there is no loss mechanism and a composite as a whole should be lossless. But in fact a composite under consideration is lossy. This can be explained as follows. Because scattering loss must always be present in a composite to satisfy energy conservation. If a short needle scatterer is able to receive energy from an incident wave then it must also be able to radiate energy back into space. Otherwise, the field intensity and the energy locked inside the needle would increase without limit with time. Thus an imaginary number $-ik_o^3/6\pi$ must be added to the real part of the inverse of the needle's polarizability due to scattering loss. This is done for the energy conservation for a lossless dipole scatterer [17]. The EA interaction constant defined by Eq. (25) is unitless contrary to the one defined in Eq. (21) which has units of m^{-3}. This EA interaction constant for a three dimensional infinite lattice can be converted into a rapidly convergent series using Poisson summation formula [19] as follows,

$$\begin{aligned}\left\langle \tilde{C}_{int} \right\rangle = d_x d_y d_z \Bigg\langle \Bigg[&\frac{1}{4\pi} \sum_{m_z=-\infty, m_z\neq 0}^{\infty} \frac{2(m_z d_z + \tilde{n}_z)^2 - \tilde{n}_y^2 - \tilde{n}_x^2}{[(m_z d_z + \tilde{n}_z)^2 + \tilde{n}_y^2 + \tilde{n}_x^2]^{5/2}} \\ &- \frac{4\pi}{d_z^3} \sum_{(m_x, m_y)\neq(0,0)} \sum_{(m_z=1)}^{\infty} m_z^2 \cos\left(\frac{2\pi m_z \tilde{n}_z}{d_z}\right) \\ &K_0\left(\frac{2\pi m_z}{d_z}\sqrt{(m_x d_x + \tilde{n}_x)^2 + (m_y d_y + \tilde{n}_y)^2}\right) \Bigg] \Bigg\rangle \end{aligned} \tag{26}$$

The EA interaction constant $\langle \tilde{C}_{int} \rangle$ can be completely known provided that if probability density functions, i.e., $p(\tilde{n}_i)$ with $i = x, y, z$ are known. It is assumed that the positioning errors of a needle along x, y and z-directions are taken to be very small, i.e., $\sigma_i \ll \lambda_o$ with $i = x, y, z$. It is desired to analyze maximum possible errors that can occur in the positioning of the short needles without violating the condition for small errors. It is known from [20] that for a random error $\tilde{n}_i$ with given variance σ_i^2, the maximum entropy, i.e., maximum positioning error occurs if $\tilde{n}_i$ is gaussian distributed. Thus, gaussian pdf is assumed for positioning errors. Also, mean values of positioning errors of needles can be taken as systematic errors. Such type of errors can easily be removed and without loss of generality, their pdfs can be assumed to have zero means.

As positioning errors along three orthogonal directions are taken to be small and independent of each other, so it can be assumed that the factor $\zeta_i = 1/2\sigma_i^2 \to \infty$ for small errors. Thus, integrals involved in the ensemble averaging of the interaction constant can be solved asymptotically using Laplace's Method [21]. Finally, the EA interaction constant can be written as,

$$\left\langle \tilde{C}_{int} \right\rangle = C_{int}^p + C_{int}^e \tag{27}$$

$$C_{int}^p = d_x d_y d_z \left[\frac{1}{\pi d_z^3}\zeta(3) - \frac{4\pi}{d_z^3} \sum_{(m_x,m_y)\neq(0,0)} \sum_{(m_z=1)}^{\infty} m_z^2 \right.$$
$$\left. K_0\left(\frac{2\pi m_z}{d_z}\sqrt{(m_x d_x)^2 + (m_y d_y)^2}\right)\right] \tag{28}$$

$$C_{int}^e = d_x d_y d_z [C_{xyz}^1 - C_{xyz}^2] \tag{29}$$

$$C_{xyz}^1 = \frac{1}{\pi d_z^3}\left[3\zeta(5)\left\{2\left(\frac{\sigma_z}{d_z}\right)^2 - \left(\frac{\sigma_x}{d_z}\right)^2 - \left(\frac{\sigma_y}{d_z}\right)^2\right\}\right.$$
$$+\frac{45}{4}\zeta(7)\left\{\left(\frac{\sigma_x}{d_z}\right)^2\left(\frac{\sigma_y}{d_z}\right)^2 - 4\left(\frac{\sigma_x}{d_z}\right)^2\left(\frac{\sigma_z}{d_z}\right)^2\right.$$
$$\left.\left. -4\left(\frac{\sigma_y}{d_z}\right)^2\left(\frac{\sigma_z}{d_z}\right)^2\right\} + 630\zeta(9)\left(\frac{\sigma_x}{d_z}\right)^2\left(\frac{\sigma_y}{d_z}\right)^2\left(\frac{\sigma_z}{d_z}\right)^2\right]$$
$$+\frac{8\pi^3}{d_z^3}\left(\frac{\sigma_z}{d_z}\right)^2 \sum_{(m_x,m_y)\neq(0,0)} \sum_{(m_z=1)}^{\infty} m_z^4 K_0$$
$$\left(\frac{2\pi m_z}{d_z}\sqrt{(m_x d_x)^2 + (m_y d_y)^2}\right) \tag{30}$$

