Progress In Electromagnetics Research B, Vol. 33, 69–82, 2011

A SIMPLE SOLUTION FOR THE DAMPED WAVE EQUATION WITH A SPECIAL CLASS OF BOUNDARY CONDITIONS USING THE LAPLACE TRANSFORM

N. Yener*

Technical Education Faculty, Kocaeli University, Izmit Kocaeli 41380, Turkey

Abstract—It is proven that for the damped wave equation when the Laplace transforms of boundary value functions $\psi(0,t)$ and $(\frac{\partial\psi(z,t)}{\partial z})_{z=0}$ of the solution $\psi(z,t)$ have no essential singularities and no branch points, the solution can be constructed with relative ease. In such a case while computing the inverse Laplace transform, the integrals along the segments on the real line are shown to always cancel. The integrals along the circles $C_{\tilde{\varepsilon}}$ and $C'_{\tilde{\varepsilon}}$ about the point $s=-\sigma/\varepsilon$ determined by the coefficient of the time derivative in the differential equation and point $s=0$ are shown to vanish unless Laplace transforms of mentioned boundary value functions have poles at these points. If such poles do exist, the problem is nevertheless one of integration along circles about these poles and then setting the radii of these circles equal to zero in the limit.

1. INTRODUCTION

Wave equation is an extremely important evolution model, and it is widely used by physicists in describing the propagation of water waves, sound waves, electromagnetic waves, seismic waves, gravity waves and oscillatory waves, etc. [1].

To gain an insight to the physical background about the damped wave equation we refer to [2] where it is stated that when the neural fields are formulated to predict neural activity using brain anatomy, one is led to the damped wave equation. In Figure 1, the explicit solution for a special case of the differential equation of this reference repeated in (1) is displayed.

$$u_{tt}+au_t=\alpha u_{zz},\ u(0,\cdot)=\sin(\pi z),\ u_t(0,\cdot)=0,\ u(t,0)=u(t,1)=0 \quad (1)$$

Received 9 April 2011, Accepted 11 July 2011, Scheduled 27 July 2011

* Corresponding author: Namik Yener (nyener123@gmail.com).

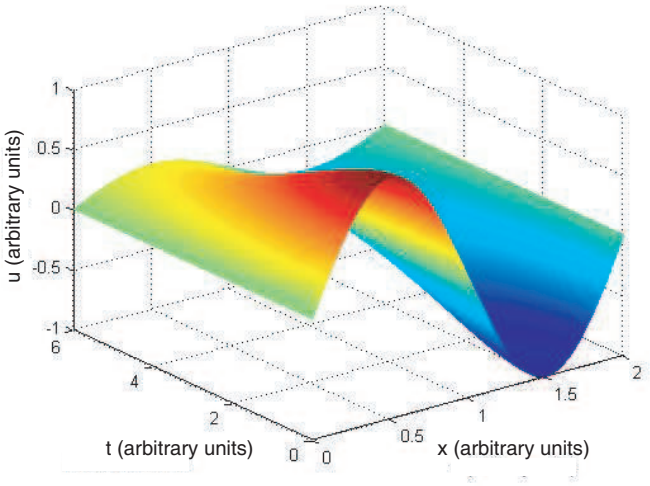

Figure 1. Explicit solution for the equation in [2]. $a = 1$.

In particular, $a^2 = 4\alpha\pi$ was selected for the figure.

We consider the damped wave equation

$$-\frac{1}{\mu\varepsilon}\frac{\partial^2\psi}{\partial z^2} + \frac{\sigma}{\varepsilon}\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial t^2} = 0, \tag{2}$$

with the boundary conditions

$$\psi(0,t) = f(t), \tag{3a}$$

$$\left(\frac{\partial\psi}{\partial z}\right)_{z=0} = F(t). \tag{3b}$$

This problem has been taken up by Stratton [3]. However, in this reference a solution has been attempted using the Fourier integral for these boundary conditions. A numerical discussion has also been included in this reference on an example for this solution. The Laplace transform is also employed for (2) and (3), but the method is applied to a medium which is unbounded in the positive-z direction so that the component wave in the negative-z direction is disregarded.

The problem of a lossy wave equation has been investigated from many different aspects. To cite some of them, one can note [4] where the Green's function has been sought for the problem. In [5], asymptotic simplification of the problem and asymptotic approximation methods for Fourier integrals for the subject equation have been discussed. In [6] Riemann's solution of the Cauchy problem

for the linear wave equation is used to find a closed form solution for the problem of transient non-sinusoidal waves in a lossy media. In [7], an unconditionally stable pseudo-spectral time-domain method is implemented by virtue of a weighted Laguerre polynomial expansion for the problem consisting in (2) with the addition of a forcing function on the right hand side. The general Maxwell's equations in a lossy medium have been treated in [8] using a marching-on in degree finite difference method.

