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ELECTROMAGNETOSTATIC CHARGES AND FIELDS IN A ROTATING CONDUCTING SPHERE

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Abstract—Charges and fields in a rotating non-magnetic conducting sphere under stationary conditions are investigated by using Minkowski's electrodynamics of moving media and the Lorentz force equation, taking into account the electric permittivity of the sphere. Starting from the assumption that the magnetic field inside the sphere is constant, exact solutions of the corresponding field equations are obtained in a first-order theory. However, it is found that there is a range of values of the sphere's net charge for which the physical interpretation of the results is difficult within a continuum model. Outside that range, our solution to the classic electromagnetostatic problem appears plausible.

1. A CLASSIC ELECTROMAGNETOSTATIC PROBLEM

Despite the fact that more than a century has passed since the advent of Einstein's special theory of relativity and Minkowski's electrodynamics of moving media, which grew out of the Maxwell electromagnetic equations, the conquest of a relativistic mentality appears to be a slow and painful process. For example, in the vast literature dealing with conductors that move in externally applied or self-excited magnetic fields, in the framework of a first-order theory, the results of Minkowski's electrodynamics are often misinterpreted or ignored. Therefore, a solution to a problem in the electrodynamics of moving bodies, in addition to its possible applications in engineering, may provide us with a better understanding of these fundamental physical theories as well as be a test of their validity. Consider an isolated conducting body of revolution with a net electric charge that rotates uniformly around its axis of revolution. What are the charge

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distribution and electromagnetic fields in the rotating conductor under stationary conditions?

Some time ago, Grøn and Vøyenli [1] solved the problem both in the inertial rest frame of the rotational axis and in the rotating rest frame of the conductor, starting from Maxwell's equations *in vacuo* and the Lorentz force equation. The authors pointed out that in the general case a space charge may appear inside the rotating conductor when a steady state is reached, contrary to what happens in the equilibrium situation of the same conductor at rest. The redistribution of charge is due to the inertia of the conduction electrons as well as to the axisymmetric magnetic field which is produced by the azimuthal convection current of the rotating charges. Grøn and Vøyenli estimated that for solid conductors under laboratory conditions the effect of the inertia of the conduction electrons cannot be neglected.

The authors, however, did not take into account the electric permittivity of the rotating conductor. (It appears that the electric permittivity of good conductors, while unknown, should be included not only in description of electromagnetic waves in stationary conductors (see, for example, [2, 3]) but also in analyses of steady-state situations in rotating conductors [4–10].) The shortcoming was remedied by Redžić [9], who retraced the analysis given in [1], considering the relativistic case in the frame of the rotational axis, putting the reflection in the framework of Minkowski's electrodynamics of moving media. The author assumed a homogeneous, non-magnetic conductor with relative permittivity ε_r and found that the **E**- and **B**-fields of the rotating conductor are independent of ε_r and coincide with those calculated in [1]; the space and surface free charge densities, however, depend on ε_r.

Both the Grøn and Vøyenli [1] and Redžić [9] analyses were applied to the case of an infinite cylindrical conductor with a given net charge rotating around its own axis. While possessing the advantage of allowing the exact relativistic solution to the problem, the system considered is obviously impracticable. A discussion of a more realistic system seems to be lacking in the literature.

In this work, we present an analysis of the electromagnetic field and the charge distribution of a rigid, non-magnetic conducting sphere of finite conductivity with a net electric charge that rotates uniformly around a diameter. We consider non-relativistic speeds and we restrict ourselves to a first-order theory, in which the classical concept of rigid body is meaningful. Our key assumption is that the **B**-field inside the sphere is constant. We show that that simple assumption leads to mathematically correct results within an apparently consistent first-order theory. There is, however, a range of values of the sphere's net charge Q for which the physical interpretation of our results seems to be difficult in the framework of a continuum model. For the values of Q outside that range, our solution to the problem appears plausible.

Finding the fields and charge distributions of a uniformly rotating conducting sphere under stationary conditions is a classic electromagnetostatic problem. A clear solution to the problem taking into account the sphere's permittivity and the inertia of the conduction electrons is not available in the literature. The only attempt in this direction seems to have been made long time ago by Swann [11] who discussed a rotating conducting sphere with no net charge. Unfortunately, his analysis takes into account only the electric force on a rotating conduction electron, without even mentioning the corresponding magnetic force, which turns out to be incorrect in the general case of a rotating conductor with a net electric charge [1]. (To do justice to Swann, it should be noted that, apart from the wrong starting Equation (9) of Swann [11], the analysis given in ([11], Section 2, 157–162), is exact and was exploited in his long paper on unipolar induction [12].) In this work, we present a generalization of Swann's solution to the problem from the perspective of Minkowski's electrodynamics of moving media, highlighting the limits of validity of the continuum model. Our solution, which is based on the method developed in [9] and an *ansatz* leading to a problem with the Dirichlet boundary condition, appears to be new. Hopefully, it could contribute to the continuous interest in application-oriented relativistic electrodynamics (cf, e.g., [13–21]).

