NUMERICAL OPTIMIZATION OF THE METHOD OF AUXILIARY SOURCES BY USING LEVEL SET TECHNIQUE

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Abstract—It is well-known that the choice of the auxiliary surface and the arrangement of radiation centers play a decisive role for ensuring accuracy and stability of the method of auxiliary sources (MAS). Using level set technique, a numerical scheme is proposed to determine the optimal location and amplitudes of the auxiliary sources for three-dimensional scattering problems.

1. INTRODUCTION

The method of auxiliary sources was introduced by a Georgian research group [1] for resolving two-dimensional electrodynamics problems. Later, it has been widely used to solve the boundary value problems arising in scattering analysis, see [2, 3]. The key idea behind this method is interchanging the differential equation and boundary conditions [4]. This allows to remove the singularities of the singular integral equation by shifting the auxiliary sources contour relative to integration one [5]. Proceeding from this, the electromagnetic fields are expanded over the basis generated by the particular solutions of Helmoltz-equation [6]. Then, the boundary value problem is solved by imposing the boundary condition at the physical scatterer surface in the same way as the standard SIE method (surface integral equation). Previous investigations [7, 8] have shown that the correct choice of the auxiliary surface and the placement of the radiation centers(auxiliary sources) are important factors to achieve efficiency of the MAS. The optimal choice of MAS parameters (auxiliary surface, radiation centers) is an open issue that was discussed in the works of many authors [9, 10]. In this paper, we propose a novel iterative scheme based on the level set technique to find the optimal MAS parameters.

The level-set method is one computational technique for tracking a propagating interface over time. The central idea of level set method is to implicitly represent the interface of interest as the zero level set of a smooth enough function φ, known as level set function. So, to identify the desired interface, we just need to identify the appropriate level set function. Using the level-set representation of the auxiliary surface, we then create a numerical scheme to search the optimal auxiliary parameters for three-dimensional scattering problems.

2. THREE-DIMENSIONAL MAS FORMULATION

Let consider Ω an open subset of $\mathbb{R}^3$, occupied by a homogeneous conducting medium, and let Γ the boundary of Ω. ε and σ denote respectively the relative electric permittivity and the conductivity of the medium. The conductor is illuminated by a plan linearly polarized electromagnetic wave $(\vec{E}_i, \vec{H}_i)$:

$$\vec{E^i} = \exp(jK\hat{k} \cdot \vec{r})\hat{a} \tag{1}$$

$$\vec{H^i} = \frac{1}{Z_0} \exp(jK\hat{k} \cdot \vec{r})\hat{b} \tag{2}$$

where $\hat{k}$, $\hat{a}$ and $\hat{b}$ are unit vectors specifying the directions of incidence, the electric field and the magnetic field, respectively.

The propagation constant, permittivity and intrinsic impedance of the surrounding medium are K, ε_0 and Z_0 respectively, and a time factor $\exp(-j\omega t)$ has been assumed and suppressed.

The total electric and magnetic fields are written as the sum of the incident and scattered fields.

$$\vec{E} = \vec{E^i} + \vec{E^s} \tag{3}$$

$$\vec{H} = \vec{H^i} + \vec{H^s} \tag{4}$$

The tangential components of the surface field are interpreted as electric and magnetic currents:

$$\vec{J} = \hat{n} \times \vec{H} = \hat{n} \times \vec{H^i} + \hat{n} \times \vec{H^s} = \vec{J^i} + \vec{J^s} \tag{5}$$

$$\vec{M} = -\hat{n} \times \vec{E} = -\hat{n} \times \vec{E^i} - \hat{n} \times \vec{E^s} = \vec{M^i} + \vec{M^s} \tag{6}$$

For a perfect conductor object, the surface impedance boundary conditions can be written in the form:

$$\vec{n} \times \vec{E^s} = \vec{M^i} \tag{7}$$

According to the MAS method, let S denotes the auxiliary surface, $M_{1 \leq k \leq n}$ the radiation sources positioned at points $\vec{r}_{1 \leq k \leq n}$ as shown
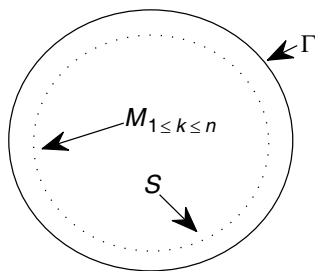
Figure 1. MAS geometry.

in Figure 1 and $\vec{U}(|\vec{r}_k - \vec{r}|)_{1 \le k \le \infty}$ the Helmholtz equation solution associated with elementary sources.

$$\vec{U}(|\vec{r}_k - \vec{r}|) = \frac{e^{jK|\vec{r}_k - \vec{r}|}}{4\pi |\vec{r}_k - \vec{r}|^2} (\vec{r}_k - \vec{r}) \tag{8}$$

Kupradze [4] proved that the set of functions $\vec{U}(|\vec{r}_k - \vec{r}|)_{1 \le k \le \infty}$ is complete and linearly independent on the surface Γ in L^2 space. So, there are coefficients $a_{1 \le k \le n}$ such that, using the n first functions of the aforementioned system, the scattered electric field can be approximated as follows.

