Progress In Electromagnetics Research M, Vol. 16, 171–184, 2011

ACCURACY OF APPROXIMATE FORMULAS FOR INTERNAL IMPEDANCE OF TUBULAR CYLINDRICAL CONDUCTORS FOR LARGE PARAMETERS

D. Lovrić

Faculty of Electrical Engineering, Mechanical Engineering
and Naval Architecture
University of Split
Ruđera Boškovića 32, HR-21000 Split, Croatia

V. Boras

Faculty of Natural Sciences
University of Split
Teslina 12, HR-21000 Split, Croatia

S. Vujević

Faculty of Electrical Engineering, Mechanical Engineering
and Naval Architecture
University of Split
Ruđera Boškovića 32, HR-21000 Split, Croatia

Abstract—Exact formulas for internal impedance per unit length of tubular cylindrical conductors energized by time-harmonic current involve Bessel functions. These functions are defined by infinite series, which yield unstable and often erroneous results for complex arguments of large magnitudes. Although it is well known how to evaluate Bessel functions numerically and many routines are now available to perform the actual computation, the available software routines often fail when computing equations that consist of a product and a quotient of Bessel functions under large complex or real arguments. For such cases, different approximate formulas can be used. In this paper, three types of approximate formulas for internal impedance of tubular cylindrical conductors are compared with respect to numerical stability and accuracy.

Received 15 December 2010, Accepted 10 January 2011, Scheduled 19 January 2011
Corresponding author: Dino Lovrić (dlovric@fesb.hr).

1. INTRODUCTION

In numerical analysis of various electromagnetic problems, computation formulas are often based on Bessel functions. Typical example of such a problem is the computation of internal impedance per unit length of tubular cylindrical conductors, which are energized by time-harmonic current. This impedance, sometimes called surface impedance [1], needs to be computed in the numerical analysis of various electromagnetic problems such as time-harmonic and transient grounding grid analysis [2], time-harmonic and transient analysis of electric power lines [3], electrical interference from electric power lines to gas pipelines [4] and analysis of wire antennas [5].

The exact formulas for internal impedance per unit length of solid and tubular cylindrical conductors, which take the skin effect into account but ignore the proximity effect, can be expressed by Bessel functions, by modified Bessel functions, by a combination of Bessel and modified Bessel functions or by Kelvin functions [6–11]. Kelvin functions represent the real and imaginary parts of modified Bessel functions. All of these functions are defined by infinite series, which are rapidly convergent only for low function arguments. Therefore, in case of low frequency values, magnitudes of Bessel function arguments are low and the exact formulas for internal impedance can be successfully employed. However, if the magnitudes of Bessel functions arguments are large, convergence difficulties appear. In such cases, which generally occur at high frequencies, exact formulas for the computation of internal impedance with infinite Bessel series yield unstable and often erroneous results. Excluding high frequency values, other parameters that can lead to large magnitudes of Bessel functions arguments are high permeability and large dimensions of the conductor.

The problem of Bessel functions computation, especially for large parameters, has engaged the attention of a number of authors in recent years, and consequently, a variety of methods is now available. Although it is well known how to evaluate Bessel functions numerically and many routines are now available to perform the actual computation [12, 13], the available software routines fail when computing equations that consist of a product and a quotient of Bessel functions under large complex or real arguments. An additional numerical problem is numerical overflow, which appears when the large arguments are complex and approximate formulas include sine and cosine functions.

The internal impedance of solid conductors for large parameters can be computed by a number of approximate formulas that can be categorized as low accuracy formulas [14–16] and high accuracy formulas [17–19].

On the other hand, the expression for internal impedance of tubular cylindrical conductors is more complicated to approximate accurately for large parameters. In this paper, three approximate formulas for internal impedance of tubular cylindrical conductors will be compared: Vujević et al. formula [17] which is based on Hankel asymptotic approximations of complex-valued Bessel functions, Mingli and Yu formula [18] which is based on polynomial approximation of complex-valued Bessel functions and formula based on Amos subroutines that approximate Bessel functions [12, 13].