2. ELECTROMAGNETOSTATICS OF ROTATING CONDUCTORS

For the convenience of the reader, in this section we briefly summarize our earlier argument [9] and give some exact relativistic results for rotating conductors under stationary conditions.

Consider an isolated conducting body of revolution that rotates with constant angular velocity $\boldsymbol{\omega} = \omega \mathbf{u}_z$ around its axis of revolution. Under steady-state conditions, in the inertial rest frame Σ_I of the axis (the lab frame), according to Minkowski's electrodynamics of moving media ([22–25], cf also [26–28]), for the system considered Maxwell's equations reduce to

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \varrho \tag{1}$$

$$\nabla \times \mathbf{E} = 0 \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 (\varrho \mathbf{v} + \nabla \times \mathbf{M}) \tag{4}$$

where **E**, **B**, **P** and **M** are the electric field strength, magnetic flux density, polarization and magnetization, respectively, $\mathbf{v} = \boldsymbol{\omega} \times \rho \mathbf{u}_\rho$ is the velocity of a material point of the conductor and ϱ ('varrho') is the space charge density. In writing Equation (4), we assumed that the conduction electrons are at rest relative to the crystal lattice, i.e., that the conduction current density vanishes. As a consequence, the equation of motion of the *conduction electrons* inside the conductor with respect to the lab is

$$-e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \gamma m \boldsymbol{\omega} \times \mathbf{v}, \quad \gamma \equiv (1 - v^2/c^2)^{-1/2}, \tag{5}$$

where $-e$ and m are the charge and mass of the electron, respectively.

Equation (5) can be recast as

$$\mathbf{E} + \mathbf{v} \times (\mathbf{B} + \gamma \mathbf{B}_m) = 0 \tag{6}$$

where

$$\mathbf{B}_m \equiv -\frac{m}{e}\boldsymbol{\omega}. \tag{7}$$

With $\omega = 10^3 s^{-1}$ one gets $B_m = (m/e)\omega = 5.7 \times 10^{-9}\,\mathrm{T}$.

Now for a linear, isotropic and non-magnetic rotating conductor of relative permittivity ε_r, using the Lorentz-covariance of Minkowski's electrodynamics, we get simple expressions for **P** and **M**:

$$\mathbf{P} = \gamma^2 \varepsilon_0 (\varepsilon_r - 1)(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{8}$$

$$\mathbf{M} = \gamma^2 \varepsilon_0 (\varepsilon_r - 1)(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \times \mathbf{v} \tag{9}$$

and consequently

$$\mathbf{M} = \mathbf{P} \times \mathbf{v} \tag{10}$$

$$\nabla \times \mathbf{M} = -(\nabla \cdot \mathbf{P})\mathbf{v}. \tag{11}$$

Thus, the magnetization current density in our rotating conductor reduces *exactly* to the convection current density of bound charges, since $-\nabla \cdot \mathbf{P}$ equals the space charge density of bound charges inside a polarized medium.

One might wonder why the term $-(\nabla \cdot \mathbf{P})\mathbf{v}$ was not included explicitly from the beginning in Equation (4). It should be noted however that like all good physical theories (that make us able 'to leap ahead of the empirical frontier') Minkowski's electrodynamics too must contain some non-obvious steps, i.e., postulates, whose validity can be checked only indirectly, by experimental verification of the consequences of the theory. So the Ampère-Maxwell law well known from the electrodynamics of bodies at rest, $\nabla \times \mathbf{H} = \mathbf{J}_c + \partial \mathbf{D}/\partial t$, where $\mathbf{H} \equiv \mathbf{B}/\mu_0 - \mathbf{M}$, $\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}$, and $\mathbf{J}_c$ is the conduction current density, is replaced in the Minkowski electrodynamics by equation $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ where **J** is now the total current density of free charges, convection plus conduction, $\mathbf{J} = \varrho\mathbf{v} + \mathbf{J}_c$. (We remind the reader that in the framework of relativistic electrodynamics the conduction current density $\mathbf{J}_c$ is *defined* by the preceding equation inside a moving medium (cf [23, 25]).) Obviously, a tacit assumption is hidden in the above postulate that, if the theory is valid, the term $\nabla \times \mathbf{M}$ should generally describe the contribution to $\nabla \times \mathbf{B}$ from the 'true' magnetization of the moving medium (due to its own magnetization in a locally co-moving frame) as well as the contribution from the convection current of bound charges. It turns out that this is indeed so (cf [25]).

