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DIFFERENTIAL ELECTROMAGNETIC EQUATIONS IN FRACTIONAL SPACE

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Abstract—The present study deals with a novel approach for fractional space generalization of the differential electromagnetic equations. These equations can describe the behavior of electric and magnetic fields in any fractal media. A new form of vector differential operator Del, and its related differential operators, is formulated in fractional space. Using these modified vector differential operators, the classical Maxwell equations have been worked out for fractal media. The Laplace, Poisson and Helmholtz equations in fractional space are derived by using modified vector differential operators. Also a new fractional space generalization of the potentials for static and time varying fields is presented.

1. INTRODUCTION

There has been much interest to study different physical phenomenon in fractional dimensional space [1–14] during the last few decades. The concept of fractional space is used to replace the real anisotropic confining structure with an isotropic fractional space, where the measurement of this confinement is given by fractional dimension [1, 2]. It is also important to mention that the experimental measurement of the dimension of real world is 3 ± 10^{-6}, not exactly 3 [1, 4].

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Among several methods, a methodology to describe the fractional dimension is fractional calculus [15], which is also used by different authors [16–23] in studying various electromagnetic problems. Axiomatic basis for spaces with fractional dimension have been provided by Stillinger [1], along with a fractional space generalization of Laplacian operator and a solution of Schrödinger wave equation in fractional dimensional space. Equations of motion in a non-integer dimensional space have been formulated in [3]. Recently Muslih [24] provided a dimensional regularization technique in order to convert any integral of a function from fractional dimensional space to a regular dimensional space along with a description of differential geometry of fractional dimensional space. The electromagnetic field on fractals was studied in [9]. The radiation phenomenon of fractal geometries have also been studied by different authors recently [25–31].

The generalization of electromagnetic theory in fractional space is of much importance to study the phenomenon of wave propagation and scattering in an anisotropic fractal media [14]. Fractal models of media are becoming popular due to relatively small number of parameters that define a medium of greater complexity and rich structure [9]. In general, the fractal media cannot be considered as continuous media, because some of points and domains are not filled by the medium particles. These unfilled domains are called porous [10]. The fractal media can be treated as continuous media for the scales much larger than average pore size. In order to describe the fractal media, the continuous medium model for fractal media reported in [10], suggests to use the space with fractional dimension. An introductory work on fractional multipoles and electromagnetic field in fractional space is reported in [11–13].

In this work we provide a novel generalization of differential electromagnetic equations in fractional space. Firstly, basic vector differential operators are generalized in fractional space and then using these fractional operators Maxwell, Laplace, Poisson and Helmholtz equations have been worked out in fractional space. The differential electromagnetic equations in fractional space, established in this work, provide a basis for application of the concept of fractional space in practical electromagnetic wave propagation and scattering problems in fractal media.

In Section 2 a review of already existing study to construct a generalized Laplacian operator using integration in $\mathfrak{D}$-dimensional fractional space is briefly described. In Section 3, fractional space generalization of the Del operator, written as $\nabla_{\mathfrak{D}}$, and its related differential operators (i.e., gradient, divergence and curl) in vector calculus is obtained. In Section 4, a novel fractional space generalization of differential Maxwell equations is presented. In Section 5, fractional space generalization of the Laplace and Poisson equations is established in addition to fractional space generalization of potentials for static field. In Section 6, potentials for time varying fields in fractional space are derived. In Section 7, the Helmholtz equation in fractional space is established. Finally, conclusions are drawn in Section 8.

2. FRACTIONAL SPACE GENERALIZATION OF LAPLACIAN OPERATOR

In [1] a formalism is provided for integration on $\mathfrak{D}$-dimensional fractional space. According to this formalism, the integration of radially symmetric function $f(r)$ in a $\mathfrak{D}$-dimensional fractional space is given by [1]:

$$\int dx_0 f(r(\mathbf{x_0}, \mathbf{x_1})) = \int_0^\infty dr W(r) f(r) \tag{1}$$

where $r(\mathbf{x_0}, \mathbf{x_1})$ is the distance between two points $\mathbf{x_0}$ and $\mathbf{x_1}$, and weight $W(r)$ given by

$$W(r) = \sigma(\mathfrak{D}) r^{\mathfrak{D}-1} \tag{2}$$

with

$$\sigma(\mathfrak{D}) = \frac{2\pi^{\mathfrak{D}/2}}{\Gamma(\mathfrak{D}/2)} \tag{3}$$