2. EXACT FORMULAS FOR INTERNAL IMPEDANCE OF TUBULAR CYLINDRICAL CONDUCTORS

Exact formulas for internal impedance per unit length of tubular cylindrical conductors, which take the skin effect into account but ignore the proximity effect, can be expressed by Bessel functions of the first and second kind, by modified Bessel functions of the first and second kind, by a combination of Bessel and modified Bessel functions or by Kelvin functions of the first and second kind. Schelkunoff first developed the exact formula for internal impedance of tubular conductors in 1934 [20]. The formula was expressed by modified Bessel functions of the first and second kind:

$$\bar{Z} = \frac{\bar{k}}{2\pi\sigma r_e} \frac{\bar{I}_0(\bar{k}_e)\bar{K}_1(\bar{k}_i) + \bar{I}_1(\bar{k}_i)\bar{K}_0(\bar{k}_e)}{\bar{I}_1(\bar{k}_i)\bar{K}_1(\bar{k}_e) - \bar{I}_1(\bar{k}_e)\bar{K}_1(\bar{k}_i)} \tag{1}$$

where $\bar{I}_0$ and $\bar{I}_1$ are the complex-valued modified Bessel functions of the first kind of order zero and order one respectively [7–9], $\bar{K}_0$ and $\bar{K}_1$ are the complex-valued modified Bessel function of the second kind of order zero and order one respectively, σ is the electrical conductivity of the conductor material, r_e is the external radius of the conductor and $\bar{k}$ is the complex wave number defined by the following expression:

$$\bar{k} = ke^{-j\frac{\pi}{4}} = \sqrt{\omega\mu\sigma}e^{-j\frac{\pi}{4}} \tag{2}$$

with μ being the permeability of the conductor material, j the imaginary unit, $\omega = 2\pi f$ the circular frequency and f the time-harmonic current frequency. In (1), $\bar{k}_e = \bar{k}r_e$ and $\bar{k}_i = \bar{k}r_i$ where r_i represents the internal radius of the conductor.

The exact formula (1) can be rewritten using Bessel functions of the first and second kind [17]:

$$\bar{Z} = \frac{\bar{k}}{2\pi\sigma r_e} \frac{\bar{J}_1(\bar{k}_i)\bar{Y}_0(\bar{k}_e) - \bar{Y}_1(\bar{k}_i)\bar{J}_0(\bar{k}_e)}{\bar{J}_1(\bar{k}_i)\bar{Y}_1(\bar{k}_e) - \bar{Y}_1(\bar{k}_i)\bar{J}_1(\bar{k}_e)} \tag{3}$$

where $\bar{J}_0$ and $\bar{J}_1$ are the complex-valued Bessel function of the first kind of order zero and order one respectively [7–9] whereas $\bar{Y}_0$ and $\bar{Y}_1$ are the complex-valued Bessel function of the second kind of order zero and order one respectively.

Another widely used exact formula for internal impedance of tubular cylindrical conductors is based on Kelvin functions of the first and second kind [18].

3. APPROXIMATE FORMULAS FOR INTERNAL IMPEDANCE OF TUBULAR CYLINDRICAL CONDUCTORS

3.1. Approximate Formula Proposed by Vujević et al.

In paper [17], a highly accurate formula for computation of internal impedance per unit length of tubular cylindrical conductors was developed. If the magnitudes of the argument $k_e < 8$, then the internal impedance of tubular cylindrical conductors is computed using the exact formula given by (3) defined by infinite Bessel series. In this case, infinite Bessel series are rapidly convergent and internal impedance is computed with high accuracy if the infinite series are approximated with 17 terms. For arguments of large magnitudes $k_e \geq 8$ and $k_e - k_i \geq 5$, the internal impedance of tubular cylindrical conductors can be accurately computed using the following equation:

$$\bar{Z} = \frac{\bar{k}}{2\pi\sigma r_e} \frac{\bar{P}_0\left(\bar{k}_e\right) + j\bar{Q}_0\left(\bar{k}_e\right)}{\bar{Q}_1\left(\bar{k}_e\right) - j\bar{P}_1\left(\bar{k}_e\right)} \tag{4}$$

where the functions $\bar{P}_0\left(\bar{k}_e\right)$, $\bar{P}_1\left(\bar{k}_e\right)$, $\bar{Q}_0\left(\bar{k}_e\right)$ and $\bar{Q}_1\left(\bar{k}_e\right)$ are given in Appendix A.