Equations (1), (4) and (11) obviously imply

$$\nabla \cdot \mathbf{E} = \varrho_t/\varepsilon_0 \tag{12}$$

$$\nabla \times \mathbf{B} = \mu_0 \varrho_t \mathbf{v} \tag{13}$$

where $\varrho_t = \varrho - \nabla \cdot \mathbf{P}$ is the total, free plus bound, space charge density inside the rotating conductor. From the preceding results, after a somewhat cumbersome but in every step simple calculus, we obtain

$$\varrho_t = -\varepsilon_0 \gamma^2 \boldsymbol{\omega} \cdot [2\mathbf{B} + \gamma(1+\gamma^2)\mathbf{B}_m] \tag{14}$$

$$\varrho = -\varepsilon_0 \gamma^2 \boldsymbol{\omega} \cdot [2\mathbf{B} + \gamma \mathbf{B}_m (3\varepsilon_r \gamma^2 - \varepsilon_r + 2 - 2\gamma^2)]. \tag{15}$$

$$\varrho_b = -\nabla \cdot \mathbf{P} = \varepsilon_0(\varepsilon_r - 1)\gamma^3(3\gamma^2 - 1)\, \boldsymbol{\omega} \cdot \mathbf{B}_m. \tag{16}$$

For details of derivation, and a more complete argument, we refer the reader to the original Reference [9].

3. A ROTATING CONDUCTING SPHERE

Specialize now to the case of a solid, non-magnetic conducting sphere of radius R and uniform relative permittivity ε_r, carrying a net charge Q.

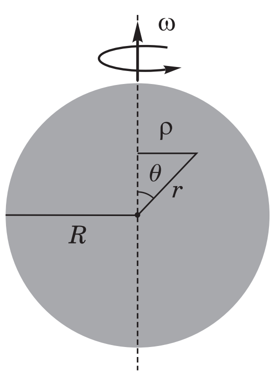

Figure 1. Conducting sphere rotating around a diameter.

The sphere rotates around a diameter with constant angular velocity $\boldsymbol{\omega} = \omega \mathbf{u}_z$, Figure 1. We consider non-relativistic speeds, $\omega R \ll c$, and we restrict ourselves to a first-order theory. The non-relativistic case is obtained by setting $\gamma = 1$ in equations of the preceding section.

After many fruitless attempts to find an exact solution of the corresponding equations for $\mathbf{E}$ and $\mathbf{B}$, it has been recognized that an apparently consistent solution is obtained by assuming that the $\mathbf{B}$-field inside the sphere is constant. In what follows the solution will naturally unfold starting from this crucial assumption.

Setting $\gamma = 1$ in Equation (6) we obtain

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}^* = \mathbf{0}, \tag{17}$$

where

$$\mathbf{B}^* \equiv \mathbf{B} + \mathbf{B_m}. \tag{18}$$

As was explained above, Equation (6), and thus Equation (17), is a consequence of our starting assumption that the conduction current vanishes inside the rotating conducting body of revolution. It should be pointed out, however, that *in the framework of a first-order theory and under stationary conditions*, that assumption is unnecessary. Namely, the vanishing of the conduction current can be derived using the constitutive equation for the conduction current density $\mathbf{J}_c$ for a rigid ohmic non-magnetic conductor in arbitrary motion. A derivation is given in Appendix A.

Assume now that inside the sphere $\mathbf{B}$ is constant, $\mathbf{B}_{\text{inside}} = B_z \mathbf{u}_z$, where B_z is a constant to be determined. Since both $\mathbf{B}_{\text{inside}}$ and $\mathbf{B}_m$ are constant and collinear with the z-axis, the same applies to $\mathbf{B}^*$. Thus $\mathbf{B}^* = B_z^* \mathbf{u}_z$.

Now we have arrived at the well trodden path. Namely, the same type of the problem arises in calculating the electric field of a conducting sphere rotating in a constant externally applied magnetic field (e.g., [4, 5, 8, 29–31]), or a permanently magnetized rotating conducting sphere ([12, 32], [24, pp. 152–157]). So we exploit the well known successful strategy for finding the electric potential of our non-magnetic, rotating sphere.

From Equation (17) and our crucial assumption we have, in cylindrical coordinates,

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}^* = -\omega \rho B_z^* \mathbf{u}_\rho, \tag{19}$$

i.e., in spherical coordinates,

$$\mathbf{E} = -\omega r B_z^* \sin\theta (\sin\theta \mathbf{u}_r + \cos\theta \mathbf{u}_\theta), \tag{20}$$

Thus, the radial component of the electric field inside the sphere is

$$E_{r\,\text{inside}} = -\omega r B_z^* \sin^2\theta. \tag{21}$$

In addition, the potential *inside* the sphere along the axis of rotation is constant, since the electric field vanishes at those points. Denote the constant potential by V_A. Then the potential over the sphere's surface is given by

$$V_{\text{surface}}(R,\theta) = V_A + \omega B_z^* R^2 (\sin^2\theta)/2, \tag{22}$$

since the potential inside the sphere is

$$V_{\text{inside}} = V_A + \omega B_z^* \rho^2/2 = V_A + \omega B_z^* r^2 (\sin^2\theta)/2, \tag{23}$$

using Equation (19). Taking into account that Legendre polynomial of second order equals

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2} = 1 - \frac{3}{2}\sin^2\theta, \tag{24}$$

the potential over the surface can be written as

$$V_{\text{surface}}(R,\,\theta) = V_A + \frac{\omega B_z^* R^2}{3} - \frac{\omega B_z^* R^2}{3} P_2(\cos\theta). \tag{25}$$

There is no charge outside the sphere, the potential satisfies the Laplace equation and, since the problem has azimuthal symmetry, for V_{outside} we have

$$V_{\text{outside}}(r,\,\theta) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta). \tag{26}$$

choosing $V(r=\infty) = 0$.