From this a single variable Laplacian operator is derived in $\mathfrak{D}$-dimensional fractional space as:

$$\nabla^2_{\mathfrak{D}} f(r) = \left[\frac{\partial^2}{\partial r^2} + \frac{\mathfrak{D}-1}{r}\frac{\partial}{\partial r}\right] f(r), \quad 0 < \mathfrak{D} \leq 1 \tag{4}$$

In Equation (4) and throughout the discussion, the subscript $\mathfrak{D}$ is used to emphasize the dimension of space in which this operator is defined. An extension of formalism in Equation (1) to two variable integration yields an expression for a two-coordinate Laplacian operator in fractional space.

$$\nabla^2_{\mathfrak{D}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\mathfrak{D}-2}{y}\frac{\partial}{\partial y}, \quad 0 < \mathfrak{D} \leq 2 \tag{5}$$

In [3] the results from [1] are generalized to n orthogonal coordinates and Laplacian operator in $\mathfrak{D}$-dimensional fractional space in three-spatial coordinates is given as:

$$\nabla^2_{\mathfrak{D}} = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z} \tag{6}$$

where, three parameters ($0 < \alpha_1 \leq 1, \quad 0 < \alpha_2 \leq 1$ and $0 < \alpha_3 \leq 1$) are used to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $\mathfrak{D} = \alpha_1 + \alpha_2 + \alpha_3$. It is obvious that for three dimensional space ($\mathfrak{D} = 3$), if we set $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in (6), the fractional Laplacian operator $\nabla^2_{\mathfrak{D}}$ reduces to the classical Laplacian operator ∇^2 [32] in Euclidean space.

3. FRACTIONAL SPACE GENERALIZATION OF DEL OPERATOR AND RELATED DIFFERENTIAL OPERATORS

In this section we wish to develop a generalization of vector differential operators in fractional space using scalar Laplacian operator described in previous section.

3.1. Del Operator in Fractional Space

From Equation (6), we consider single variable Laplacian operator in fractional space:

$$\nabla^2_{\mathfrak{D}} = \frac{\partial^2}{\partial x^2} + \frac{\mathfrak{D}-1}{x}\frac{\partial}{\partial x}, \quad 0 < \mathfrak{D} \leq 1 \tag{7}$$

We wish to find an expression for Del operator $\nabla_{\mathfrak{D}}$ in fractional space. As

$$\nabla_{\mathfrak{D}} = |\nabla_{\mathfrak{D}}| \cdot \hat{\nabla}_{\mathfrak{D}} \tag{8}$$

In single variable case we assume $\hat{\nabla}_{\mathfrak{D}} = \hat{x}$ also $|\nabla_{\mathfrak{D}}| = \sqrt{\nabla^2_{\mathfrak{D}}}$, because $\nabla_{\mathfrak{D}}.\nabla_{\mathfrak{D}} = \nabla^2_{\mathfrak{D}}$, where $\nabla^2_{\mathfrak{D}}$ is given in (7).

$$|\nabla_{\mathfrak{D}}| = \sqrt{\frac{\partial^2}{\partial x^2} + \frac{\mathfrak{D}-1}{x}\frac{\partial}{\partial x}} \tag{9}$$

Expansion of (9) using Binomial series expansion [32] for $|\ x\ | \gg 1$, ignoring terms involving second or higher degree of x in denominator, leads to the following form:

$$|\nabla_{\mathfrak{D}}| = \frac{\partial}{\partial x} + \frac{1}{2}\frac{\mathfrak{D}-1}{x} \tag{10}$$

From (8) and (10), Del operator in single variable x with fractional dimension $\mathfrak{D}$ is given by:

$$\nabla_{\mathfrak{D}} = \left(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\mathfrak{D}-1}{x}\right)\hat{x} \tag{11}$$