The objective of our work is to solve the problem defined by (2) and (3) using the Laplace transform. Our problem also incorporates initial conditions given by (6) below, and we shall look for a solution satisfying boundary values (3) and initial conditions (6). The crux of the subject matter of this work is the proof that while computing the inverse Laplace transform, line integrals that have to be carried out along part of the real axis always vanish for a class of boundary conditions (3) that is detailed in Section 2. Nevertheless, we give a detailed account of the full computation to solve this class of problems by the Laplace transform technique.

We take the Laplace transform of (2) with respect to t, then

$$-\frac{1}{\mu\varepsilon}\frac{d^2\Psi}{dz^2}+\frac{\sigma}{\varepsilon}[s\Psi-\psi(z,0)]+\left[s^2\Psi-s\psi(z,0)-\left(\frac{\partial\psi}{\partial t}\right)_{t=0}\right]=0 \quad (4)$$

can be obtained. Here Ψ stands for the Laplace transform of ψ. I.e.,

$$\Psi(z,s)=\int_0^\infty \psi(z,t)\exp(-st)dt.$$

Here the Laplace transforms of $f(t)$ and $F(t)$ in (3a) and (3b) will be denoted by $f(s)$ and $F(s)$ respectively. Rearranging terms in (4),

$$-\frac{1}{\mu\varepsilon}\frac{d^2\Psi}{dz^2}+s^2\Psi+s\frac{\sigma}{\varepsilon}\Psi=\left(s+\frac{\sigma}{\varepsilon}\right)\psi(z,0)+\left(\frac{\partial\psi}{\partial t}\right)_{t=0} \quad (5a)$$

can be written. Furthermore, (5a) can be cast into the following form as well:

$$\frac{d^2\Psi}{dz^2}-h^2\Psi=-\frac{h^2}{s}f_1(z)-\mu\varepsilon f_2(z)=Z(z,s) \quad (5b)$$

where

$$f_1(z)=\lim_{t\to 0}\psi(z,t), \quad (6a)$$

$$f_2(z)=\lim_{t\to 0}\frac{\partial\psi(z,t)}{\partial t}, \quad (6b)$$

represent the *initial state of the solution* and $h^2 = \mu\varepsilon s^2 + \sigma\mu s$ as in [3, p.318]. The general solution of (5b) is a solution of the homogeneous equation

$$\frac{d^2\Psi}{dz^2} - h^2\Psi = 0. \tag{7}$$

The particular solution of (5b) will then be [3]:

$$\Psi_p(z,s) = \frac{\exp(hz)}{2h}\int \exp(-hz)Z(z,s)dz - \frac{\exp(-hz)}{2h}\int \exp(hz)Z(z,s)dz. \tag{8}$$

2. THE GENERAL SOLUTION OF (5B) SATISFYING THE BOUNDARY CONDITIONS

The general solution of (5b) will be in the form

$$\Psi(z,s) = A(s)\exp(hz) + B(s)\exp(-hz). \tag{9}$$

Hence imposing the boundary conditions (3a) and (3b) on (9), one gets

$$A(s) + B(s) = f(s), \tag{10a}$$

$$A(s) - B(s) = \frac{F(s)}{h(s)}. \tag{10b}$$

Hence one gets in view of (9) and (10),

$$A(s) = \frac{1}{2}\left[f(s) + \frac{F(s)}{h}\right], \tag{11a}$$

$$B(s) = \frac{1}{2}\left[f(s) - \frac{F(s)}{h}\right]. \tag{11b}$$

Therefore (9) can now be written as

$$\Psi(z,s) = \frac{1}{2}\left[f(s)+\frac{F(s)}{h}\right]\exp(hz)+\frac{1}{2}\left[f(s)-\frac{F(s)}{h}\right]\exp(-hz). \tag{12}$$

Recalling that $h(s) = \sqrt{\mu\varepsilon s^2 + \mu\sigma s}$, and using the Bromwich integral, the inverse Laplace transform of (12) will be for $t > 0$:

$$\begin{aligned}\psi(z,t) = \frac{1}{2\pi i}\int_L \Bigg\{ &\frac{1}{2}\left[f(s)+\frac{F(s)}{\sqrt{\mu\varepsilon s^2+\mu\sigma s}}\right]\exp\left(z\sqrt{\mu\varepsilon s^2+\mu\sigma s}\right)\times\exp(st)\\ &+\frac{1}{2}\left[f(s)-\frac{F(s)}{\sqrt{\mu\varepsilon s^2+\mu\sigma s}}\right]\exp\left(-z\sqrt{\mu\varepsilon s^2+\mu\sigma s}\right)\\ &\times\exp(st)\Bigg\}ds.\end{aligned} \tag{13a}$$