From the continuity of the potential at $r = R$, using Equations (25) and (26) and the properties of Legendre polynomials, we find that

$$\frac{B_0}{R} = V_A + \frac{\omega B_z^* R^2}{3}, \quad B_2 = -\frac{\omega B_z^* R^5}{3}, \tag{27}$$

and that all other B_i vanish. Since the monopole term in expansion (26) must coincide with the potential of a point charge Q located at the centre of the sphere it follows that

$$B_0 = \frac{Q}{4\pi\varepsilon_0} \tag{28}$$

and consequently

$$V_A = \frac{Q}{4\pi\varepsilon_0 R} - \frac{\omega B_z^* R^2}{3}. \tag{29}$$

Thus the potential outside the sphere is the sum of a monopole and a quadrupole terms

$$V_{\text{outside}}(r,\,\theta) = \frac{Q}{4\pi\varepsilon_0 r} - \frac{\omega B_z^* R^5}{3r^3} P_2(\cos\theta), \tag{30}$$

and the radial component of the electric field is given by

$$E_{r\,\text{outside}} = \frac{Q}{4\pi\varepsilon_0 r^2} - \frac{\omega B_z^* R^5}{r^4} P_2(\cos\theta). \tag{31}$$

The *total* surface charge density over the sphere $r = R$ must satisfy the relation

$$\frac{\varsigma_t}{\varepsilon_0} = E_r(r = R^+) - E_r(r = R^-). \tag{32}$$

From Equations (32), (31), (21) and (24) we find that

$$\varsigma_t = \sigma - \varepsilon_0 \omega B_z^* R \left[\frac{5}{3} P_2(\cos\theta) - \frac{2}{3}\right] = \sigma + \varepsilon_0 \omega B_z^* R \left(\frac{5}{2}\sin^2\theta - 1\right). \tag{33}$$

where

$$\sigma \equiv \frac{Q}{4\pi R^2} \tag{34}$$

The *total* space charge density inside the sphere is obtained by taking the divergence of Equation (17), and then using Equations (12), (13), the vectorial identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{B}), \tag{35}$$

and the relation $\nabla \times \mathbf{v} = 2\boldsymbol{\omega}$. In this way, neglecting second-order quantities, we get the following result for the total space charge density

$$\varrho_t = -2\varepsilon_0 \omega B_z^*. \tag{36}$$

The same expression is obtained setting $\gamma = 1$ in the general result (14), as it should be. (Note that in the above derivation of ϱ_t it was not assumed that $\mathbf{B}$ is constant inside the sphere.) One can easily verify that, when $\mathbf{B}$ inside the sphere is constant, the Poisson equation

$$\nabla^2 V_{\text{inside}} = -\frac{\varrho_t}{\varepsilon_0} \tag{37}$$

is identically satisfied with expressions (23) and (36), as it should be. One can also verify that the total charge inside the sphere, $-2\varepsilon_0 \omega B_z^* (4/3)\pi R^3$, plus the total charge on the surface of the sphere obtained by integration of Equation (33) is Q, as it should be. (It is straightforward to find the free and bound charge densities inside and over the sphere.) As can be seen, the redistribution of charge due to rotation, as well as to the inertia of the conduction electrons, produces the electric field inside the sphere which in combination with the $\mathbf{v} \times \mathbf{B}$-field provides the necessary centripetal force on the conduction electrons. Needless to say, the electric field due to σ given by Equation (34) vanishes inside the sphere.

Thus our rotating conducting sphere produces a stationary electric field whose potential inside and outside the sphere is given by

Equations (23) and (29), and (30), respectively. The potential outside the sphere is obtained by finding the solution of the Laplace equation that reduces to the prescribed function given by Equation (25) over the sphere's surface, and that matches the sphere's net charge Q. As can be seen, uniqueness theorem applies to this case too. Namely, for any given B_z^* and V_A we have a problem with the Dirichlet boundary condition for the potential outside the sphere. The same applies to the potential inside the sphere. Thus the potentials given by Equations (23) and (29), and (30), are the only solution to our problem for a given B_z^* and the given sphere's net charge Q. The sources of the stationary electric field are (free and bound) surface and space charges of the rotating sphere whose total charge densities are given by Equations (33) and (36), respectively. Note that these results for the electric field are valid for *any* constant B_z^*.

