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FLEXIBLE ARRAY BEAMPATTERN SYNTHESIS USING HYPERGEOMETRIC FUNCTIONS

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Abstract—For array beampattern synthesis, it is possible to simplify the model and reduce the computational load by formulating it to be a Quadratic Programming (QP) problem. Moreover, the QP method is conceptually simple and also flexible and convenient for the constrained problems. In the QP method, a key component is the template function which describes the desired beampattern as a deterministic function of direction. However, so far this method has only found its application for the Dolph-Chebyshev arrays but not for other arrays. In this paper, the template functions in the form of Hypergeometric Function corresponding to Legendre arrays and Gegenbauer arrays, namely, Legendre Hypergeometric Function (LHF) and Gegenbauer Hypergeometric Function (GHF), are derived and the synthesis procedures are also presented. The proposed generalized template form using Hypergeometric Function works for the Dolph-Chebyshe arrays as well. From the simulation results, it can be shown that when the proposed template functions are used in the QP method, the exactly synthesized beampatterns can be obtained and they can provide good performance in the design of the constrained problems for both Legendre arrays and Gegenbauer arrays. In addition, some discussion results about the application of Gegenbauer arrays in the QP method are presented.

1. INTRODUCTION

It is well-known that orthogonal polynomials such as Chebyshev polynomials are useful in array beampattern synthesis. In [1], Dolph has proposed the well-known Dolph-Chebyshev method to synthesize the array beampattern of a Uniform Linear Array (ULA) based on

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the properties of Chebyshev polynomials. For a given Side-Lobe Level (SLL), it has been proven that the Dolph-Chebyshev array provides narrowest mainbeam width in the array beampattern, while for a given mainbeam width, it achieves the lowest SLL. During decades, several methods [2–7] have been proposed to make some improvements to the Dolph-Chebyshev synthesis method, and they have either complex concepts or heavy computational load. In particular, some synthesis approaches based on the convex programming such as [8–11] are considered to be used owing to their good performance. In such cases, the excitations of the array elements are computed to "match" a desired mask using the convex optimization method. Bucci [8] proposed a general approach to form the synthesis problem to a intersections finding problem to determine both antenna geometry and the excitation using segment procedures to make a good compromise between speed and complexity, Lebret [9] synthesized the array pattern by minimizing the level over a given zone, and Rocca [11] obtained optimal compromise among sum and difference patterns through sub-arraying. However, these methods using convex programming are solved by brute force, more or less, since some numerical techniques and iterative work are required and sometimes the convergence problem needs to be taken into account in the application. To deal with it, Ng [12] proposed the Quadratic Programming (QP) flexible synthesis method (Appendix A) which provides the one-step solution without the numerical or iterative work and complex mathematical concepts. By formulating the array pattern synthesis to be a QP problem, The QP method only needs to have a deterministic mathematical function, e.g., Dolph-Chebyshev Function (DCF), as the template function to compute the coefficients which minimize the mean-square error between the synthesized array pattern and the template function. It is shown in Appendix A that the QP method not only has very low computational load and conceptually simple procedures, but also flexibility to solve the constrained problems which the classical Dolph-Chebyshev method cannot deal with.

However, so far this method has only found its application for the Dolph-Chebyshev arrays but not for other arrays. As known, besides Chebyshev polynomials, Legendre polynomials and Gegenbauer polynomials have also been employed to synthesize the array beampattern [13, 14] which also produces narrow mainbeam, low and controllable level of side-lobes. Legendre arrays may not provide the optimum equi-ripple array beampattern as Doph-Chebyshev array, but they still have the advantages over Chebyshev polynomials such as lower far out side-lobes and higher beam efficiency. Moreover, Gegenbauer arrays are so generalized that Legendre arrays and Dolph-

Chebyshev arrays are just the special cases of them, and in the design, the beam efficiency and the directivity can be further adjusted with the SLL and the element number specified. Therefore, we hope to apply the QP method in the beampattern synthesis of both Legendre arrays and Gegenbauer arrays, and then the proper mathematical functions, which can express their beampatterns exactly so that it can be hired as templates for the QP method, are desired. In this paper, based on the features of Hypergeometric Series (HS) and Hypergeometric Functions (HF), Legendre Hypergeometric Function (LHF) and Gegenbauer Hypergeometric Function are proposed as the template functions for the QP method and the corresponding synthesis procedures are also illustrated. The proposed generalized template form using Hypergeometric Function works for the Dolph-Chebyshe arrays as well. From the simulation results, it can be shown that when the proposed template functions are used in the QP method, the exactly synthesized beampatterns can be obtained and they can provide good performance in the design of the constrained problems for both Legendre arrays and Gegenbauer arrays.