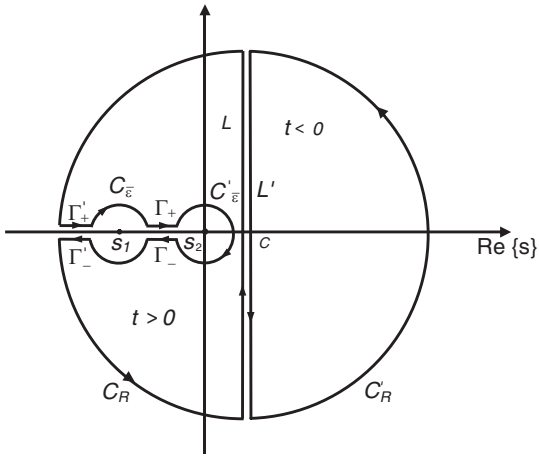

Figure 2. Integration paths for the inverse Laplace transform.

For $t < 0$ the inverse Laplace transform will be:

$$\psi(z,t) = \frac{1}{2\pi i}\int_{L'}\left\{\frac{1}{2}\left[f(s)+\frac{F(s)}{\sqrt{\mu\varepsilon s^2+\mu\sigma s}}\right]\exp\left(z\sqrt{\mu\varepsilon s^2+\mu\sigma s}\right)\times\exp(st)\right.$$
$$\left.+\frac{1}{2}\left[f(s)-\frac{F(s)}{\sqrt{\mu\varepsilon s^2+\mu\sigma s}}\right]\exp\left(-z\sqrt{\mu\varepsilon s^2+\mu\sigma s}\right)\times\exp(st)\right\}ds. \tag{13b}$$

The paths L and L' involved are indicated in Figure 2.

Suppose now that $f(s)$ and $F(s)$ are single valued functions of s not incorporating any branch points. Furthermore, suppose that $f(s)$ and $F(s)$ have no essential singularities. Then the only branch points in (13) will be $s_1 = -\sigma/\varepsilon$ and $s_2 = 0$ due to the function $h(s) = \sqrt{\mu\varepsilon s^2 + \mu\sigma s}$.

The branch cut is chosen between $s_1 = -\sigma/\varepsilon$ and $s_2 = 0$ on the real axis. This choice entails:

$$-\pi < \arg(s - s_1) < \pi$$
$$-\pi < \arg(s - s_2) < \pi$$

In the above, $\arg(\cdot)$ represents the argument of the complex number in parentheses.

On the other hand, the real constant c that appears in Figure 2 is so chosen that all singularities of the integrand in (13) lie to the left of the straight line $s = c$ in the complex s plane.

Figure 2 shows the paths of the integrals in (13) and the singularities $s_1 = -\sigma/\varepsilon$ and $s_2 = 0$. Notice that because the integrand in (13a) does not possess a branch point at infinity, it will be continuous along the paths Γ'_+ and Γ'_-. Therefore, the integrals along them will cancel. Hence in the following these integrals will be disregarded.

On the other hand, keeping the integrand in (13a) also in the integrals below,

$$\frac{1}{2\pi i}\int_L ds + \lim_{R\to\infty}\frac{1}{2\pi i}\int_{C_R} ds + \lim_{\bar{\varepsilon}\to 0}\frac{1}{2\pi i}\int_{C_{\bar{\varepsilon}}} ds + \lim_{\bar{\varepsilon}\to 0}\frac{1}{2\pi i}\int_{C'_{\bar{\varepsilon}}} ds + \frac{1}{2\pi i}\int_{\Gamma_+} ds + \frac{1}{2\pi i}\int_{\Gamma_-} ds = 0 \quad (14a)$$

will hold due to Residue Theorem unless $f(s)$ and $F(s)$ have poles within the domain defined by the contour made up of L, C_R, $C_{\bar{\varepsilon}}$, $C'_{\bar{\varepsilon}}$, Γ'_+, Γ'_-, Γ_+ and Γ_- when R (radius of C_R) approaches ∞ and $\bar{\varepsilon}$ (radius of $C_{\bar{\varepsilon}}$ and $C'_{\bar{\varepsilon}}$) tends to zero. If $f(s)$ and $F(s)$ have poles within this domain, the right hand side of (14a) will be equal to the sum of the residues corresponding to these poles [9].

Also similar to (14a), we have

$$\frac{1}{2\pi i}\int_{L'} ds + \frac{1}{2\pi i}\int_{C'_R} ds = 0. \quad (14b)$$

The integrands of both integrals in (14b) are equal to the integrand of (13b). First we focus on the integrals along C_R and C'_R in (14a) and (14b) and prove that they vanish when R approaches infinity.