In this case, the skin depth is relatively small, which makes the internal impedance of a tubular conductor approximately equal to the internal impedance of a solid cylindrical conductor [17].

In the remaining case when $k_e \geq 8$ and $k_e - k_i < 5$, due to numerical reasons, novel approximations were introduced, which have lead to the next approximate formula:

$$\bar{Z} = \frac{j\bar{k}}{2\pi\sigma r_e} \frac{\cos a + j\sin a + e^{-a}}{\cos a + j\sin a - e^{-a}} + \bar{F}\left(\bar{k}_e\right) \tag{5}$$

where:

$$a = \sqrt{2}\left(k_e - k_i\right) \tag{6}$$

The function $\bar{F}\left(\bar{k}_e\right)$ is a linear correction function defined by:

$$\bar{F}\left(\bar{k}_e\right) = \bar{c}_1 + \frac{k_e - k_{e1}}{k_{e2} - k_{e1}}\left(\bar{c}_2 - \bar{c}_1\right) \tag{7}$$

with:

$$\bar{k}_{e1} = \bar{k}_1 \cdot r_e = (1-j)\frac{8}{\sqrt{2}}; \quad \bar{k}_{e2} = \bar{k}_2 \cdot r_e = (1-j)\frac{5r_e}{\sqrt{2}(r_e - r_i)} \quad (8)$$

$$\bar{c}_1 = \bar{Z}_A(\bar{k}_{e1}) - \bar{Z}_B(\bar{k}_{e1}); \quad \bar{c}_2 = \bar{Z}_C(\bar{k}_{e2}) - \bar{Z}_B(\bar{k}_{e2}) \quad (9)$$

The function $\bar{Z}_A(\bar{k}_{e1})$ represents the internal impedance per unit length of a tubular cylindrical conductor computed using the exact formula (3) and infinite Bessel series; in this case the magnitude of the argument $k_{e1} = 8$. The function $\bar{Z}_C(\bar{k}_{e2})$ represents the internal impedance per unit length of a tubular cylindrical conductor computed using the approximate formula (4); in this case the magnitude of the argument $k_2 \cdot (r_e - r_i) = 5$.

Finally, the internal impedances per unit length of a tubular cylindrical conductor $\bar{Z}_B(\bar{k}_{e1})$ and $\bar{Z}_B(\bar{k}_{e2})$ are computed using next approximate formula:

$$\bar{Z}_B = \frac{j\bar{k}}{2\pi\sigma r_e}\frac{\cos a + j\sin a + e^{-a}}{\cos a + j\sin a - e^{-a}} \quad (10)$$

where a is defined by (6).

3.2. Approximate Formula Proposed by Mingli and Yu

In paper [18], using the polynomial approximations of complex-valued Bessel functions, the following approximate formula for the computation of internal impedance of tubular cylindrical conductors under large complex parameters was developed:

$$\bar{Z} = \frac{jk}{2\pi r_e \sigma}\frac{1 + e^{\bar{A}}\frac{\bar{\Phi}(k_i)}{\bar{\Phi}(-k_i)}}{\bar{\Phi}(k_e) - e^{\bar{A}}\frac{\bar{\Phi}(k_i)}{\bar{\Phi}(-k_i)}\bar{\Phi}(-k_e)} \quad \text{if } \mathrm{k_i} \geq 8 \quad (11)$$

where the functions $\bar{\Phi}$ and $\bar{A}$ are given in Appendix B.

If the magnitudes of the argument $k_i < 8$, then the internal impedance of tubular cylindrical conductors is computed using the exact formula given by (3) and infinite Bessel series.