Extending above procedure to three variable case for $| x |, | y |, | z | \gg 1$ we get Del operator $\nabla_{\mathfrak{D}}$ in fractional space as follows:

$$\nabla_{\mathfrak{D}} = \left(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\alpha_1 - 1}{x}\right)\hat{x} + \left(\frac{\partial}{\partial y} + \frac{1}{2}\frac{\alpha_2 - 1}{y}\right)\hat{y} + \left(\frac{\partial}{\partial z} + \frac{1}{2}\frac{\alpha_3 - 1}{z}\right)\hat{z} \quad (12)$$

where, parameters ($0 < \alpha_1 \leq 1$, $0 < \alpha_2 \leq 1$ and $0 < \alpha_3 \leq 1$) are used to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $\mathfrak{D} = \alpha_1 + \alpha_2 + \alpha_3$. It is important to mention that Equation (12) and all differential operators presented in later sections are valid in far-field region only (i.e., $| x |, | y |, | z | \gg 1$) because of the first order approximation given by (10). Clearly, for three dimensional space ($\mathfrak{D} = 3$), if we set $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in (12), the fractional Del operator $\nabla_{\mathfrak{D}}$ reduces to the classical Del operator ∇ [32] in Euclidean space.

3.2. Gradient Operator in Fractional Space

The gradient of a scalar field ψ in fractional space is a vector that represents both the magnitude and the direction of maximum space rate of increase of ψ in fractional space. Using (12) the modified form of the gradient of scalar field ψ, written as $\mathfrak{grad}_{\mathfrak{D}}\psi$, in far-field region in the fractional space is given as:

$$\mathfrak{grad}_{\mathfrak{D}}\psi = \nabla_{\mathfrak{D}}\psi = \left(\frac{\partial\psi}{\partial x} + \frac{1}{2}\frac{(\alpha_1 - 1)\psi}{x}\right)\hat{x} + \left(\frac{\partial\psi}{\partial y} + \frac{1}{2}\frac{(\alpha_2 - 1)\psi}{y}\right)\hat{y} + \left(\frac{\partial\psi}{\partial z} + \frac{1}{2}\frac{(\alpha_3 - 1)\psi}{z}\right)\hat{z} \quad (13)$$

3.3. Divergence Operator in Fractional Space

From (12) a generalized form of divergence of a vector $\mathbf{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$ at point $P(x_0, y_0, z_0)$ in far-field region in the fractional space is written as $\mathfrak{div}_{\mathfrak{D}}\mathbf{F}$ and is given by

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{F} = \nabla_{\mathfrak{D}} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{1}{2}\frac{(\alpha_1 - 1)F_x}{x} + \frac{\partial F_y}{\partial y} + \frac{1}{2}\frac{(\alpha_2 - 1)F_y}{y} + \frac{\partial F_z}{\partial z} + \frac{1}{2}\frac{(\alpha_3 - 1)F_z}{z} \quad (14)$$

3.4. Curl Operator in Fractional Space

The modified form of curl of a vector $\mathbf{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$ at point $P(x_0, y_0, z_0)$ in far-field region in the fractional space is written as

$\mathfrak{curl}_{\mathfrak{D}}\mathbf{F}$ and using (12) it is given by

$$\begin{aligned}\mathfrak{curl}_{\mathfrak{D}}\mathbf{F} &= \nabla_{\mathfrak{D}} \times \mathbf{F}\\ &= \left[\left(\frac{\partial F_z}{\partial y}+\frac{1}{2}\frac{(\alpha_2-1)F_z}{y}\right)-\left(\frac{\partial F_y}{\partial z}+\frac{1}{2}\frac{(\alpha_3-1)F_y}{z}\right)\right]\hat{x}\\ &\quad+\left[\left(\frac{\partial F_x}{\partial z}+\frac{1}{2}\frac{(\alpha_3-1)F_x}{z}\right)-\left(\frac{\partial F_z}{\partial x}+\frac{1}{2}\frac{(\alpha_1-1)F_z}{x}\right)\right]\hat{y}\\ &\quad+\left[\left(\frac{\partial F_y}{\partial x}+\frac{1}{2}\frac{(\alpha_1-1)F_y}{x}\right)-\left(\frac{\partial F_x}{\partial y}+\frac{1}{2}\frac{(\alpha_2-1)F_x}{y}\right)\right]\hat{z}\end{aligned} \tag{15}$$

or

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{F} = \nabla_{\mathfrak{D}} \times \mathbf{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x}+\frac{1}{2}\frac{\alpha_1-1}{x} & \frac{\partial}{\partial y}+\frac{1}{2}\frac{\alpha_2-1}{y} & \frac{\partial}{\partial z}+\frac{1}{2}\frac{\alpha_3-1}{z} \\ F_x & F_y & F_z \end{vmatrix} \tag{16}$$

4. FRACTIONAL SPACE GENERALIZATION OF DIFFERENTIAL MAXWELL EQUATIONS

The Maxwell equations are the fundamental equations describing the behavior of electric and magnetic fields. In classical electromagnetic theory following quantities are dealt with:

$\mathbf{E}$= electric field intensity (V/m)
$\mathbf{B}$= magnetic field intensity (A/m)
$\mathbf{D}$= electric flux density (C/m^2)
$\mathbf{B}$= magnetic flux density (W/m^2)
$\mathbf{J}$= electric current density (A/m^2)
$\rho_{\mathbf{v}}$= electric charge density (C/m^3)
with $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$, where μ and ϵ are permeability and permittivity of the medium, respectively.