For the integral along C_R in (14a) to represent the points on the semi-circle C_R, we use the relation

$$s - c = p, \quad (15)$$

where s is a point on the semi-circle C_R; p is the vector emanating from the point $Re\{s\} = c$; $Im\{s\} = 0$ and directed toward the point s. Hence we can write $p = R\exp(i\theta)$ where R is the radius of the semi-circle C_R, and θ is the angle that p makes with the $Re\{s\}$ axis. (See Figure 3 for an illustration of s, p, c, θ, R and $\bar{\varepsilon}$.) We can substitute $p = R\exp(i\theta)$ and (15) in (13a) to get for the first and second terms in the integrand in (13a)

$$\frac{1}{2}\left\{ f(c+R\exp(i\theta)) + \frac{F(c+R\exp(i\theta))}{R\exp(i\theta)\left[\mu\varepsilon\left(\frac{c^2}{p^2}+2\frac{c}{p}+1\right)+\mu\sigma\left(\frac{c}{p^2}+\frac{1}{p}\right)\right]^{1/2}} \right\}$$
$$\times\exp\left\{ R\exp(i\theta)\left[\mu\varepsilon\left(\frac{c^2}{p^2}+2\frac{c}{p}+1\right)+\mu\sigma\left(\frac{c}{p^2}+\frac{1}{p}\right)\right]^{1/2} z \right\}$$
$$\times\exp\{[c+R\exp(i\theta)]t\}$$

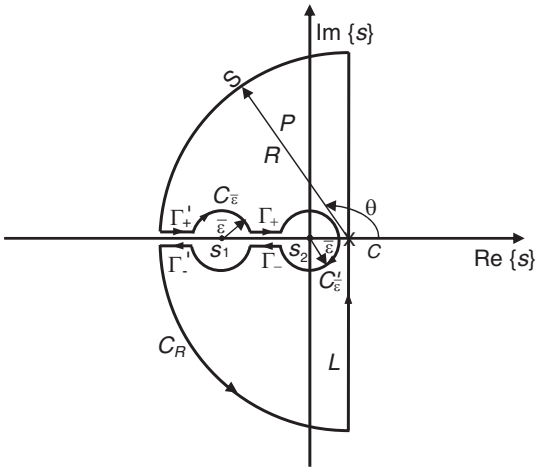

Figure 3. Depiction of quantities needed in evaluating integrals along C_R, $C_{\bar{\varepsilon}}$, and $C'_{\bar{\varepsilon}}$.

$$+\frac{1}{2}\left\{f(c+R\exp(i\theta))-\frac{F(c+R\exp(i\theta))}{R\exp(i\theta)\left[\mu\varepsilon\left(\frac{c^2}{p^2}+2\frac{c}{p}+1\right)+\mu\sigma\left(\frac{c}{p^2}+\frac{1}{p}\right)\right]^{1/2}}\right\} \cdot (16)$$
$$\times\exp\left\{-R\exp(i\theta)\left[\mu\varepsilon\left(\frac{c^2}{p^2}+2\frac{c}{p}+1\right)+\mu\sigma\left(\frac{c}{p^2}+\frac{1}{p}\right)\right]^{1/2}z\right\}$$
$$\times\exp\{[c+R\exp(i\theta)]t\}$$

Here we have factored out $R\exp(i\theta)$ from the radical involved with the expression for h. Note that for $\pi/2 < \theta < 3\pi/2$ which is the applicable interval for θ along C_R, $R\exp(i\theta)$ has a real negative part so that when R approaches infinity, the two exponential factors post multiplying each term yield a finite limit value which is the zero value for *both* terms only under the condition $-z\sqrt{\mu\varepsilon}R\cos\theta + t(c + R\cos\theta) < 0$. If we divide this inequality by R and then set $R \to \infty$, we obtain the condition $t > z\sqrt{\mu\varepsilon} > 0$. So we conclude that $\frac{1}{2\pi i}\int\limits_{C_R} ds$ in (14a) converges (to the value zero) as $R \to \infty$ only if $t > z\sqrt{\mu\varepsilon}$. Our solution will therefore be subject to this constraint so that when the final $\psi(z,t)$ is obtained it should be remembered that this inequality is satisfied. Our entire analysis hinges on this assumption. Otherwise, the inverse Laplace transform integral will simply not converge. In passing, it should be noted that the same remark takes place in [10] too and that this fact is attributed to the hyperbolicity of (2).