To illustrate the whole contents, this paper is outlined as follows: Section 2 illustrates the mathematical concepts of the HS and HF and how to find the mathematical form of Legendre polynomials and Gegenbauer polynomials. Section 3 shows how the LHF and GHF are derived and how a Legendre array and a Gegenbauer array are synthesized using the HF form template function in the QP method. Some discussion about the Gegenbauer arrays are included as well. Section 4 shows how the proposed template functions are used in the QP method for the constrained synthesis problems.

2. HYPERGEOMETRIC SERIES AND HYPERGEOMETRIC FUNCTIONS

In this section, the concepts of Hypergeometric Series (HS) and Hypergeometric Function (HF) are introduced, and our objective is to find a mathematical function form of Legendre polynomials and Gegenbauer polyomials.

From [15], the explicit expression of the Legendre polynomials is

$$T_N(x) = \frac{1}{2^N} \sum_{n=0}^{[N/2]} (-1)^n \frac{(2N-2n)!}{n!(N-n)!(N-2n)!} (2x)^{N-2n}, \qquad (1)$$

while the explicit expression of the Gegenbaure polynomials is

$$T_N'^{(\alpha)}(x) = \sum_{n=0}^{[N/2]} (-1)^n \frac{\Gamma(N-n+\alpha)}{\Gamma(\alpha)n!(N-2n)!} (2x)^{N-2n}, \qquad (2)$$

where α is a constant. It can be observed that both Chebyshev polynomials and Legendre polynomials have the same form as

$$P_N(x) = \sum_{n=0}^{[N/2]} a_n x^{N-2n}. \tag{3}$$

Therefore, obviously, the beampattern of a ULA can be expressed by a Legendre polynomial or a Gegenbauer polynomial as well as Chebyshev polynomial. That is why both Legendre polynomials and Gegenbauer polynomials can be used to synthesize the beampattern of a ULA. From [12], it is proven that when using the QP method, once the template function that can express the beampattern exactly is found, the current excitations can be obtained easily. As a result, we are motivated to find a mathematical function form of Chebyshev polynomials or Legendre polynomials.

In mathematics [16], a power series in which the ratio of successive coefficients indexed by n is a rational function of n is defined as a Hypergeometric Series (HS), in the most general sense. The series, if convergent, will define a Hypergeometric Function (HF), which may then turn out to be defined over a wider domain of the argument by analytic continuation. A HF is defined as ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ if the ratio of successive coefficients of the corresponding HS can be written as

$$\frac{a_{n+1}}{a_n} = \frac{(n+a_1)(n+a_2)\ldots(n+a_p)}{(n+b_1)(n+b_2)\ldots(n+b_q)(n+1)}. \tag{4}$$

From (1), it is noticed that the ratio of the successive coefficients of an N order Legendre polynomials is

$$\frac{a_{n+1}}{a_n} = -\frac{(N-2n)(N-2n-1)}{8(n+1)(2N-2n-1)}. \tag{5}$$

Obviously, it is a rational functions of n. Therefore, Legendre polynomials are certainly a kind of HS and can be expressed by a HF in the form of ${}_pF_q(a, b; c; x)$. The similar conclusion can be obtained for Gegenbauer polynomials. Even then, it is still difficult to derive the HF forms corresponding to Legendre polynomials and Gegenbauer polynomials, so an alternative way from the viewpoint of differential equations is considered.

As known, orthogonal polynomials always originate from the solutions of the differential equation in such a form as [15]

$$Q(x)\frac{d^2y}{dx^2} + U(x)\frac{dy}{dx} + \lambda y = 0, \tag{6}$$

where $Q(x)$ is a given quadratic (at most) polynomial, $U(x)$ is a given linear polynomial, and λ is a constant. As a kind of orthogonal polynomials, Legendre function is the solution of Legendre's differential equation given as

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+N(N+1)y=0, \tag{7}$$

or

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+N(N+1)y=0. \tag{8}$$

When n is a nonnegative integer, e.g., $N = 0, 1, 2, \ldots$, its solution, Legendre function, is reduced to an N-order polynomial, which is defined as a Legendre polynomial. Similarly, an N-order Gegenbauer polynomial is the solution of Gegenbauer's differential equation given by