All of these quantities are functions of space variables x, y, z and tim t. The basic classical Maxwell equations in differential form in Euclidean space are [33]:

$$\nabla \cdot \mathbf{D} = \rho_v \tag{17}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{18}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{19}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{20}$$

Also the continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{21}$$

is implicit in Maxwell equations.

Now we wish to have a generalized form of Maxwell equations in $\mathfrak{D}$-dimensional fractional space. From the results of Section 3, we are now able to write differential form of Maxwell equations in far-field region in the fractional space as follows:

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{D} = \rho_v \tag{22}$$

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{B} = 0 \tag{23}$$

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{24}$$

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{25}$$

and the continuity equation in fractional space as:

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{26}$$

where, $\mathfrak{div}_{\mathfrak{D}}$ and $\mathfrak{curl}_{\mathfrak{D}}$ are defined in Equation (14) through (16). Equation (22) through (25) provide generalization of classical Maxwell equations form integer dimensional Euclidean space to a non-integer dimensional fractional space. For $\mathfrak{D} = 3$, these fractional equations can be reduced to classical Maxwell equations in Euclidean space.

In phasor form, assuming a time factor $e^{j\omega t}$, Maxwell equations in fractional space are given by replacing $\frac{\partial}{\partial t}$ with $j\omega$ [33] as below:

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{D_s} = \rho_{vs} \tag{27}$$

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{B_s} = 0 \tag{28}$$

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{E_s} = -j\omega\mathbf{B_s} \tag{29}$$

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{H_s} = \mathbf{J_s} + j\omega\mathbf{D_s} \tag{30}$$

and the phasor form of continuity equation in fractional space as:

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{J_s} = -j\omega\rho_{vs} \tag{31}$$

where, $\mathbf{D_s}, \mathbf{B_s}, \mathbf{E_s}, \mathbf{H_s}, \mathbf{J_s}, \rho_{vs}$ represent the phasor form of instantaneous quantities $\mathbf{D}, \mathbf{B}, \mathbf{E}, \mathbf{H}, \mathbf{J}$ and ρ_v, respectively.

5. FRACTIONAL SPACE GENERALIZATION OF POTENTIALS FOR STATIC FIELDS, POISSON AND LAPLACE EQUATIONS

From Maxwell equations in previous section, it is shown that the behavior of electrostatic field in fractional space can be described by two differential equations:

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{E} = \frac{\rho_v}{\epsilon_0} \tag{32}$$

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{E} = 0 \tag{33}$$

where, ϵ_0 is permittivity of free space. Equation (33) being equivalent to the statement that $\mathbf{E}$ is the gradient of a scalar function, the scalar potential for electric filed ψ. Because

$$\mathfrak{curl}_{\mathfrak{D}}(-\mathfrak{grad}_{\mathfrak{D}}\psi) = 0 \tag{34}$$

so,

$$\mathbf{E} = -\mathfrak{grad}_{\mathfrak{D}}\psi \tag{35}$$

A detailed proof of Equation (34) is provided in Appendix A. Equations (32) and (35) can be combined into one partial differential equation for the single function $\psi(x, y, z)$ as follows:

$$\mathfrak{div}_{\mathfrak{D}}\mathfrak{grad}_{\mathfrak{D}}\psi = \frac{\rho_v}{\epsilon_0} \tag{36}$$

As $\mathfrak{div}_{\mathfrak{D}}\mathfrak{grad}_{\mathfrak{D}}\psi = \nabla^2_{\mathfrak{D}}\psi$, so finally we get

$$\nabla^2_{\mathfrak{D}}\psi = \frac{\rho_v}{\epsilon_0} \tag{37}$$

where $\nabla^2_{\mathfrak{D}}$ is scalar Laplacian operator in fractional space given by (6). Equation (37) is called *Poisson equation* in fractional space. In regions of space that lack a charge density, the scalar potential ψ satisfies the *Laplace equation* given by:

$$\nabla^2_{\mathfrak{D}}\psi = 0 \tag{38}$$

Equation (37) through (38) are important in solving practical electrostatic problems in fractional space.