For the integral along C'_R in (14b), to represent the points on the semi-circle C'_R, we again use the relation (15) where s, p and c have the same meanings described above except that now s is a point on the semi-circle C'_R. We again write $p = R\exp(i\theta)$ where R is the radius of the semi-circle C'_R, and θ is the angle p makes with the $Re\{s\}$ axis. We can substitute $p = R\exp(i\theta)$ and (15) in (13b) to get for the first and second terms in the integrand in (13b) the same expression as given by (16).

Suppose that there exists a real number γ, such that

$$\lim_{T\to\infty}\int_0^T |\exp(-\gamma\, t)\psi(z,t)|dt < \infty. \tag{17}$$

Then $\psi(z,t)$ is Laplace transformable, and the expression, consisting of (16) less the factor of $\exp(st)$ that multiplies the two terms present, has to be analytic for $Re\{s\} > \gamma_a$ where γ_a is the abscissa of absolute convergence. This means that when $t < 0$, because $\exp[(c + R\cos\theta)t]$ becomes small without bound as R tends to infinity, we can assume that

$$\frac{1}{2\pi i}\int_{C'_R} ds \to 0 \text{ as } R \to \infty, \tag{18}$$

where the integrand is the same as that of (13b) or just the full expression in (16).

Note that $\frac{1}{2\pi i}\int\limits_{C_R} ds$ in (14a) and (18) tended to zero also under the assumption that $f(s)$ and $F(s)$ have no essential singularities. If for instance they had an essential singularity at infinity, and the integrands had become non-zero due to such a singularity, these integrals might not have vanished.

Now let us proceed to determine the integrals along Γ_+ and Γ_- and the circles $C_{\bar\varepsilon}$ and $C'_{\bar\varepsilon}$ in (14a).

On the circle

$$C'_{\bar\varepsilon}: \quad s = \bar\varepsilon\exp(i\phi) \text{ with } |s| = \bar\varepsilon. \tag{19}$$

On the segment

$$\Gamma_+ : -\frac{\sigma}{\varepsilon} + \bar\varepsilon < x < -\bar\varepsilon,\ y = 0,\ \arg(s) = \pi \quad \arg\Big(s + \frac{\sigma}{\varepsilon}\Big) = 0. \tag{20}$$

On the circle

$$C_{\bar\varepsilon}: \quad s + \frac{\sigma}{\varepsilon} = \bar\varepsilon\exp(i\phi) \text{ with } \Big|s + \frac{\sigma}{\varepsilon}\Big| = \bar\varepsilon. \tag{21}$$

On the segment

$$\Gamma_- : \; -\frac{\sigma}{\varepsilon} + \bar{\varepsilon} < x < -\bar{\varepsilon}, \; y = 0, \; \arg(s) = -\pi \quad \arg\left(s + \frac{\sigma}{\varepsilon}\right) = 0. \quad (22)$$

In the above, $x = Re\{s\}$ and $y = Im\{s\}$.

Now in order to evaluate the integral along the circle $C'_{\bar{\varepsilon}}$ consider the below quantity (23) obtained from the integrand in (13b) observing (19):

$$\begin{aligned} &\bar{\varepsilon} \max_{C'_{\bar{\varepsilon}}} \left| \left[f(s) + \frac{F(s)}{\sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) + \sigma/\varepsilon]}} \right] \right. \\ &\times \exp\left\{ \sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) + \sigma/\varepsilon]}\, z + \bar{\varepsilon}\exp(i\phi)t \right\} \\ &+ \left[f(s) - \frac{F(s)}{\sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) + \sigma/\varepsilon]}} \right] \\ &\left. \times \exp\left\{ -\sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) + \sigma/\varepsilon]}\, z + \bar{\varepsilon}\exp(i\phi)t \right\} \right| \end{aligned} \quad (23)$$

Notice that observing (19) and so long as $f(s)$ or $F(s)$ has no poles at $s = 0$, the above quantity (23) will vanish as $\bar{\varepsilon}$ approaches zero. Hence the corresponding integral will vanish. In case that $F(s)$ or $f(s)$ has a pole at $s = 0$, the subject integral may be evaluated using a variable transformation $\bar{\varepsilon}\exp(i\phi) = s$ and integrating between $\phi = \pi$ and $\phi = -\pi$ and taking then limit $\bar{\varepsilon} \to 0$ [11].