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-(2\alpha+1)x\frac{dy}{dx}+N(N+2\alpha)y=0, \tag{9}$$

where α is a positive real number. Meanwhile, the differential equation in such a form as

$$\left(x\frac{d}{dx}+a\right)\left(z\frac{d}{dx}+b\right)y=\left(x\frac{d}{dx}+c\right)\frac{dy}{dx}, \tag{10}$$

or

$$x(1-x)\frac{d^2y}{dx^2}+[c-(a+b+1)x]\frac{dy}{dx}-aby=0, \tag{11}$$

is known as a Hypergeometric Differential Equation (HDE). It is proved that a HDE has its solution ${}_2F_1(-a, b; c; x)$ which is the HF mentioned above. For a HDE, it is proven that when a is a non-positive integer, e.g., $-N$, its solutions are reduced to polynomials. Up to constant factors and scales, these are special cases of orthogonal polynomials such as Legendre polynomials, Gegenbauer polynomials. etc..

Letting $z = \frac{1-x}{2}$, obviously, a Legendre's differential equation in the form of (7) can be transformed to be a HDE as

$$z(1-z)\frac{d^2y}{dz^2}+(1-2z)\frac{dy}{dz}+N(N+1)y=0. \tag{12}$$

Comparing (11) and (12), the parameters of the HDE can be determined as $a = -N$, $b = N + 1$, and $c = 1$. Therefore, the HF representing the Legendre polynomials is ${}_2F_1(-N, N + 1; 1; z)$. After transformation the solution turns out to be ${}_2F_1(-N, N + 1; 1; \frac{1-x}{2})$.

As the same reason, a Gegenbauer polynomial can be written in the form of a HF as

$$T_N^{(\alpha)}(x)=\frac{(2\alpha)^{(N)}}{N!}{}_2F_1\left(-N,2\alpha+N;\alpha+\frac{1}{2};\frac{1-x}{2}\right), \tag{13}$$

where $a^{(N)}$ is the rising factorial computed by

$$a^{(N)} = \frac{(a+N-1)!}{(a-1)!} = \frac{\Gamma(a+N)}{\Gamma(a)}. \tag{14}$$

In addition, an interesting thing is that, as another kind of orthogonal polynomial, Chebyshev polynomial can be expressed by a HF as well. Letting $z = \frac{1-x}{2}$, a Chebyshev's differential equation can be transformed to be a HDE given by

$$z(1-z)\frac{d^2y}{dz^2} + \left(\frac{1}{2} - z\right)\frac{dy}{dz} + N^2by = 0. \tag{15}$$

Then the parameters of the HDE can be determined as $a = -N, b = N$, and $c = \frac{1}{2}$. Therefore, the HF representing the Chebyshev polynomials is ${}_2F_1(-N, N; \frac{1}{2}; \frac{1-x}{2})$.

As a result, we find a generalized way to express Legendre polynomials, Gegenbauer polynomials and Chebyshev polynomials in the form of a mathematical function using a HF.

3. BEAMPATTERN SYNTHESIS USING HYPERGEOMETRIC FUNCTIONS

In this section, we try to use the HF as the template function and apply it in the QP method to synthesize the beampatterns of a Legendre array and a Gegenbauer array.

Recall that in the paper [12], the Dolph-Chebyshev Function (DCF) is used as a template function to synthesize a Dolph-Chebyshev array, which is given by

$$H_{DCF}(\theta) = \cos\left[(L-1)\cos^{-1} u \cos\left(\frac{\pi d}{\lambda}(\cos\theta - \cos\theta_0)\right)\right], \tag{16}$$

where L is the number of array elements, d is the inter-element spacing, λ is the wavelength corresponding to the operating frequency of the array, θ is the azimuth angle measured w.r.t the array axis, and θ_0 is the looking direction. The parameter u, which is used to control the SLL of the synthesized Dolph-Chebyshev array beampattern, is given by

$$u = \cosh\left[\frac{1}{L-1}\cosh^{-1}\left(10^{-sll/20}\right)\right], \tag{17}$$

where sll is the specified SLL in dB scale.

From the discussion in the previous section, the DCF can also be rewritten in the form of a HF as

$$H_{DCHF}(\theta) ={}_2 F_1\left(-L+1, L-1; \frac{1}{2}; Z\right), \tag{18}$$

where

$$Z = \frac{1 - u\cos\left(\frac{\pi d}{\lambda}\left(\cos\theta - \cos\theta_0\right)\right)}{2}. \tag{19}$$

That is, the function can also be defined as the Dolph-Chebyshev Hypergeometric Function (DCHF). Before the simulations of Legendre arrays and Gegenbauer arrays using the proposed template function in the QP method, we can make a test to use the DCHF as the template.