[It is perhaps worthwhile to answer the following query: how do we know that charge distributions (33) and (36) indeed produce the electric field whose potential is given by Equations (23) and (29), and (30)? The answer is simple. Namely, as was pointed out above, this potential is the only function complying with the conditions of our problem for a given B_z^* and Q. On the other hand, on the basis of the well known representation formula for the electrostatic potential which is obtained using Green's second identity (cf, e.g., [33, pp. 750–751], [34, Section 1.8], and also [3, Section 1.12]) we have that the potential of our rotating sphere is everywhere equal to

$$V(\mathbf{r}_0) = \frac{1}{4\pi\varepsilon_0}\int \frac{\varrho_t(\mathbf{r})}{|\mathbf{r}_0-\mathbf{r}|}d^3\mathbf{r} + \frac{1}{4\pi\varepsilon_0}\oint \frac{\varsigma_t(\mathbf{r})}{|\mathbf{r}_0-\mathbf{r}|}da,$$

where $\mathbf{r}_0$ is the observation point, $\mathbf{r}$ is the integration variable, and the second term is the surface integral over the sphere $r = R$ (which is the only surface of discontinuity of the gradient of V). Of course, one can verify by direct calculation that charge distributions (33) and (36) indeed have the potential given by Equations (23) and (29), and (30). A proof is sketched in Appendix B.

The preceding considerations imply the following interesting result. If the surface charge density over the sphere $r = R$ is $\sigma^* = A\sin^2\theta$, where A is a constant, the electric field outside and inside the sphere is given by

$$\begin{aligned}\mathbf{E}_{\text{outside}} &= \nabla\left[\frac{2AR^4}{15\varepsilon_0 r^3}P_2(\cos\theta)\right] + \frac{2AR^2}{3\varepsilon_0 r^2}\frac{\mathbf{r}}{r},\\ \mathbf{E}_{\text{inside}} &= \frac{4A}{15\varepsilon_0 R}\mathbf{r}P_2(\cos\theta) - \frac{2A}{5\varepsilon_0 R}r\sin\theta\cos\theta\mathbf{u}_\theta.\end{aligned}$$

The above result, well known from the theory of the spherical harmonics (cf, e.g., [35, pp. 157–160]), was essential in Schlomka and

Schenkel's analysis of electric fields due to free and bound charges in the case of a uniformly magnetized rotating sphere [32]. (A simplified version of the problem, involving total charge densities only, is found in [34, pp. 285–286].)]

The field outside the sphere is the sum of a monopole (due to constant σ of Equation (34)) and a quadrupole field (due to ω-dependent term in Equation (33) and to ϱ_t of Equation (36)); the electric field inside the sphere is obviously due to the rotation-induced distribution only.

There still remains the problem of finding the magnetic field produced by our rotating conducting sphere. The sources of the field are the azimuthal convection current of the rotation induced distribution (ω-dependent term in Equation (33) and ϱ_t of Equation (36)) as well as the azimuthal convection current of the uniform surface charge distribution (the σ-term in Equation (33)). The magnetic field due to the σ-source is well known (cf, e.g., [33, pp. 505–506], [36, pp. 236–237]). Inside the sphere it is given by

$$\mathbf{B}_\sigma = B_\sigma \mathbf{u}_z = \frac{2}{3}\mu_0 \sigma \omega R \mathbf{u}_z; \tag{38}$$

outside the sphere the field is identical to that of an ideal magnetic dipole of magnetic moment $\mathbf{P}_m = (4/3)\pi R^3 \sigma \omega R \mathbf{u}_z$ located at the centre of the sphere. (The vector potential of the ideal dipole field is of course $\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi)\mathbf{P}_m \times \mathbf{r}/r^3$.)

A somewhat cumbersome but in every step simple calculus reveals that the magnetic field due to the rotation-induced charge distribution is a second-order quantity and thus negligible, consistent with our crucial assumption above. (The calculus is outlined in Appendix C.) Consequently, since Equation (18) and our crucial assumption imply

$$\mathbf{B}^* = \mathbf{B} + \mathbf{B}_m = (B_z + B_{mz})\mathbf{u}_z, \tag{39}$$

using Equations (38) and (7) we have

$$B_z^* = \frac{2}{3}\mu_0 \sigma \omega R - B_m. \tag{40}$$

Thus it appears that we have arrived at a consistent first-order theory giving us fields and charge distribution of our rotating conducting sphere, Figure 2. Namely, the total $\mathbf{B}^*$ field is the sum of $\mathbf{B}_\sigma$, $\mathbf{B}_m$ and the rotation induced fields $\mathbf{B}^<_{\omega-sur.ch.}$ and $\mathbf{B}^<_{sp.ch.}$ calculated in Appendix C. It is clear that the only choice for $\mathbf{B}^*$ consistent with our starting assumption, in the first-order theory, is the one expressed by Equation (40), i.e., $\mathbf{B}^* = \mathbf{B}_\sigma + \mathbf{B}_m$. (Note that in our analysis we had to deduce the value of B_z^* as a vital part of the solution, whereas the corresponding quantities in References [4, 5, 8, 12, 29–32], [24, pp. 152–157], were simply given.)

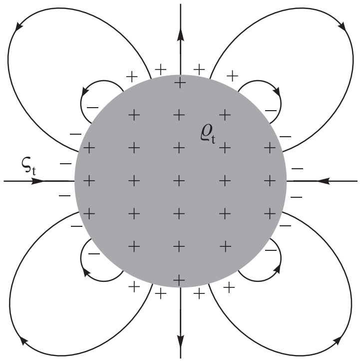

Figure 2. Induced charges and electric field of the rotating sphere in the special case of vanishing net charge. The electric field outside the sphere is a linear quadrupole field.