In order to evaluate the integral along the circle $C_{\bar{\varepsilon}}$, consider the below quantity (24) obtained from the integrand in (13b) observing (21):

$$\begin{aligned} &\bar{\varepsilon} \max_{C_{\bar{\varepsilon}}} \left| \left[f(s) + \frac{F(s)}{\sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) - \sigma/\varepsilon]}} \right] \right. \\ &\times \exp\left\{ \sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) - \sigma/\varepsilon]}\, z + [\bar{\varepsilon}\exp(i\phi) - \sigma/\varepsilon]t \right\} \\ &+ \left[f(s) - \frac{F(s)}{\sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) - \sigma/\varepsilon]}} \right] \\ &\left. \times \exp\left\{ -\sqrt{\bar{\varepsilon}\exp(i\phi)\mu\varepsilon[\bar{\varepsilon}\exp(i\phi) - \sigma/\varepsilon]}\, z + [\bar{\varepsilon}\exp(i\phi) - \sigma/\varepsilon]t \right\} \right| \end{aligned} \quad (24)$$

Notice that observing (21) and so long as $f(s)$ or $F(s)$ has no poles at $s = -\sigma/\varepsilon$, the above quantity (24) will vanish as $\bar{\varepsilon}$ approaches zero. Hence the corresponding integral will vanish. The case, when $F(s)$ has a pole at $s = -\sigma/\varepsilon$, will be treated as an example.

If $f(s)$ and/or $F(s)$ has a pole at $s = -\sigma/\varepsilon$ the subject integral may be evaluated using above mentioned method for integral along $C'_{\bar{\varepsilon}}$ when its integrand has a pole at the origin.

If we consider the integrals along Γ_+ and Γ_- while observing (20) and (22), we shall see that the first one of these will be

$$\int\Bigg\{\left[f(x)+\frac{F(x)}{\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(i\pi)}}\right]\exp\Big[\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(i\pi)}x\Big]$$
$$\times\exp(-xt)+\left[f(x)-\frac{F(x)}{\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(i\pi)}}\right]$$
$$\times\exp\left[-\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(i\pi)}x\right]\times\exp(-xt)\Bigg\}\,dx, \tag{25a}$$

whereas the second one will be

$$\int\Bigg\{\left[f(x)+\frac{F(x)}{\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(-i\pi)}}\right]\exp\Big[\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(-i\pi)}x\Big]$$
$$\times\exp(-xt)+\left[f(x)-\frac{F(x)}{\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(-i\pi)}}\right]$$
$$\times\exp\left[-\sqrt{(\mu\varepsilon x^2+\sigma\mu x)\exp(-i\pi)}x\right]\times\exp(-xt)\Bigg\}\,dx. \tag{25b}$$

When the second integral is subtracted from the first one because the same integral path lies in opposite directions along Γ_+ and Γ_-, it can be seen that the difference is zero. This is the crux of the point of this paper. These line integrals, along Γ_+ and Γ_-, will always yield a zero contribution to the inverse Laplace transform integral in (13a).

3. EXAMPLE

As an example we take up the case $f(s)=0$ and $F(s)=\frac{1}{(s+\sigma/\varepsilon)^2}$. Because if we consider $f(s)=0$ and $F(s)\neq 0$ or $f(s)\neq 0$ and $F(s)=0$ we shall have either a sine hyperbolic function or a cosine hyperbolic function in the expression to compute the inverse Laplace transform of. The structure of the problem is the same. If we have both functions non-zero, then superposition applies. In fact, this choice of an example is more complicated than that $f(s)$ and $F(s)$ have poles only within the domain defined by the contour made up of L, C_R, $C_{\bar{\varepsilon}}$, $C'_{\bar{\varepsilon}}$, Γ'_+, Γ'_-, Γ_+ and Γ_- in Figure 2. If $f(s)$ and $F(s)$ have poles only within this domain, then one can simply reach the result by residue computation alone.

Hence

$$\psi(z,t) = \frac{1}{2\pi i}\int\limits_{L}\frac{1}{2(s+\sigma/\varepsilon)^2}\left\{\frac{1}{\sqrt{\mu\varepsilon s^2+\mu\sigma s}}\left[\exp(\sqrt{\mu\varepsilon s^2+\mu\sigma s})z - \exp(-\sqrt{\mu\varepsilon s^2+\mu\sigma s})z\right]\right\}\times\exp(st)ds \tag{26}$$

will be true. We shall find (26) by computing

$$\lim_{\bar{\varepsilon}\to 0}\frac{1}{2\pi i}\int\limits_{C_{\bar{\varepsilon}}} ds \tag{27}$$

in (14a). Indeed the integrand in (26) (which is equal to that of (27)) has branch points at $s=0$ and $s=-\sigma/\varepsilon$ and a pole of order two at $s=-\sigma/\varepsilon$. The branch cut is chosen between $s_1=-\sigma/\varepsilon$ and $s_2=0$ on the real axis. Therefore, we select the integration path for $t>0$ depicted in Figure 2. The two circles $C_{\bar{\varepsilon}}$ and $C'_{\bar{\varepsilon}}$ encircle the branch points, but the circle $C_{\bar{\varepsilon}}$ also encircles the pole at $s=-\sigma/\varepsilon$. Also because then all terms in (14a) other than the first and third vanish, we can compute (26) by finding the negative of (27). In Section 2, we set (27) equal to zero. In this example however (27) does not vanish because of the pole that $F(s)$ has at $s=-\sigma/\varepsilon$.