3.3. Formula that Utilizes the Bessel Functions Approximation Based on Amos Subroutines

The previous two approximate formulas evaluated the entire expression for the internal impedance of tubular conductors to obtain a solution for large parameters. On the other hand, Amos subroutines approximate Bessel functions themselves [12, 13]. This approximation of Bessel functions for large arguments is also employed in the program package MATLAB.

Although the Amos subroutines yield stable and accurate results when computing only Bessel functions, they fail when computing the formula for internal impedance that consists of a product and a quotient of Bessel functions under large complex or real arguments.

4. NUMERICAL COMPARISON OF FORMULAS

In this section, numerical comparison of presented formulas will be conducted. Skin effect internal impedance ratios will be computed according to the next equation:

$$\bar{f}_z = f_z e^{j\varphi_z} = \bar{f}_z\left(\bar{k}_e, s\right) = \frac{\bar{Z}}{R_{DC}} \tag{12}$$

where R_{DC} is the per unit length direct current internal resistance of a cylindrical conductor and $s = r_i/r_e$ represents the radius ratio. In a general case, the skin effect internal impedance ratio $\bar{f}_z$ can be presented as a function of two arguments: $\bar{k}_e$ and the radius ratio s, where $s = 0$ for a solid conductor. The argument k_e is connected with the conductor skin depth δ [1] by the following equation:

$$k_e = \sqrt{2}\,\frac{r_e}{\delta} \tag{13}$$

The magnitudes and phase angles of the skin effect internal impedance ratios will be computed using the approximate formulas based on Amos subroutines, by Mingli and Yu and by Vujević et al. and then compared to the truncated exact formula where the Bessel infinite series will be approximated with 100 terms. The truncated exact formula based on infinite Bessel series with 100 terms yields erroneous results for large parameters. Increasing the number of infinite Bessel series terms cannot improve numerical accuracy, but can, in fact, cause numerical overflow. The approximate formulas were compared for multiple values of the radius ratio s. For this paper, three typical values of s were chosen to demonstrate the behavior of different approximate formulas: $s \in \{0.1,\ 0.4,\ 0.95\}$.

4.1. Comparison of Formulas for $s = 0.1$

The magnitudes and phase angles of the skin effect internal impedance ratios of tubular cylindrical conductors with a radius ratio $s = 0.1$ are computed using the presented approximate formulas and the truncated exact formula where the Bessel infinite series are approximated with 100 terms. The results are plotted relative to the argument k_e (Figure 1).

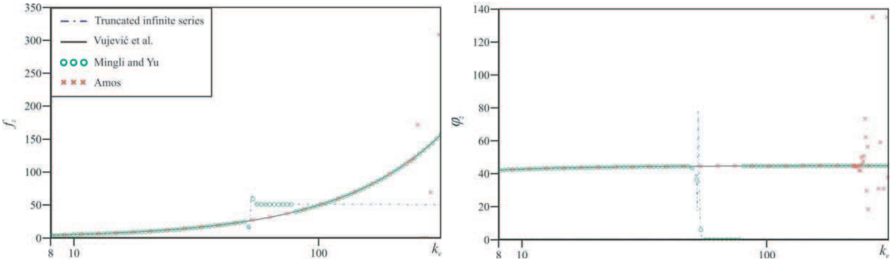

Figure 1. Comparison of skin effect internal impedance ratios for tubular cylindrical conductors with a radius ratio $s = 0.1$.

Observing Figure 1, one can conclude that the truncated exact formula exerts significant numerical instability and gives erroneous results at function arguments as low as $k_e = 45$. The approximate formula proposed by Mingli and Yu displays similar instabilities as the truncated exact formula, but results eventually restabilize for extra large arguments. The formula based on Amos subroutines which are also used in MATLAB becomes unstable at $k_e = 225$. Only the Vujević et al. formula is stable throughout the entire interval of observation.

In the interval of observation where the truncated exact formula is stable and accurate, all approximate formulas yield highly accurate results with a maximum percent error of 0.005% for the magnitude of the skin effect internal impedance ratios (Vujević et al. formula).