4. DISCUSSION

From Equations (19), (36) and (40) we see that the electric field and the total charge density inside the sphere have one component due to the charge σ and one due to the inertia of the electrons. To estimate the magnitude of these components recall that under normal laboratory conditions the dielectric strength of air is $3 \times 10^6\,\mathrm{Vm}^{-1}$. As can be seen (cf Equations (30) and (31)), the magnitude of the sphere's electric field is maximum just outside the surface, and is of the order

$$\frac{\sigma + \varepsilon_0 \omega B_z^* R}{\varepsilon_0}$$

which using Equation (40) and neglecting second-order quantities reduces to

$$\frac{\sigma - \varepsilon_0 \omega B_m R}{\varepsilon_0}.$$

Choosing as earlier $\omega = 10^3\,\mathrm{s}^{-1}$ we find that for a sphere with a radius $R = 0.1\,\mathrm{m}$, the maximum value of σ that the sphere will keep before the breakdown of air is of the order $10^{-5}\,\mathrm{Cm}^{-2}$, the term $\varepsilon_0 \omega B_m R$ being many orders smaller than that. Then a corresponding value of B_σ is of the order $10^{-9}\,\mathrm{T}$, which is of the order of B_m calculated earlier under the same conditions. Hence the contribution to $\mathbf{E}$ and ϱ_t inside the sphere from the charge σ and from the inertia of the conduction electrons may be of the same order under laboratory conditions in this case too. It is perhaps worthwhile to note that there is the potential difference between the pole of the rotating sphere and its equator, given by

$$U = -\frac{\omega B_z^* R^2}{2}, \tag{41}$$

as follows, e.g., from Equation (22). With $\omega = 10^3\,\mathrm{s}^{-1}$, $R = 0.1\,\mathrm{m}$, for a sphere with zero net charge this gives $U = 2.8 \times 10^{-8}\,\mathrm{V}$.

The theory seems to be wholly satisfactory up to know so it was somewhat surprising to recognize that there is a problematic point. Namely, using Equations (36) and (40) we find that the total space charge inside the sphere Q_V is

$$Q_V = \frac{4}{3}\pi R^3 2\varepsilon_0 \omega B_m - \frac{4}{9}\left(\frac{\omega R}{c}\right)^2 Q. \tag{42}$$

For the sphere of radius $R = 0.1\,\mathrm{m}$, and with $\omega = 10^3 s^{-1}$, we find that the total space charge inside the sphere is about $4 \times 10^{-19}\,\mathrm{C}$ under laboratory conditions ($\sigma = 10^{-5}\,\mathrm{Cm}^{-2}$, $Q \approx 10^{-6}\,\mathrm{C}$). That would mean that a redistribution of several electrons would be enough for settling an equilibrium state inside the rotating sphere. This however seems to be hardly reconcilable with a continuum model, taking into account that the volume of our sphere of radius $R = 0.1\,\mathrm{m}$ is about $4000\,\mathrm{cm}^3$.

Pursuing this line of thought, we find that when Q decreases from $10^{-3}\,\mathrm{C}$ to $-10^{-3}\,\mathrm{C}$ (assuming that the sphere is *in vacuo*), Q_V increases from (approximately) $-5 \times 10^{-17}\,\mathrm{C}$ to $5 \times 10^{-17}\,\mathrm{C}$, which is about 300 and -300 electron charges, respectively. These values of Q_V also seem to be hardly reconcilable with a continuum model. On the other hand, when $Q = 10^{-2}\,\mathrm{C}$, we have that $Q_V \approx -5 \times 10^{-16}\,\mathrm{C}$, or about 3000 electrons, which is perhaps consistent with a continuum model, for our sphere with volume of about $4000\,\mathrm{cm}^3$.

Durand [33, p. 506] claims: "With a conducting sphere where the charges can move freely over the surface, the centrifugal force would modify the distribution of the charges at rest but feebly." Our analysis shows that generally things are not so simple.

It is difficult to see what the correct interpretation of the small values of the space charge Q_V is, when the sphere's net charge Q is in the range of $10^{-3}\,\mathrm{C}$ to $-10^{-3}\,\mathrm{C}$. Excluding the possibility that Minkowski's electrodynamics of moving media was incorrectly used in the above analysis of the simplest electromagnetostatic problem, and if we wish to retain the continuum model, what remains is that our very starting assumption, namely that there is a stationary situation in our system, is incorrect for those values of Q. The most radical possibility would be of course that, in the general case, there is no equilibrium and also that Minkowski's electrodynamics is inadequate for description *in the lab frame* of the electrodynamics of accelerated systems, such as our simple uniformly rotating sphere. (This radical alternative would perhaps require a revision of fundamental concepts (cf, e.g., [37, 38]).) It should be noted, however, that outside the 'problematic' range of values of the sphere's net charge Q our solution to the problem appears plausible.