From Maxwell equations in last section, it is shown that the behavior of magnetostatic field in fractional space can be described by two differential equations:

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{B} = 0 \tag{39}$$

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{H} = \mathbf{J} \tag{40}$$

From Equation (40) we say that in problems concerned with finding the magnetic fields in a current free region, the $\mathfrak{curl}_{\mathfrak{D}}$ of magnetic field $\mathbf{H}$ is zero. Any vector with zero $\mathfrak{curl}_{\mathfrak{D}}$ may be represented as the $\mathfrak{grad}_{\mathfrak{D}}$ of a scalar (see e.g., Equation (34)). Thus, the magnetic field for points in such regions can be expressed as

$$\mathbf{H} = -\mathfrak{grad}_{\mathfrak{D}}\psi_m \tag{41}$$

where, ψ_m (in amperes) is the magnetic scalar potential and the minus sign is taken to complete the analogy with electrostatic field in (35).

From (39), the divergence of $\mathbf{B}$ is zero everywhere, so using (39) and (41)

$$\mathfrak{div}_{\mathfrak{D}}(\mu\mathfrak{grad}_{\mathfrak{D}}\psi_m) = 0 \tag{42}$$

Thus for a homogenous medium in fractional space the magnetic scalar potential ψ_m satisfies the Laplace equation:

$$\nabla_{\mathfrak{D}}^2 \psi_m = 0 \tag{43}$$

From (39) we know that for magnetostatic filed $\mathfrak{div}_{\mathfrak{D}} \mathbf{B} = 0$. Also we know that

$$\mathfrak{div}_{\mathfrak{D}} \mathfrak{curl}_{\mathfrak{D}} \mathbf{A} = 0 \tag{44}$$

In order to satisfy (39) and (44) simultaneously, we can define vector magnetic potential $\mathbf{A}$ (in webers/meter) such that

$$\mathbf{B} = \mathfrak{curl}_{\mathfrak{D}} \mathbf{A} \tag{45}$$

Now if we substitute (45) into (40) we get

$$\mathfrak{curl}_{\mathfrak{D}} \mathfrak{curl}_{\mathfrak{D}} \mathbf{A} = \mu \mathbf{J} \tag{46}$$

This may be considered as differential equation relating $\mathbf{A}$ to the current density $\mathbf{J}$. Using vector identity

$$\mathfrak{curl}_{\mathfrak{D}} \mathfrak{curl}_{\mathfrak{D}} \mathbf{A} = \mathfrak{grad}_{\mathfrak{D}} (\mathfrak{div}_{\mathfrak{D}} \mathbf{A}) - \nabla_{\mathfrak{D}}^2 \mathbf{A} \tag{47}$$

with

$$\mathfrak{div}_{\mathfrak{D}} \mathbf{A} = 0 \tag{48}$$

in (46) we get

$$\nabla_{\mathfrak{D}}^2 \mathbf{A} = -\mu \mathbf{J} \tag{49}$$

This is a vector equivalent of Poisson equation in (37). It includes three component scalar equations which are exactly of the poisson form.

6. FRACTIONAL SPACE GENERALIZATION OF POTENTIALS FOR TIME-VARYING FIELDS

A we have seen, in Maxwell equations fields are related to each other and sources as well. But sometimes it is helpful to introduce some intermediate functions, known as potential functions, which are directly related to sources and from which we can drive fields [33]. Such function are found useful for static fields as well (see e.g., Equations (35), (41), (45)).

From (45) we have $\mathbf{B} = \mathfrak{curl}_{\mathfrak{D}} \mathbf{A}$. This relation may now be substituted into Maxwell Equation (24) to get

$$\mathfrak{curl}_{\mathfrak{D}} \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right] = 0 \tag{50}$$

Equation (50) states that $\mathfrak{curl}_{\mathfrak{D}}$ of a certain quantity is zero. But this condition allows a vector to be derived as a $\mathfrak{grad}_{\mathfrak{D}}$ of a scalar ψ.