Assuming a 32-element broadside ULA with half-wavelength inter-element spacing, if the desired SLL is −40 dB, the synthesized beampattern and the template function DCF are shown in Fig. 1. It is shown that the proposed template function DCHF works as well as the DCF applied in the QP method. Therefore, it indicates that the HF form template function also works for Legendre arrays and Gegenbauer arrays. The proposed template functions, Legendre-Hypergeometric Function (LHF) and Gegenbauer-Hypergeometric Function (GHF), are given by

$$H_{LHF}(\theta) = {}_2F_1\left(-L+1, L; 1; Z\right), \tag{20}$$

and

$$H_{GHF}(\theta) = \frac{(2\alpha)^{(N)}}{N!}\,{}_2F_1\left(-L+1, 2\alpha+L-1; \alpha+\frac{1}{2}; Z\right), \tag{21}$$

where Z is the same as given in (17) and u is used to control the SLL of the synthesized beampattern as well.

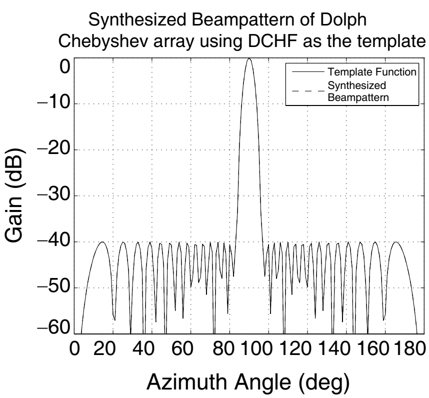

Figure 1. Synthesized beampattern of a 32-element Dolph-Chebyshev array using DCHF as the template.

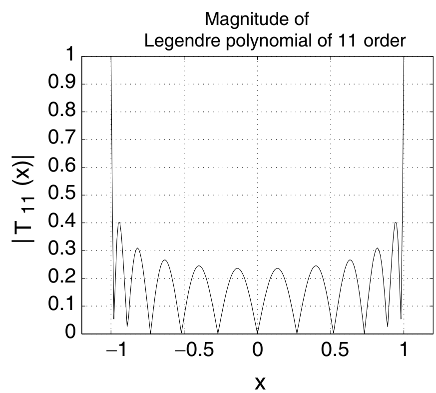

Figure 2. Magnitude of Legendre polynomial of 11-order.

To determine the value of the parameter u, the procedures for Legendre arrays and Gegenbauer arrays are similar:

(1) Find the order of the corresponding Legendre (or Gegenbauer) polynomial by subtracting the element number by one.
(2) Find the maximum value of the side-lobes, which is actually the peak value of the first side-lobe.
(3) Multiply the peak value of the first side-lobe according to the desired SLL, equate the Legendre (or Gegenbauer) polynomial with this value and solve for the roots of the equation.

Taking a 12-element half-wavelength spacing Legendre array as an example, the order of the corresponding Legendre polynomial is 11 and its plot is shown in Fig. 2. In the figure, it can be observed that the peak value of the first side-lobe is 0.4065. For this example, if the desired SLL is $-40\,\text{dB}$, we equate the Legendre polynomial with 40.65 and obtain $u = 1.115$ by solving for the roots of the equation. For the example above, if the element number is changed to 32, by the same procedures, the parameter u is determined to be 1.0149, and the synthesized beampattern is shown in Fig. 3. It is shown that the beampattern of the Legendre array can be synthesized exactly by the LHF function with the QP method. The various values of u obtained by the procedures above for different SLLs and element numbers are shown in Fig. 4.

Comparing the Fig. 1 and Fig. 3, it is observed that Dolph-Chebyshev arrays can obtain better directivity (Appendix B) while

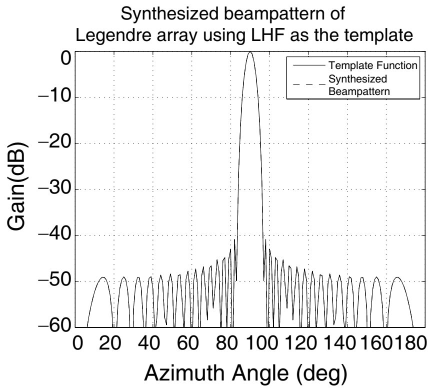

Figure 3. Synthesized beampattern of a 32-element Legendre array using LHF as the template.