5. CONCLUSIONS

In this paper, the fields and charges in a rotating non-magnetic conducting sphere with a net electric charge under stationary conditions are investigated, in a first-order theory, using Minkowski's electrodynamics of moving media and the Lorentz force equation. Starting from the assumption that the magnetic field inside the sphere is constant and solving the Laplace equation for the potential outside the sphere with the Dirichlet boundary condition, the exact solutions of the corresponding field equations are found, deducing the value of the constant magnetic field inside as a vital part of the complete solution. However, the physical interpretation of the obtained results is in the general case difficult. Namely, there is a range of values of the sphere's net charge Q for which the solution is problematic in the framework of a continuum model. Outside the 'problematic' range of values of Q the solution appears plausible.

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APPENDIX A.

The constitutive equation for the conduction current density $\mathbf{J}_c$ for a rigid ohmic non-magnetic conductor in arbitrary motion reads

$$\mathbf{J}_c = \sigma_c \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{m}{e} \dot{\mathbf{v}} \right), \tag{A1}$$

where $\mathbf{v}$ and $\dot{\mathbf{v}}$ are the velocity of a *material point* of the conductor and its acceleration, respectively, and σ_c is the conductivity of the medium (cf, e.g., [39, p. 222]). In the case of our uniformly rotating body of revolution

$$\mathbf{v} = \omega \rho \mathbf{u}_\phi, \quad \dot{\mathbf{v}} = -\omega^2 \rho \mathbf{u}_\rho \tag{A2}$$

and thus

$$\mathbf{J}_c = \sigma_c \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{m}{e}\omega^2 \rho \mathbf{u}_\rho \right) \tag{A3}$$

Then under stationary conditions, taking into account the fact that $\nabla \cdot \mathbf{v} = 0$ and the azimuthal symmetry of the problem, assuming that the conductor is homogeneous and using the identity

$$\nabla \times (\mathbf{a} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{B}, \tag{A4}$$

we find that the curl of $\mathbf{J}_c$ vanishes

$$\nabla \times \mathbf{J}_c = 0. \tag{A5}$$

Also, from the local conservation of electric charge for media in arbitrary motion (cf [10]) we obtain that in our case, when $\nabla \cdot \mathbf{v} = 0$ and $(\mathbf{v} \cdot \nabla)\varrho = 0$,

$$\nabla \cdot \mathbf{J}_c = 0. \tag{A6}$$

In addition, from the continuity of the normal component of $\mathbf{J}_c$ across the surface of our rotating conductor, and from the fact that there is no conduction current outside, it follows that $J_{cn} = 0$ at the surface. Now since the interior of the rotating conductor is a simply connected space, and since the curl of $\mathbf{J}_c$ vanishes, it follows that $\mathbf{J}_c$ can be expressed as the gradient of a single-valued potential Ψ, $\mathbf{J}_c = -\nabla\Psi$. Then from Green's first identity we have

$$\oint_S \Psi \frac{\partial \Psi}{\partial n} dS = \int_\tau [(\nabla\Psi)^2 + \Psi\nabla^2\Psi] d\tau \tag{A7}$$

where τ is the volume of the conductor, and S is its surface. The left-hand side of the above equation vanishes ($J_{cn} = 0$), and also $\nabla^2\Psi = 0$, which implies that $\nabla\Psi$ vanishes inside the conductor, i.e., Ψ is a constant. Thus $\mathbf{J}_c$ vanishes inside our rotating conductor under stationary conditions. The above elegant way of proving that $\mathbf{J}_c$ vanishes seems to be due to van Bladel [31, pp. 285–286]. Needless to say, the vanishing of $\mathbf{J}_c$ and equation (A3) imply Equation (17).

APPENDIX B.

Let there be a circular ring *in vacuo* carrying a charge Q^* distributed uniformly on it, the ring being of such a size that its diameter subtends an angle $2\theta_0$ at the centre of a sphere of radius R. The potential of the charged ring is well known and is given by

$$V^> = \frac{Q^*}{4\pi\varepsilon_0 R} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta_0) P_n(\cos\theta) \tag{B1}$$

$$V^< = \frac{Q^*}{4\pi\varepsilon_0 R} \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^{n} P_n(\cos\theta_0) P_n(\cos\theta) \tag{B2}$$

in the regions $r > R$ and $r < R$, respectively (cf, e.g., [35, pp. 101–104]). Replacing Q^* by $\varsigma_t 2\pi R^2 \sin\theta_0 d\theta_0$ and integrating over θ_0 from 0 to π, using well-known properties of Legendre polynomials, we get the potential of the surface charge distribution (33)

$$V^{>}_{sur.ch.} = \frac{Q}{4\pi\varepsilon_0 r} + \frac{2\omega B^*_z R^3}{3r} - \frac{\omega B^*_z R^5}{3r^3} P_2(\cos\theta) \quad \text{(B3)}$$

$$V^{<}_{sur.ch.} = \frac{Q}{4\pi\varepsilon_0 R} + \frac{2\omega B^*_z R^2}{3} - \frac{\omega B^*_z r^2}{3} P_2(\cos\theta) \quad \text{(B4)}$$

outside and inside the conducting sphere, respectively. (For an alternative way of proving this cf [36, pp. 142–143].)