4.2. Comparison of Formulas for $s = 0.4$

The magnitudes and phase angles of the skin effect internal impedance ratios of tubular cylindrical conductors with a radius ratio $s = 0.4$ are computed using the presented approximate formulas and the truncated exact formula where the Bessel infinite series are approximated with 100 terms. The results are plotted relative to the argument k_e (Figure 2).

Again, the truncated exact formula exerts significant numerical instability and gives erroneous results at function arguments as low as $k_e = 32$. The approximate formula employing Amos subroutines becomes unstable at $k_e = 60$. The approximate formulas proposed by Mingli and Yu and by Vujević et al. yield stable results throughout the entire interval of observation.

In the interval of observation where the truncated exact formula is stable and accurate, all approximate formulas yield highly accurate results with a maximum percent error of 0.125% for the phase angle of the skin effect internal impedance ratios (Vujević et al. formula).

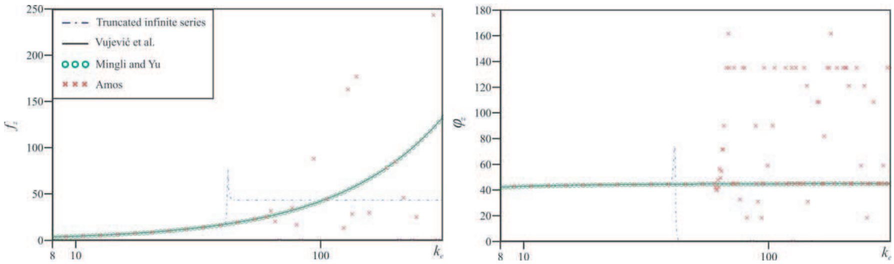

Figure 2. Comparison of skin effect internal impedance ratios for tubular cylindrical conductors with a radius ratio $s = 0.4$.

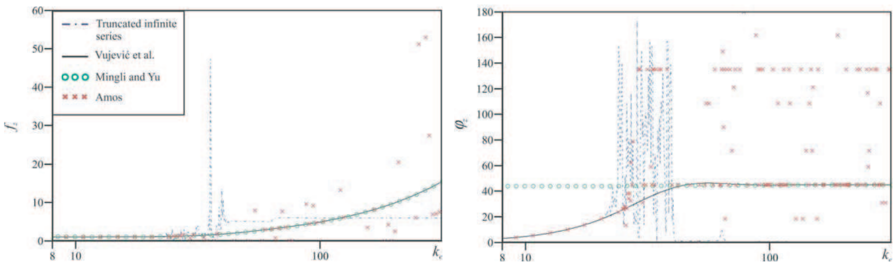

Figure 3. Comparison of skin effect internal impedance ratios for tubular cylindrical conductors with a radius ratio $s = 0.95$.

4.3. Comparison of Formulas for $s = 0.95$

The magnitudes and phase angles of the skin effect internal impedance ratios of tubular cylindrical conductors with a radius ratio $s = 0.95$ are computed using the presented approximate formulas and the truncated exact formula where the Bessel infinite series are approximated with 100 terms. The results are plotted relative to the argument k_e (Figure 3).

The truncated exact formula exerts significant numerical instability and gives erroneous results at function arguments as low as $k_e = 20$. The approximate formula employing Amos subroutines becomes unstable at $k_e = 25$. The approximate formulas proposed by Mingli and Yu and Vujević et al. yield stable results throughout the entire interval of observation, but the inaccuracy of the Mingli and Yu formula is significant for this s (Figure 3).