We shall compute this contribution by expanding the factors $\exp(\sqrt{\mu\varepsilon s^2+\mu\sigma s}z)$ and $\exp(st)$ that appear in (26) into power series about the point $s=-\sigma/\varepsilon$. We expand about the point $s=-\sigma/\varepsilon$ because this is necessary to evaluate the mentioned integral under the limit $\bar{\varepsilon}\to 0$, for due to (21), setting $s=-\sigma/\varepsilon$ is equivalent to effecting this limit. The two power series mentioned will be as follows:

$$\exp(st)=\exp(-\sigma t/\varepsilon)\left[1+t\bar{\varepsilon}\exp(i\phi)+t^2[\bar{\varepsilon}\exp(i\phi)]^2/2+\ldots\right], \tag{28a}$$

$$\begin{aligned}\exp\left(\sqrt{\mu\varepsilon s^2+\mu\sigma s}z\right) &= (1+\sqrt{\mu\varepsilon[\bar{\varepsilon}\exp(i\phi)][\bar{\varepsilon}\exp(i\phi)-\sigma/\varepsilon]}z\\ &+\mu\varepsilon[\bar{\varepsilon}\exp(i\phi)]\,[\bar{\varepsilon}\exp(i\phi)-\sigma/\varepsilon]\,z^2/2\\ &+\{\mu\varepsilon[\bar{\varepsilon}\exp(i\phi)][\bar{\varepsilon}\exp(i\phi)-\sigma/\varepsilon]\}^{3/2}\,z^3/6+\ldots)\end{aligned} \tag{28b}$$

Then we make the change of variables $s+\frac{\sigma}{\varepsilon}=\bar{\varepsilon}\exp(i\phi)$ in (27). Now ds becomes $\bar{\varepsilon}i\exp(i\phi)d\phi$. Next we multiply the product of these two series by $\frac{1}{[\bar{\varepsilon}\exp(i\phi)]^2\sqrt{\mu\varepsilon[\bar{\varepsilon}\exp(i\phi)][\bar{\varepsilon}\exp(i\phi)-\sigma/\varepsilon]}}$ which is multiplying by $\frac{1}{(s+\sigma/\varepsilon)^2\sqrt{\mu\varepsilon s^2+\mu\sigma s}}$.

Before the integration in (27) is performed, we express the integrand in a simpler form also making use of (28) as follows:

$$\frac{id\phi}{\bar{\varepsilon}\exp(i\phi)}\left\{2z+2[\mu\varepsilon\bar{\varepsilon}\exp(i\phi)(\bar{\varepsilon}\exp(i\phi)-\sigma/\varepsilon)]z^3/6+\ldots\right\}$$
$$\times\exp(-\sigma t/\varepsilon)\left\{1+t\bar{\varepsilon}\exp(i\phi)+t^2/2[\bar{\varepsilon}\exp(i\phi)]^2+\ldots\right\}.$$

Here we have also subtracted the two terms present in (26). When i) the integration is performed between the limits $\phi = 2\pi$ and $\phi = 0$, and ii) next when the limit $\bar{\varepsilon} \to 0$ is effected, terms other than those in (29) will vanish. Therefore, the net value of the quantity $\lim_{\bar{\varepsilon}\to 0}\frac{1}{2\pi i}\int_{C_{\bar{\varepsilon}}} ds$ in (14a) becomes:

$$-\exp(-\sigma t/\varepsilon)(tz-\mu\sigma z^3/6). \tag{29}$$

This method of computation of the integral $\lim_{\bar{\varepsilon}\to 0}\frac{1}{2\pi i}\int_{C_{\bar{\varepsilon}}} ds$ is the same method employed in Section 61 of [11]. When this value is substituted in (14a) one gets for $\psi(z,t)$:

$$\psi(z,t)=\exp(-\sigma t/\varepsilon)\left(tz-\mu\sigma z^3/6\right). \tag{30}$$