On the other hand, a simple calculus reveals that the potential of the uniform space charge distribution (36) is given by

$$V^{>}_{sp.ch.} = -\frac{2\omega B^*_z R^3}{3r} \quad \text{(B5)}$$

$$V^{<}_{sp.ch.} = \frac{\omega B^*_z r^2}{3} - \omega B^*_z R^2 \quad \text{(B6)}$$

outside and inside the sphere, respectively. Adding up Equations (B3) and (B5) we get expression (30); also, adding up Equations (B4) and (B6) we get the potential inside the sphere expressed by Equations (23) and (29).

APPENDIX C.

Let there be a circular current loop with current I^* *in vacuo*, the circle being of such a size that its diameter subtends an angle $2\theta_0$ at the centre of a sphere of radius R. The magnetic field of the current loop is well known and is given by

$$B^{>}_r = \frac{\mu_0 I^* \sin\theta_0}{2R} \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^{n+2} P^1_n(\cos\theta_0) P_n(\cos\theta) \quad \text{(C1)}$$

$$B^{>}_\theta = \frac{\mu_0 I^* \sin\theta_0}{2R} \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{R}{r}\right)^{n+2} P^1_n(\cos\theta_0) P^1_n(\cos\theta) \quad \text{(C2)}$$

and

$$B^{<}_r = \frac{\mu_0 I^* \sin\theta_0}{2R} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^{n-1} P^1_n(\cos\theta_0) P_n(\cos\theta) \quad \text{(C3)}$$

$$B^{<}_\theta = -\frac{\mu_0 I^* \sin\theta_0}{2R} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{R}\right)^{n-1} P^1_n(\cos\theta_0) P^1_n(\cos\theta) \quad \text{(C4)}$$

in the regions $r > R$ and $r < R$, respectively (cf, e.g., [40, 41]). Now replace I^* by $\omega R^2 \sigma_\omega(\theta_0) \sin\theta_0 d\theta_0$, where

$$\sigma_\omega(\theta_0) = \frac{2}{3}\varepsilon_0 \omega R B_z^* - \frac{5}{3}\varepsilon_0 \omega R B_z^* P_2(\cos\theta_0) \tag{C5}$$

is ω-dependent term in Equation (33), and integrate over θ_0 from 0 to π. Using the recurrence formula for Legendre polynomials,

$$P_n'(1-\mu^2) = \frac{n(n+1)}{2n+1}(P_{n-1} - P_{n+1}) \tag{C6}$$

where $\mu \equiv \cos\theta_0$, the definition

$$P_n^1(\cos\theta_0) = \sin\theta_0 P_n'(\mu), \tag{C7}$$

and properties of Legendre polynomials (cf, e.g., [42, pp. 146–147, 158]) we obtain that the magnetic field due to the azimuthal current of ω-dependent surface charge in Equation (33), inside our rotating conducting sphere, is given by

$$\begin{aligned} \mathbf{B}^<_{\omega-sur.ch.} &= \left(\frac{\omega R}{c}\right)^2 B_z^* \left\{ \frac{4}{9}\mathbf{u}_z + \left[\frac{2}{9}P_1(\cos\theta) - \frac{4}{7}\left(\frac{r}{R}\right)^2 P_3(\cos\theta)\right]\mathbf{u}_r \right. \\ &\quad \left. - \left[\frac{2}{9}P_1^1(\cos\theta) - \frac{4}{21}\left(\frac{r}{R}\right)^2 P_3^1(\cos\theta)\right]\mathbf{u}_\theta \right\}, \end{aligned} \tag{C8}$$

which is a second-order quantity. It can be verified that the curl of $\mathbf{B}^<_{\omega-sur.ch.}$ vanishes identically, as it should.

Similarly, replacing I^* by $\omega \varrho_t r^2 dr \sin\theta_0 d\theta_0$, it is found that the magnetic field due to the azimuthal current of the uniform space charge distribution (36) is given by

$$\begin{aligned} \mathbf{B}^<_{sp.ch.} &= -2\left(\frac{\omega R}{c}\right)^2 B_z^* \left\{ P_1(\cos\theta)\left[\frac{1}{3} - \frac{1}{5}\left(\frac{r}{R}\right)^2\right]\mathbf{u}_r \right. \\ &\quad \left. + P_1^1(\cos\theta)\left[-\frac{1}{3} + \frac{2}{5}\left(\frac{r}{R}\right)^2\right]\mathbf{u}_\theta \right\}, \end{aligned} \tag{C9}$$

inside the rotating sphere.

A simple calculus reveals that the second-order field (C9) satisfies Equation (13), i.e.,

$$\nabla \times \mathbf{B}^<_{sp.ch.} = \mu_0 \varrho_t \mathbf{v}, \tag{C10}$$

with ϱ_t given by Equation (36), as it should.

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