$$C_{xyz}^2 = \frac{4\pi}{d_z^3} \sum_{(m_x,m_y)\neq(0,0)} \sum_{(m_z=1)}^{\infty} m_z^2 \left[1 - 2\pi^2 m_z^2 \left(\frac{\sigma_z}{d_z}\right)^2\right]$$
$$\left[C_0 K_0\left(\frac{2\pi m_z}{d_z}\sqrt{(m_x d_x)^2 + (m_y d_y)^2}\right)\right.$$
$$\left. +C_1 K_1\left(\frac{2\pi m_z}{d_z}\sqrt{(m_x d_x)^2 + (m_y d_y)^2}\right)\right] \tag{31}$$

$$\begin{aligned}
C_0 \;=\; & \frac{2\pi^2 m_z^2}{[(m_x d_x)^2+(m_y d_y)^2]}\left\{\left(m_x d_x \frac{\sigma_x}{d_z}\right)^2+\left(m_y d_y \frac{\sigma_y}{d_z}\right)^2\right\} \\
& -3\pi^2\left(\frac{\sigma_x}{d_z}\right)^2\left(\frac{\sigma_y}{d_z}\right)^2 \frac{(m_z d_z)^2}{[(m_x d_x)^2+(m_y d_y)^2]^3} \\
& \Bigg[(m_x d_x)^4+(m_y d_y)^4-6(m_x d_x)^2(m_y d_y)^2 \\
& -\frac{4\pi^2}{3}\left(\frac{m_z}{d_z}\right)^2(m_x d_x)^2(m_y d_y)^2\left\{(m_x d_x)^2+(m_y d_y)^2\right\}\Bigg] \qquad (32)
\end{aligned}$$

$$\begin{aligned}
C_1 \;=\; & \frac{\pi(m_z d_z)[(m_x d_x)^2-(m_y d_y)^2]}{[(m_x d_x)^2+(m_y d_y)^2]^{3/2}}\left\{\left(\frac{\sigma_x}{d_z}\right)^2-\left(\frac{\sigma_y}{d_z}\right)^2\right\} \\
& +3\pi^2\left(\frac{\sigma_x}{d_z}\right)^2\left(\frac{\sigma_y}{d_z}\right)^2 \frac{(m_z d_z)^2}{[(m_x d_x)^2+(m_y d_y)^2]^{7/2}} \\
& \Bigg[\frac{6}{\pi}(m_x d_x)^2(m_y d_y)^2\left(\frac{d_z}{m_z}\right)+\frac{10\pi}{3}(m_x d_x)^2(m_y d_y)^2\left(\frac{m_z}{d_z}\right) \\
& \left[(m_x d_x)^2+(m_y d_y)^2\right]-\frac{1}{\pi}\left(\frac{d_z}{m_z}\right)\left[(m_x d_x)^4+(m_y d_y)^4\right] \\
& -\frac{2\pi}{3}\left(\frac{m_z}{d_z}\right)\left[(m_x d_x)^6+(m_y d_y)^6\right]\Bigg] \qquad (33)
\end{aligned}$$

where $\zeta(.)$ is a zeta function given by [22]. The factor C_{int}^p is the periodic interaction constant of a three dimensional infinite lattice of dipole scatterers given by [19]. Effects of positioning errors are described by the factor C_{int}^e. It is dependent upon periodic locations of needles and positioning error variances σ_i^2 with $i = x, y, z$. If there exist no errors in positioning of needles then we can take $\sigma_i = 0$, which shows that all needles are at their mean positions. Thus, the factor $C_{int}^e = 0$ and the EA interaction constant $\langle \tilde{C}_{int} \rangle$ becomes equal to C_{int}^p. Likewise, if positioning errors are increased then $\langle \tilde{C}_{int} \rangle$ reduces showing less interaction between the needles.

5. EFFECTS OF ORIENTATIONAL ERRORS

Effects of orientational errors are described by the EA polarizability tensor $\langle \tilde{\bar{\boldsymbol{\Upsilon}}} \rangle$ given by Eq. (22). This factor is dependent upon random variables $\tilde{\theta}$ and $\tilde{\phi}$ and their associated pdfs. In order to simplify the analysis, it is assumed that all short needles are at their mean positions, i.e., having no positioning errors. The EA polarizability tensor models effects of orientational errors within the yz-plane and out of the plane. Orientational errors within the plane are handled by random variable $\tilde{\theta}$ and its associated pdf. Likewise, out of the plane orientational errors are dependent upon random variable $\tilde{\phi}$ and its pdf. Thus, in order to consider effects of orientational errors within the plane, we can take $\tilde{\phi} = \pi/2$. The random variable $\tilde{\theta}$ is taken to be uniformly distributed in the interval $[-\xi_1, \xi_1]$ with zero mean and variance $\sigma_\theta^2 = \xi_1^2/3$. Thus, EA effective permittivity $\langle \epsilon_z(\omega) \rangle$ can be expressed in terms of an orientational error standard deviation σ_θ as follows,

$$\langle \epsilon_z(\omega) \rangle = 1 + \frac{n f(\sigma_\theta)}{\epsilon_o \mathrm{Re}\left(\frac{1}{\Lambda}\right) - i\left(\frac{k_o^3}{6\pi}\right) - n C_{int}^{p} f(\sigma_\theta)} \tag{34}$$

with

$$f(\sigma_\theta) = \frac{1}{2}\left[1 + \frac{\sin(2\sqrt{3}\sigma_\theta)}{2\sqrt{3}\sigma_\theta}\right] \tag{35}$$