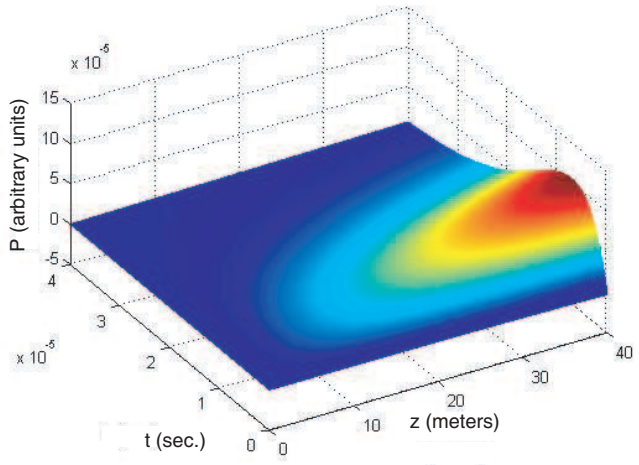

Figure 4. Solution in Equation (30) with $1/\sqrt{\mu\varepsilon} = 1/3 \times 10^8$ (m/s), $\sigma/(2\varepsilon) = 1.4 \times 10^5$ (sec^{-1}), the vertical axis representing $\psi(z,t)$. Parameter values are from [3].

(See Figure 4 for a three dimensional plot of this function). It is simple to verify that this result satisfies (2) subject to (3) where $f(t) = 0$ and $F(t) = t\exp(-\sigma t/\varepsilon)$, and the latter of which is the inverse Laplace transform of $F(s) = \frac{1}{(s+\sigma/\varepsilon)^2}$.

Now we are in a position compute the 'particular solution' given by (8). Indeed if the functions $f_1(z)$ and $f_2(z)$ given by (6) are generated such that (30) is used as the function $\psi(z,t)$, one finds that the 'particular solution' obtained by evaluating (8) for this choice is naturally equal to (30) itself. So the solution of (2) for the case $f(s) = 0$ and $F(s) = \frac{1}{(s+\sigma/\varepsilon)^2}$ will be equal to (30), and this result also satisfies (6) because of the coalescence of (30) with the 'particular solution' (8).

4. CONCLUSION

It has been proven that under a special class of boundary conditions the solution for the damped wave equation (1) can be constructed with relative ease. The specialty of the class of boundary conditions lies in the fact that the Laplace transforms of (3) should possess no essential singularities and no branch points. In such a case, the integrals along the segments Γ_+ and Γ_- have been shown to *always* cancel. The integrals along the circles $C_{\bar{\varepsilon}}$ and $C'_{\bar{\varepsilon}}$ have been shown to vanish unless Laplace transforms of Equations (3) have poles at $s = s_1 = -\sigma/\varepsilon$ and/or $s = s_2 = 0$. If such poles exist, the problem is one of integration along a circle about s_1 and/or s_2 and then setting the radii of these circles equal to zero in the limit. If Laplace transforms of Equation (3) have branch points, there is no off-hand solution, and the problem must be re-investigated because of the non-vanishing of the integrals along Γ_+ and Γ_-. However in case that $F(s)$ and $f(s)$ are multi-valued only through a dependence on $h(s)$ and that this dependence is an even function of h for both F and f, it can be shown that the integrals along Γ_+ and Γ_- vanish again.

REFERENCES

1. Lu, X., "The applications of microlocal analysis in σ-evolution equations," Ph.D. dissertation, Zhejiang University, Hangzhou, Zhejiang, 2010.
2. Jradeh, M., "On the damped wave equation," *12th Intern. Confer. on Hyperbolic Problems*, Maryland, USA, June 9–13, 2008.
3. Stratton, J. A., *Electromagnetic Theory*, McGraw Hill, New York, 1941.

4. Alexio, R. and E. C. de Oliveira, "Green's functions for the lossy wave equation," *Revista Brasileira de Ensino de Fisica*, Vol. 30, No. 1, Sao Paulo, 2008.
5. Zauderer, E., *Partial Differential Equations of Applied Mathematics*, 2nd edition, John Wiley & Sons Inc., New York, 1989.
6. Asfar, O. R., "Riemann-green function solution of transient electromagnetic plane waves in lossy media," *IEEE Trans. EM Compatibility*, Vol. 32, No. 3, 228–231, 1990.
7. Lin, H., et al., "A novel unconditionally stable PSTD method based on weighted Laguerrre polynomial expansion," *Journal of Electromagnetic Waves and Applications*, Vol. 23, No. 8–9, 1011–1020, 2009.
8. Jung, B. H. and T. K. Sarkar, "Solving time domain Helmholtz wave equation with MOD-FDM," *Progress In Electromagnetics Research*, Vol. 79, 339–352, 2008.
9. Levinson, N. and R. M. Redheffer, *Complex Variables*, Holden-Day Inc., San Francisco, 1970.
10. Sonnenchein, E., I. Rutkevich, and D. Censor, "Wave packets and group velocity in absorbing media: Solutions of the telegrapher's equation," *Progress In Electromagnetics Research*, Vol. 27, 129–158, 2000.
11. Churchill, R. V., *Modern Operational Mathematics in Engineering*, McGraw Hill, New York, 1944.