To better analyze the accuracy of the presented formulas, the percent errors of the formula based on Amos subroutines, Vujević et al. formula and the Mingli and Yu formula relative to the truncated exact formula are presented in Figure 4. The interval of observation extends

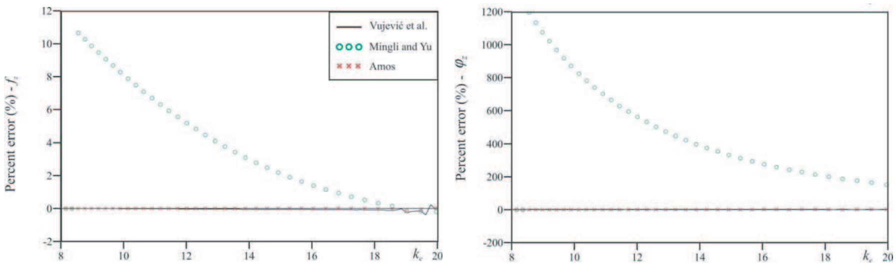

Figure 4. Percent errors of the skin effect internal impedance ratios of approximate formulas relative to the truncated exact formula for tubular cylindrical conductors with a radius ratio $s = 0.95$.

to the limits where the truncated exact formula starts to display instabilities. For this s, the Mingli and Yu formula displays the highest percent error of all approximate formulas both for the magnitude of the skin effect internal impedance ratios (11%) and for the phase angle of the skin effect internal impedance ratios (1200%). On the other hand, the results computed by the formula proposed by Vujević et al. and formula based on Amos subroutines are highly accurate in this interval although the formula based on Amos subroutines becomes unstable shortly after the truncated exact formula.

4.4. Overview of the Numerical Stability and Accuracy of the Presented Formulas

Evidently, the presented formulas yield results of varying accuracy and numerical stability depending on the k_e and s. A concise summary of the accuracy and stability features of presented formulas is given in Table 1. Naturally, the accuracy of the approximate formula is observed in the interval where the truncated exact formula yields numerically stable results.

The truncated exact formula is numerically stable up to a certain value of k_e depending on the radius ratio s. As the conductor becomes thinner (s increases), the results become numerically unstable at a smaller value of k_e (Figure 1, Figure 2 and Figure 3).

The Mingli and Yu formula is numerically stable for all k_e if the radius ratio $s > 0.1$. In the case of thick conductors ($s \leq 0.1$), it displays instabilities similar to those of the truncated exact formula (Figure 1). Furthermore, although it yields highly accurate results for most values of s, it yields highly inaccurate results for thin conductors ($s \geq 0.9$) as shown in Figure 3 and Figure 4.

The formula employing Amos subroutines, which are used in the program package MATLAB, displays similar behavior as the truncated exact formula. It becomes numerically unstable for every s but at a greater value of k_e than for the truncated exact formula. On the other hand, it yields the most accurate results in the interval where it is stable, but it does not solve the basic problem of stability under large parameters.

The Vujević et al. formula yields numerically stable and highly accurate results for all values of s and k_e, which makes it the optimal all-around method. Due to the introduction of a linear correction function, this approximate formula yields highly accurate results especially for extra-large parameters, even for extremely thin conductors.

Table 1. Overview of the numerical stability and accuracy of the presented formulas.

Formula	**Numerical stability**	**Accuracy in stable interval**
Truncated exact	stable up to a certain k_e depending on s	-
Mingli and Yu	unstable for $s \leq 0.1$; stable otherwise	high for small s; low for large s
Amos (MATLAB)	stable up to a certain k_e depending on s	High
Vujević et al.	stable for all k_e and s	High

5. CONCLUSION

The computation of internal impedance per unit length of solid and tubular cylindrical conductors energized by time-harmonic current, which takes the skin effect into account, requires the use of Bessel functions of the first and second kind. Due to the inherent instabilities of these functions under large parameters, approximation formulas are developed to circumvent this problem.

In this paper, three approximate formulas for the computation of internal impedance of tubular cylindrical conductors are presented and compared. The approximate formula proposed by Mingli and Yu are based on polynomial approximation of complex-valued Bessel functions, whereas the formulas proposed by Vujević et al. are based on

Hankel asymptotic approximations of complex-valued Bessel functions. The third formula utilizes Amos subroutines for approximation of the Bessel functions themselves, which are used in the program package MATLAB.