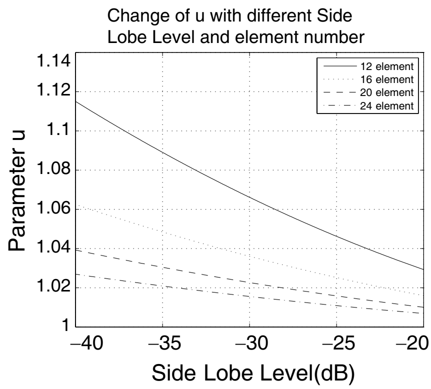

Figure 4. The values of u for different SLLs and element numbers.

Legendre arrays can provide higher beam efficiency (Appendix B) than Dolph-Chebyshev arrays. In Fig. 1, the directivity and beam efficiency of the Dolph-Chebyshev array is 13.95 dB and 99.89%, respectively, whereas in Fig. 3, the directivity and beam efficiency of the Legendre array is 13.74 dB and 99.97%. It is concluded that the choice of Legendre array and Dolph-Chebyshev array is a compromise between the beam efficiency and the directivity. However, for either Legendre arrays or Dolph-Chebyshev arrays, if the desired SLL is given, the beam efficiency and the directivity of the synthesized beampattern cannot be changed any more. To solve this problem, we can hire the Gegenbauer arrays to further adjust the beam efficiency or the directivity to meet the specified requirements.

Comparing (21) with (18) and (20), it is observed that when $\alpha = \frac{1}{2}$, the GHF is reduced to a LHF and when α is a small value, the GHF can be considered as an approximation of the DCHF. Therefore, the GHF is a very generalized function and the DCHF and the LHF are its special cases. Using the GHF as the template function can not only control the specifications of the synthesized beampattern better by changing the parameter α, but also take the advantages of the QP method, the visuality and simplicity of the operations, to make it more convenient to synthesize an array.

To illustrate how the parameter α effect the synthesized beampattern, the comparison of different α values is presented in Fig. 5, and the values of directivity and the beam efficiency are listed in Table 1. It is observed that when α increases, the beam efficiency of the Gegenbauer array increases while the directivity of Gegenbauer array decreases. Therefore, it is concluded that the GHF method is more generalized than the LHF method and the DCHF method and the QP method makes it easier to work. The curves representing the beam efficiency and the directivity of a 12-element Gegenbauer array with different values of α are shown in Fig. 6. From the figure, for the specified beam efficiency and directivity, an appropriate value of α

Table 1. Specifications of beampattern of 12-element Gegenbauer arrays with different α values.

α	Beam Efficiency	Directivity (dB)
0.05	99.97%	9.565
0.2	99.98%	9.526
0.5	99.98%	9.460
1.0	99.99%	9.374

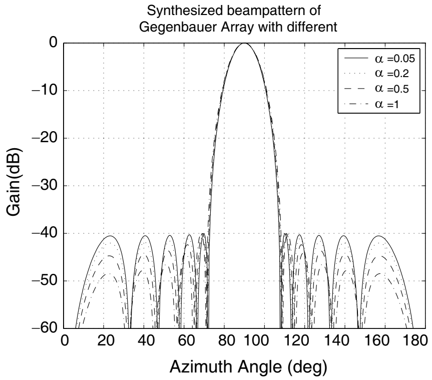

Figure 5. Synthesized beampatterns of 12-element Gegenbauer arrays with different α values.

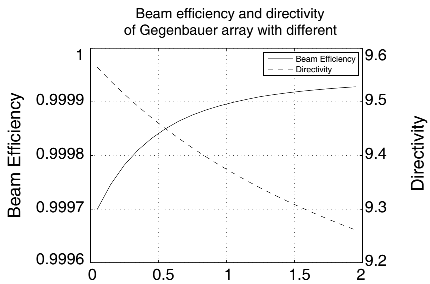

Figure 6. Beam Efficiency and Directivity of 12-element Gegenbauer arrays with different α values.

can be easily found. As a result, using GHF method makes it possible to obtain desired beam efficiency and directivity of the synthesized beampattern conveniently by adjusting the value of the parameter α without changing other parameters such as SLL and element number.