If there exist no error in orientations of needles, then $f(\sigma_\theta) = 1$ and the proposed effective permittivity become consistent with [17, 19].

For numerical results, the length l of each needle is taken to be 8 mm for the operating wavelength of $\lambda_o = 100$ mm. From [23], it can be seen that if the wire radius a lie in the range of $\lambda_o/100 = 1$ mm to $\lambda_o/10 = 10$ mm, then the wire can be taken as a relatively thick wire. In this case, the end effects of wire can not be ignored because of additional charge accumulation on the end caps of the wire which tend to increase the polarizability. Therefore, for thick needles higher order moments can not be neglected. So, the radius of short needle is taken to be $a = 0.98$ mm which is less than 1 mm, showing that the needle is thin. Also $d_x = 5$ mm, $d_y = 5.5$ mm and $d_z = l + 0.4$ mm. From Fig. 1, it can be seen that increasing positioning errors of the needles reduces negative bandwidth. A closer view of Fig. 1 is given in Fig. 2. A significant reduction in the negative bandwidth is observed for positioning errors as high as 40% along three orthogonal directions. Likewise, effects of orientational errors are given in Fig. 3, where the EA effective permittivity with no error in orientation is compared with error in orientation of short needles. Its closer view is given in Fig. 4. Again a reduction in the negative bandwidth is observed for increasing orientational errors.

6. CONCLUSIONS

Effects of positioning and orientational errors upon the ensemble averaged effective permittivity of a composite medium containing electrically short needles are analyzed. It can be concluded that increasing positioning errors reduces the negative bandwidth. Also, the reduction in the negative bandwidth can be observed by increasing orientational errors. The proposed theory is general and can be applied to any type of positioning and orientational errors for short needles. For this, one needs to know only probability density functions associated with these errors.

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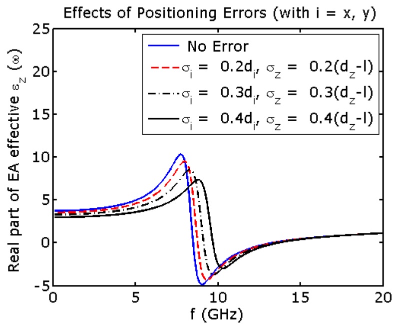

Figure 1. Effects of positioning errors upon the EA effective permittivity. Positional errors along three orthogonal directions, i.e., x, y and z are taken to be normally distributed with means, $m_i = 0$ where $i = x, y, z$. Negative bandwidth reduces significantly for $\sigma_x = 0.4d_x$, $\sigma_y = 0.4d_y$ and $\sigma_z = 0.4(d_z - l)$.

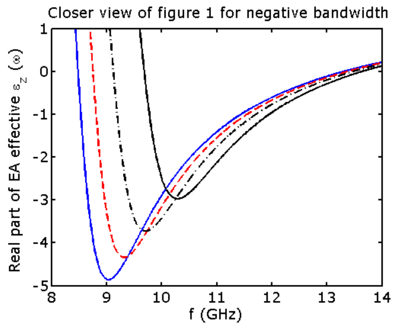

Figure 2. Closer view of the negative bandwidth for Figure 1.

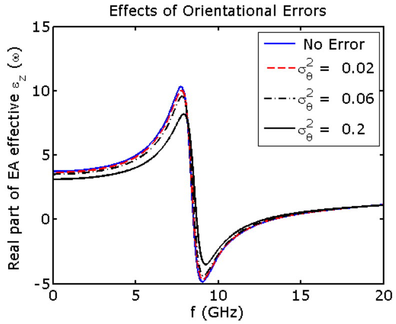

Figure 3. Effects of orientational errors upon the EA effective permittivity. Orientational error is taken to be uniformly distributed between $[-\xi_1, \xi_1]$ with zero mean and variance $\sigma_\theta^2 = \xi_1^2/3$. Negative bandwidth reduces as orientational error increases.

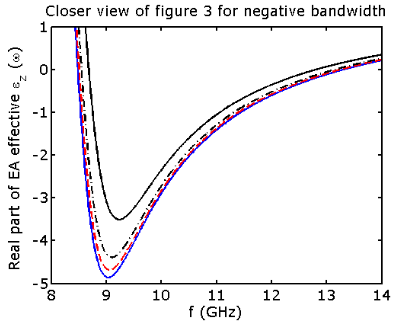

Figure 4. Closer view of the negative bandwidth for Figure 3.

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