Approximate formulas are compared to the truncated exact formula for different types of cylindrical conductors. The exact formula, the Mingli and Yu formula and the formula based on Amos subroutines all have stability issues whereas the Vujević et al. formula is numerically stable in all cases. Furthermore, the Vujević et al. formula also displays the most accurate results for thin conductors while the Mingli and Yu formula is highly inaccurate for these cases (percent error of the phase angle of the skin effect internal impedance ratios is higher than 1000%).

APPENDIX A.

The functions $\bar{P}_\nu\left(\bar{k}_e\right)$ and $\bar{Q}_\nu\left(\bar{k}_e\right)$; where $\nu = 1,\ 2$; are computed using the following expressions [17]:

$$\bar{P}_\nu\left(\bar{k}_e\right) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{m=1}^{2n}\left[4\nu^2 - (2m-1)^2\right]}{(2n)!\left(8\bar{k}_e\right)^{2n}} \tag{A1}$$

$$\bar{Q}_\nu\left(\bar{k}_e\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\prod_{m=0}^{2n}\left[4\nu^2 - (2m+1)^2\right]}{(2n+1)!\left(8\bar{k}_e\right)^{2n+1}} \tag{A2}$$

On a basis of large number of numerical tests, it has been determined that for large magnitudes of the argument $k_e \geq 8$ infinite sums in (A1) and (A2) can be accurately approximated by following expressions:

$$\bar{P}_0(\bar{k}_e) = 1 - \frac{0.0703125}{\bar{k}_e^2} + \frac{0.1121521}{\bar{k}_e^4} - \frac{0.572501421}{\bar{k}_e^6} \tag{A3}$$

$$\bar{P}_1(\bar{k}_e) = 1 + \frac{0.1171875}{\bar{k}_e^2} - \frac{0.144195557}{\bar{k}_e^4} + \frac{0.676592588}{\bar{k}_e^6} \tag{A4}$$

$$\bar{Q}_0(\bar{k}_e) = -\frac{0.125}{\bar{k}_e} + \frac{0.0732421875}{\bar{k}_e^3} - \frac{0.227108002}{\bar{k}_e^5} + \frac{1.72772750258}{\bar{k}_e^7} \tag{A5}$$

$$\bar{Q}_1(\bar{k}_e) = \frac{0.375}{\bar{k}_e} - \frac{0.1025390625}{\bar{k}_e^3} + \frac{0.277576447}{\bar{k}_e^5} - \frac{1.99353173375}{\bar{k}_e^7} \tag{A6}$$

APPENDIX B.

The function $\bar{\Phi}$ used in (11) is described by the following expression:

$$\begin{aligned}\bar{\Phi}(x) = &(0.7071068+\mathrm{j}0.7071068)+(-0.0625001-\mathrm{j}0.0000001)(8/x)\\&+(-0.0013813+\mathrm{j}0.0013811)(8/x)^2\\&+(0.0000005+\mathrm{j}0.0002452)(8/x)^3\\&+(0.0000346+\mathrm{j}0.0000338)(8/x)^4\\&+(0.0000117-\mathrm{j}0.0000024)(8/x)^5\\&+(0.0000016-\mathrm{j}0.0000032)(8/x)^6\end{aligned} \tag{B1}$$

The function $\bar{A}$, also used in (11), is given by:

$$\bar{A} = -\sqrt{2}\,(1+j)\,(k_e-k_i) - \bar{\Theta}\,(k_e) + \bar{\Theta}\,(-k_e) + \bar{\Theta}\,(k_i) - \bar{\Theta}\,(-k_i) \tag{B2}$$

with:

$$\begin{aligned}\bar{\Theta}(x) = &-\mathrm{j}0.3926991+(0.0110486-\mathrm{j}0.0110485)(8/x)-\mathrm{j}0.0009765(8/x)^2\\&+(-0.0000906-\mathrm{j}0.0000901)(8/x)^3\\&+(-0.0000252+\mathrm{j}0.0000000)(8/x)^4\\&+(-0.0000034+\mathrm{j}0.0000051)(8/x)^5\\&+(0.0000006+\mathrm{j}0.0000019)(8/x)^6\end{aligned} \tag{B3}$$