4. SYNTHESIS OF LEGENDRE ARRAY AND GEGENBAUER ARRAY FOR THE CONSTRAINED PROBLEM USING THE QP METHOD

As discussed above, the classical synthesis methods of Legendre array and Gegenbaure array fail in dealing with the constrained problems, while the QP method is flexible and convenient to solve them. Figs. 7–10 show the performance of the proposed method used with null control for Legendre arrays and Gegenbauer arrays, respetively. Note that there are several ways to impose null response on the side lobe [19, 20]. However, since the intention of the paper is not to address the problem of null constraints design but to illustrate the performance of the proposed method when it is used with null control in the pattern design, simple point constraints are used to fix a null at 50° and a broad null at $[55^\circ, 57^\circ, 59^\circ, 61^\circ]$. In these examples, a -40 dB SLL with an integration sector $[0^\circ, 180^\circ]$ is used. It can be seen from the examples that the performance of the proposed method for the constrained problems are good and the properties of both Legendre array and Gegenbauer array in the beampattern are still well preserved.

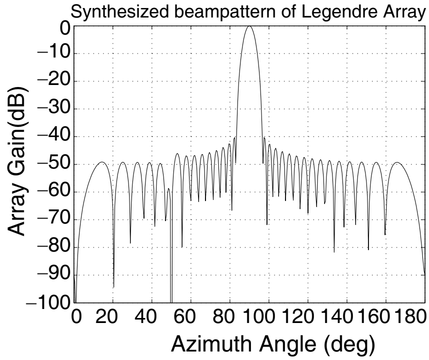

Figure 7. Synthesized beampatterns of 32-element Legendre array with null constraint imposed at 50°.

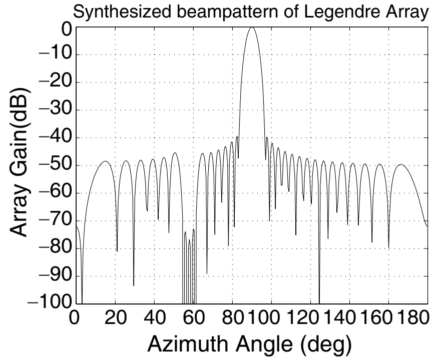

Figure 8. Synthesized beampatterns of 32-element Legendre array with null constraints imposed at 55°, 57°, 59°, 61°.

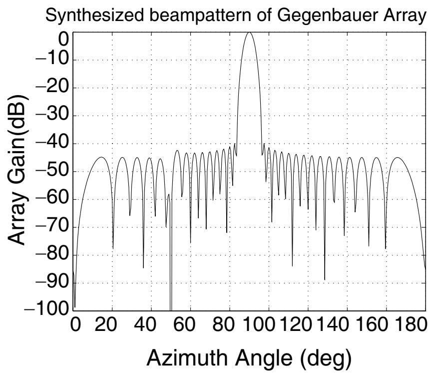

Figure 9. Synthesized beampatterns of 32-element Gegenbauer array with null constraint imposed at 50°.

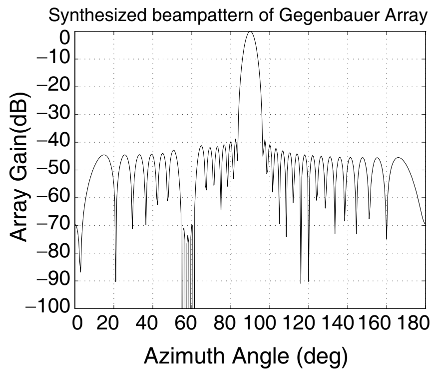

Figure 10. Synthesized beampatterns of 32-element Gegenbauer array with null constraints imposed at 55°, 57°, 59°, 61°.

5. CONCLUSION

In this paper, LHF and GHF are proposed as the template functions for the QP method and the corresponding synthesis procedures are also illustrated. The proposed generalized template form using Hypergeometric Function works for the Dolph-Chebyshe arrays as well. From the simulation results, it can be shown that when the proposed template functions are used in the QP method, the exactly synthesized beampatterns can be obtained and they can provide good performance in the design of the constrained problems for both Legendre arrays and Gegenbauer arrays. In addition, some discussion results about the application of Gegenbauer arrays in the QP method are presented.

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APPENDIX A. THE FORMULATION OF THE QP METHOD FOR ARRAY BEAMPATTERN SYNTHESIS

In array signal processing, one can define the array beampattern at a certain operating frequency f to a plane-wave front with unit amplitude arriving in direction of the azimuth angle θ as

$$H(\theta, \mathbf{w}) = \mathbf{w}^H \mathbf{a}(\theta), \tag{A1}$$

where the superscript H denotes Hermitian transpose, $\mathbf{w}$ is the L-dimensional complex vector of adjustable weights, and $\mathbf{a}(\theta)$ is the space vector given by

$$\mathbf{a}(\theta) = \left[\exp\left(j\mathbf{k}^T\mathbf{r}_1\right), \ldots, \exp\left(j\mathbf{k}^T\mathbf{r}_L\right)\right]^T \tag{A2}$$

where the wave-number vector

$$\mathbf{k}(\theta) = (2\pi f/\mathbf{v}) \cos\theta, \tag{A3}$$

and $\mathbf{r}_i$ is the ith sensor location vector.