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\mathfrak{grad}_{\mathfrak{D}}\psi \tag{51}$$

$$\mathbf{E} = -\mathfrak{grad}_{\mathfrak{D}}\psi - \frac{\partial \mathbf{A}}{\partial t} \tag{52}$$

Equations (45) and (52) are the valid relationships between fields and potential functions $\mathbf{A}$ and ψ. We substitute (52) into (22), to obtain

$$-\nabla^2_{\mathfrak{D}}\psi - \frac{\partial(\mathfrak{div}_{\mathfrak{D}}\mathbf{A})}{\partial t} = \frac{\rho_v}{\epsilon} \tag{53}$$

Then by substituting (45) and (52) into (53), we get

$$\mathfrak{curl}_{\mathfrak{D}}\mathfrak{curl}_{\mathfrak{D}}\mathbf{A} = \mu\mathbf{J} + \mu\epsilon\left[-\mathfrak{grad}_{\mathfrak{D}}\frac{\partial\psi}{\partial t} - \frac{\partial^2\mathbf{A}}{\partial t^2}\right] \tag{54}$$

Using the vector identity (45) and choosing

$$\mathfrak{div}_{\mathfrak{D}}\mathbf{A} = -\mu\epsilon\frac{\partial\psi}{\partial t} \tag{55}$$

Equations (53) and (54) can be reduced to

$$\nabla^2_{\mathfrak{D}}\psi - \mu\epsilon\frac{\partial^2\psi}{\partial t^2} = -\frac{\rho_v}{\epsilon} \tag{56}$$

$$\nabla^2_{\mathfrak{D}}\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \tag{57}$$

Thus the potential functions $\mathbf{A}$ and ψ, defined in terms of sources $\mathbf{J}$ and ρ_v by the Equations (56) and (57) in fractional space, may be used to drive electric and magnetic fields using (45) and (52).

7. FRACTIONAL SPACE GENERALIZATION OF THE HELMHOLTZ EQUATION

From Equations (24) and (25), using $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$, we finally obtain

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \tag{58}$$

$$\mathfrak{curl}_{\mathfrak{D}}\mathbf{H} = \mathbf{J} + \epsilon\frac{\partial\mathbf{E}}{\partial t} \tag{59}$$

Taking $\mathfrak{curl}_{\mathfrak{D}}$ of Equation (58) on both sides and using (59) gives

$$\mathfrak{curl}_{\mathfrak{D}}\mathfrak{curl}_{\mathfrak{D}}\mathbf{E} = -\mu\frac{\partial}{\partial t}\left[\mathbf{J} + \epsilon\frac{\partial\mathbf{E}}{\partial t}\right] \tag{60}$$

This result can be simplified using (47) and (26) in (60) as :

$$\nabla^2_{\mathfrak{D}}\mathbf{E} = \frac{1}{\epsilon}\mathfrak{grad}_{\mathfrak{D}}\rho_v + \mu\frac{\partial \mathbf{J}}{\partial t} + \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{61}$$

For source-free region ($\rho_v = 0$, $\mathbf{J} = 0$) (62) becomes

$$\nabla^2_{\mathfrak{D}}\mathbf{E} - \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{62}$$

Equation (63) is the Helmholtz equation, or wave equation, for **E** in fractional space. An identical equation for **H** in fractional space can also be derived in the same manner:

$$\nabla^2_{\mathfrak{D}}\mathbf{H} - \mu\epsilon\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \tag{63}$$

8. CONCLUSION

A novel fractional space generalization of the differential electromagnetic equations, that is helpful in studying the behavior of electric and magnetic fields in fractal media, is provided. A new form of vector differential operator Del, written as $\nabla_{\mathfrak{D}}$, and its related differential operators is formulated in fractional space. Using these modified vector differential operators, the classical Maxwell electromagnetic equations have been worked out. The Laplace, Poisson and Helmholtz equations in fractional space are derived by using modified vector differential operators. Also a new fractional space generalization of potentials for static and time-varying fields is presented. For all investigated cases, when integer dimensional space is considered, the classical results can be recovered. The provided fractional space generalization of differential electromagnetic equations is valid in far-field region only. The differential electromagnetic equations in fractional space, established in this work, provide a basis for application of the concept of fractional space in practical electromagnetic wave propagation and scattering phenomenon in far-field region in any fractal media.