REFERENCES

1. Stratton, J. A., *Electromagnetic Theory*, 532–533, IEEE Press Series on Electromagnetic Wave Theory, Wiley-IEEE Press, 2007.
2. Sarajčev, P. and S. Vujević, "Grounding grid analysis: Historical background and classification of methods," *International Review of Electrical Engineering (IREE)*, Vol. 4, No. 4, 670–683, 2009.
3. Dommel, H. W., *EMTP Theory Book*, 2nd edition, Microtran Power System Analysis Corporation, Vancouver, 1992.
4. Dawalibi, F. P. and R. D. Southey, "Analysis of electrical interference from power lines to gas pipelines — Part I: Computation methods," *IEEE Transactions on Power Delivery*, Vol. 4, No. 3, 1840–1846, 1989.
5. Moore, J. and R. Pizer (eds.), *Moment Methods in Electromagnetics — Techniques and Applications*, John Wiley & Sons, New York, 1984.
6. Stevenson, W. D., *Elements of Power System Analysis*, 2nd edition, 76–93, McGraw-Hill, New York, 1962.

7. Spiegel, M. R. and J. Liu, *Mathematical Handbook of Formulas and Tables*, 2nd Edition, Schaum's Outlines Series, 150–159, McGraw-Hill, New York, 1999.
8. Jeffrey, A. and H.-H. Dai, *Handbook of Mathematical Formulas and Integrals*, 4th edition, 289–299, Elsevier, Amsterdam, 2008.
9. Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematical Series 55', 358–385, National Bureau of Standards, 1964.
10. Paul, C. R., *Analysis of Multiconductor Transmission Lines*, 164–167, John Wiley & Sons, New York, 1994.
11. Wang, Y. J. and S. J. Liu, "A review of methods for calculation of frequency-dependent impedance of overhead power transmission lines," *Proc. Natl. Sci. Counc. ROC(A),* Vol. 25, No. 6, 329–338, 2001.
12. Amos, D. E., "A subroutine package for Bessel functions of a complex argument and nonnegative order," SAND85-1018, Sandia National Laboratories, Albuquerque, NM., 1985.
13. Amos, D. E., "Algorithm 644: A portable package for Bessel functions of a complex argument and nonnegative order," *ACM Transactions on Mathematical Software*, Vol. 12, No. 3, 265–273, 1986.
14. Nahman, N. S. and D. R. Holt, "Transient analysis of coaxial cables using the skin effect approximation $A + B\sqrt{s}$," *IEEE Transactions on Circuit Theory*, Vol. 19, No. 5, 443–451, 1972.
15. Semlyen, A. and A. Deri, "Time domain modeling of frequency dependent three phase transmission line impedance," *IEEE Transactions on Power Apparatus and Systems*, Vol. 104, No. 6, 1549–1555, 1985.
16. Wedepohl, L. M. and D. J. Wilcox, "Transient analysis of underground power transmission systems: System-model and wave propagation characteristics," *IEE Proceedings on Generation, Transmission and Distribution*, Vol. 20, No. 2, 253–260, 1973.
17. Vujević S., V. Boras, and P. Sarajčev, "A novel algorithm for internal impedance computation of solid and tubular cylindrical conductors," *International Review of Electrical Engineering (IREE),* Vol. 4, No. 6, Part B, 1418–1425, 2009.
18. Mingli, W. and F. Yu, "Numerical calculations of internal impedance of solid and tubular cylindrical conductors under large parameters," *IEE Proceedings — Generation, Transmission and*

Distribution, Vol. 151, No. 1, 67–72, 2004.

19. Knight, D. W., "Practical continuous functions and formulae for the internal impedance of cylindrical conductors," March 2010, http://www.g3ynh.info/zdocs/comps/Zint.pdf.
20. Schelkunoff, S. A., "The electromagnetic theory of coaxial transmission lines and cylindrical shields," *Bell System Technical Journal*, 532–578, 1934.