Based on the classical mathematical model above, the traditional beamforming problem can be formulated as a quadratic program as

$$\begin{aligned} &\textbf{Minimize} \quad && \sum_k \left(\frac{1}{\Delta\theta_k}\right) \int_{\theta_{l_k}}^{\theta_{u_k}} |H(\theta, \mathbf{w}) - H_0(\theta)|^2 d\theta \\ &\textbf{Subject to} \quad && H(\theta_n, \mathbf{w}) = H_c(\theta_n), \end{aligned} \tag{A4}$$

where $\Delta\theta_i = \theta_{u_i} - \theta_{l_i}$, and the function $H_0(\theta)$ is the template function which describes the desired array pattern. Generally, for convenience, commonly we use one integration sector as $[0°, 180°]$. Using the matrix notations, (A4) can also be written as a quadratic program as

$$\textbf{Minimize} \quad \mathbf{w}^H\mathbf{Q}\mathbf{w} - \mathbf{w}^H\mathbf{P} - \mathbf{P}^H\mathbf{w} + c \tag{A5}$$

$$\textbf{Subject to} \quad \mathbf{w}^H\mathbf{D}(\theta_n) = \mathbf{F}^T, \tag{A6}$$

where

$$\begin{aligned}
\mathbf{Q} &= \sum_k \left(\frac{1}{\Delta\theta_k}\right) \int_{\theta_{l_k}}^{\theta_{u_k}} \mathbf{a}\mathbf{a}^H d\theta, \\
\mathbf{P} &= \sum_k \left(\frac{1}{\Delta\theta_k}\right) \int_{\theta_{l_k}}^{\theta_{u_k}} H_0(\theta)\mathbf{a} d\theta, \\
c &= \sum_k \left(\frac{1}{\Delta\theta_k}\right) \int_{\theta_{l_k}}^{\theta_{u_k}} H_0(\theta) * H_0(\theta)^\star d\theta,
\end{aligned} \tag{A7}$$

and the $\mathbf{D}$ and $\mathbf{F}$ are the matrix and vector used to achieve the certain constrained pattern $H_c(\theta_n)$. Using the Lagrange multiplier method, (A5) can be solved to yield the following solution weight vector:

$$\mathbf{w}_0 = \mathbf{Q}^{-1}\mathbf{D}\left(\mathbf{D}^H\mathbf{Q}^{-1}\mathbf{D}\right)^{-1}\left(\mathbf{F} - \mathbf{D}^H\mathbf{Q}^{-1}\mathbf{P}\right) + \mathbf{Q}^{-1}\mathbf{P}. \tag{A8}$$

If there is no constraint in the QP method, the solution in (A8) is reduced to:

$$\mathbf{w}_0 = \mathbf{Q}^{-1}\mathbf{P}. \tag{A9}$$

From (A4) to (A9), it is shown that in the QP method, once given a desired template function, all the coefficients can be calculated in one step without any extra complex mathematical work or iterative procedures.

To analyze the computational complexity of the QP method, firstly it can be observed that for the case of unconstraint synthesis problem, the final solution can be obtained by calculation the product of a inversion of a matrix and a vector. Assuming an N-element array, the matrix $\mathbf{Q}$ in (A9) is $N \times N$ and the vector $\mathbf{P}$ is $N \times 1$, which indicates its small computational load. Moreover, the work of calculating the integration in (A7) can be further reduced significantly since the problem is actually to solve a linear system problem, the exact integration is not strictly required. If the sum of the function values for the integrand were used to calculate the integration at the sample points with small inter-spacing, there would exist avoidable redundant information in the system. The practical simulations show that generally $N+1$ sample points in the integration interval is enough for the solution of the linear system, in other words, for the integration interval $[0^\circ, 180^\circ]$, the sampling spacing can be as large as $\frac{180^\circ}{N+1}$. In conclusion, the QP method has very low computational complexity and for a symmetric array, the computational load can be half as well.

APPENDIX B. THE DIRECTIVITY AND THE BEAM EFFICIENCY OF A BROADSIDE LINEAR ARRAY

Directivity is defined as the ratio of the maximum radiation intensity of the antenna to the radiation intensity of isotropic source given as [17]

$$Directivity = \frac{|H(\frac{\pi}{2})|^2}{\int_0^\pi |H(\theta)|^2 \sin\theta d\theta}, \tag{B1}$$

where $H(\theta)$ is the function of array beampattern.