APPENDIX A. PROOF OF EQUATION (34)

According to Equation (34)

$$\mathfrak{curl}_{\mathfrak{D}}(-\mathfrak{grad}_{\mathfrak{D}}\psi) = 0$$

Here, we show that this useful fact that the curl of the gradient of scalar is zero in far-field region of any fractional space. Using (13) we write

$$\mathbf{G}=\mathfrak{grad}_{\mathfrak{D}}\psi$$
$$=\left(\frac{\partial\psi}{\partial x}+\frac{1}{2}\frac{(\alpha_1-1)\psi}{x}\right)\hat{x}+\left(\frac{\partial\psi}{\partial y}+\frac{1}{2}\frac{(\alpha_2-1)\psi}{y}\right)\hat{y}+\left(\frac{\partial\psi}{\partial z}+\frac{1}{2}\frac{(\alpha_3-1)\psi}{z}\right)\hat{z}$$

Now, using (16) we write

$$\begin{aligned}
&\mathfrak{curl}_{\mathfrak{D}}(-\mathbf{G})\\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x}+\frac{1}{2}\frac{\alpha_1-1}{x} & \frac{\partial}{\partial y}+\frac{1}{2}\frac{\alpha_2-1}{y} & \frac{\partial}{\partial z}+\frac{1}{2}\frac{\alpha_3-1}{z} \\ -G_x & -G_y & -G_z \end{vmatrix}\\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x}+\frac{1}{2}\frac{\alpha_1-1}{x} & \frac{\partial}{\partial y}+\frac{1}{2}\frac{\alpha_2-1}{y} & \frac{\partial}{\partial z}+\frac{1}{2}\frac{\alpha_3-1}{z} \\ -\left(\frac{\partial\psi}{\partial x}+\frac{1}{2}\frac{(\alpha_1-1)\psi}{x}\right) & -\left(\frac{\partial\psi}{\partial y}+\frac{1}{2}\frac{(\alpha_2-1)\psi}{y}\right) & -\left(\frac{\partial\psi}{\partial z}+\frac{1}{2}\frac{(\alpha_3-1)\psi}{z}\right) \end{vmatrix}\\
&= -\left(\frac{\partial^2\psi}{\partial y\partial z}+\frac{(\alpha_3-1)}{2z}\frac{\partial\psi}{\partial y}+\frac{(\alpha_2-1)}{2y}\frac{\partial\psi}{\partial z}+\frac{(\alpha_2-1)(\alpha_3-1)\psi}{4yz}\right)\hat{x}\\
&\quad+\left(\frac{\partial^2\psi}{\partial y\partial z}+\frac{(\alpha_2-1)}{2y}\frac{\partial\psi}{\partial z}+\frac{(\alpha_3-1)}{2z}\frac{\partial\psi}{\partial y}+\frac{(\alpha_2-1)(\alpha_3-1)\psi}{4yz}\right)\hat{x}\\
&\quad+\left(\frac{\partial^2\psi}{\partial x\partial z}+\frac{(\alpha_3-1)}{2z}\frac{\partial\psi}{\partial x}+\frac{(\alpha_1-1)}{2x}\frac{\partial\psi}{\partial z}+\frac{(\alpha_1-1)(\alpha_3-1)\psi}{4xz}\right)\hat{y}\\
&\quad-\left(\frac{\partial^2\psi}{\partial x\partial z}+\frac{(\alpha_1-1)}{2x}\frac{\partial\psi}{\partial z}+\frac{(\alpha_3-1)}{2z}\frac{\partial\psi}{\partial x}+\frac{(\alpha_1-1)(\alpha_3-1)\psi}{4xz}\right)\hat{y}\\
&\quad+\left(\frac{\partial^2\psi}{\partial x\partial y}+\frac{(\alpha_2-1)}{2y}\frac{\partial\psi}{\partial x}+\frac{(\alpha_1-1)}{2x}\frac{\partial\psi}{\partial y}+\frac{(\alpha_2-1)(\alpha_1-1)\psi}{4xy}\right)\hat{z}\\
&\quad-\left(\frac{\partial^2\psi}{\partial x\partial y}+\frac{(\alpha_1-1)}{2x}\frac{\partial\psi}{\partial y}+\frac{(\alpha_2-1)}{2y}\frac{\partial\psi}{\partial x}+\frac{(\alpha_2-1)(\alpha_1-1)\psi}{4xy}\right)\hat{z}\\
&= 0
\end{aligned}$$

All the terms in above equation get cancel and give rise to zero result. This proves Equation (34).

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