Beam efficiency is defined as the ratio of the power transmitted within the main beam to the power transmitted by the antenna. For the broadside linear array, the beam efficiency can be formulated as [17]

$$BE = \frac{\int_{\theta_1}^{\frac{\pi}{2}} |H(\theta)|^2 \sin\theta d\theta}{\int_0^{\frac{\pi}{2}} |H(\theta)|^2 \sin\theta d\theta}, \tag{B2}$$

where θ_1 is the direction where the first null occurs.

REFERENCES

1. Dolph, C. L., "A current distribution for broadside arrays which optimizes the relationship between beamwidth and side-lobe level," *Proc. IRE*, Vol. 34, 335–348, Jun. 1946.
2. Qu, L., W. Ser, and Z. H. Shao, "An efficient adaptive beamforming technique with pattern control ability," *IEEE 64th Vehicular Technology Conference*, 479–483, 2006.
3. Qu, L., W. Ser, and Z. H. Shao, "Beam pattern synthesis in the presence of interference and multipath," *International Symposium on Wireless Communication Systems*, 267–271, 2006.
4. Keizer, W. P. M. N., "Fast low-sidelobe synthesis for large planar array antennas utilizing successive fast Fourier transforms of the array factor," *IEEE Trans. Antennas Propagat.*, Vol. 55, 715–722, 2007.
5. Wang, F., V. Balakrishnan, and P. Y. Zhou, "Optimal array pattern synthesis using semidefinite programming," *IEEE Trans. Antennas Propagat.*, Vol. 51, 1172–1183, 2003.
6. Corcoles, J., M. A. Gonzalez, and J. Rubio, "Mutual coupling compensation in arrays using a spherical wave expansion of the radiated field," *IEEE Trans. Antennas Propagat.*, Vol. 8, 108–111, 2009.
7. Corcoles, J., M. A. Gonzalez, and J. Rubio, "Array design for different SLL and null directions with an interior-point

optimization method from the generalized-scattering-matrix and spherical modes," *3rd European Conference on Antannas and Propagation*, 1319–1323, 2009.

8. Bucci, O. M., D. D'Elia, G. Mazzarella, and G. Panatiello, "Antenna pattern synthesis: A new general approach," *Proc. IEEE*, Vol. 82, 358–371, Mar. 1994.
9. Lebret, H. and S. Boyd, "Antenna array pattern synthesis via convex optimization," *IEEE Trans. on Signal Processing*, Vol. 45, No. 3, 526–532, Mar. 1997.
10. Bucci, O. M., M. D'Urso, and T. Isernia, "Exploiting convexity in array antenna synthesis problems," *Radar Conference 2008, Roma, Italy, Proceedings of the Conference*, May 26–30, 2008.
11. Rocca, P., L. Manica, R. Azaro, and A. Massa, "A Hybrid approach to the synthesis of subarrayed monopulse linear arrays," *IEEE Trans. Antennas Propagat.*, Vol. 57, No. 1, 280–283, Jan. 2009.
12. Ng, B. P., M. H. Er, and C. Kot, "A flexible array synthesis method using quadratic programming," *IEEE Trans. Antennas Propagat.*, Vol. 41, 1541–1550, Nov. 1993.
13. Phongcharoenpanich, C., M. Krairiksh, K. Meksamoot, and T. Wakabayashi, "Legendre array," *Proc. TJSAP*, 197–201, May 1997.
14. Goto, N., "A synthesis of array antennas for high directivity and low sidelobes," *IEEE Trans. Antennas Propagat.*, Vol. 20, No. 4, 427–431, Jul. 1972.
15. Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965.
16. Bailey, W. N., *Generalised Hypergeometric Series*, University Press, Cambridge, England, 1935.
17. Balanis, C. A., *Antenna Theory Analysis and Design*, John Wiley and Sons, New York, 1997.
18. Suetin, P. K., *Encyclopaedia of Mathematics*, Springer, 2001.
19. Er, M. H. and A. Cantoni, "A new set of linear constraints for broadband time domain element space processors," *IEEE Trans. Antennas Propagat.*, Vol. 34, No. 3, 320–329, Mar. 1986.
20. Ng, B. P., "Generalised approach to the design of narrowband antenna array processors with broadband capability," *IEE Proc. F*, Vol. 137, No. 1, 41–47, Feb. 1990.