$$\vec{E}^s = \sum_{k=1}^{n} a_k \vec{U}(|\vec{r}_k - \vec{r}|) \tag{9}$$

Let

$$\vec{G}(|\vec{r}_k - \vec{r}|) = \vec{n}(\vec{r}) \times \vec{U}(|\vec{r}_k - \vec{r}|) \tag{10}$$

The boundary conditions can be written as:

$$\sum_{k=1}^{n} a_k \vec{G}(|\vec{r}_k - \vec{r}|) = \vec{M}^i(\vec{r}) \tag{11}$$

By matching the boundary condition at m collocation points $\vec{x}_p$ $1 \le p \le m$, the problem can be formulated as follows:

Find $(a_{1 \le k \le n})$ such that

$$\sum_{k=1}^{n} a_k \vec{G}(|\vec{r}_k - \vec{r}_p|) = \vec{M}^i(\vec{x}_p) \quad 1 \le p \le m \tag{12}$$

The expansion coefficients $(a_{1 \le k \le n})$ can be interpreted as the amplitudes of auxiliary sources. Once the coefficients are calculated, the approximate solution of the boundary problem outside Ω is

$$\vec{E}^s(\vec{r}) \approx \sum_{k=1}^{n} a_k \vec{U}(|\vec{r}_k - \vec{r}|) \tag{13}$$

which will approach exact solution as $n \to \infty$.

It should be noted, that the stability of the obtained algebraic system (12) depends on the proper choice of auxiliary parameters which are the shape of auxiliary surface S and distribution of the radiation centers $\vec{r}_{1 \leq k \leq n}$. Ignoring this point leads to a weakening of convergence and even to a diverging of solution with increasing n. Therefore, locate the main singularities of scattered field is an essential part of the scheme to construct the optimal solution by means of the MAS [10]. To make the method more effective we must locate the auxiliary sources in the singularity region as shown in Figure 2.
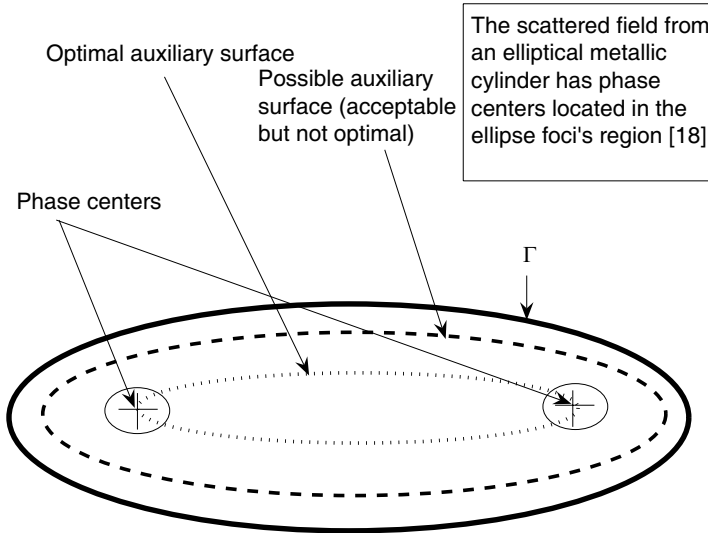

Figure 2. MAS geometry of an elliptical cylinder.

The matter is that the necessary number of terms of the series (12) strongly depends on the relative distance between the real surface Γ and the auxiliary surface S on which the auxiliary sources are placed. When the auxiliary surface moves away from the real one the number of the terms in (12) decreases strongly and consequently, the computational cost decrease also [10].

3. THE LEVEL SET METHOD

3.1. An Overview of Level Set Method

The level set method was introduced by Osher and Sethian [11] in the fields of fluid dynamics, for tracing interfaces between different

phases of fluid flows. Later, it has been used for many different kind of physical problems, see [12–14, 16]. The main idea behind this method is to represent the interface at each time t as the zero level set of a function φ. Thus, given a surface S in $\mathbb{R}^3$ bounding an open region $D \subset \Omega$, we wish to compute and study its motion under a velocity field F. The level set idea consists in defining a smooth function $\varphi(\vec{r}, t) : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ to implicitly represent the interface S as the set of points $\vec{r} \in \mathbb{R}^3$ where $\varphi(\vec{r}, t)$ vanishes. That is $S = \{\vec{r} \in \mathbb{R}^3 / \varphi(\vec{r}, t) = 0\}$. The function φ is called the level set function, and it has the following properties

$$
\begin{aligned}
\varphi < 0 \quad &\text{for} \quad \vec{r} \in D \\
\varphi > 0 \quad &\text{for} \quad \vec{r} \notin D \\
\varphi = 0 \quad &\text{for} \quad \vec{r} \in S
\end{aligned}
\tag{14}
$$
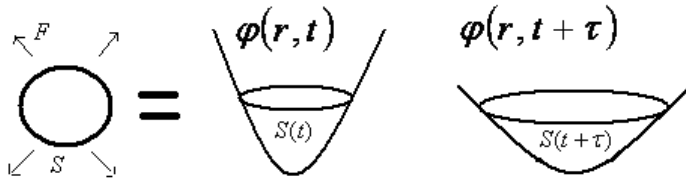
This concept illustrated by the Figure 3.

Figure 3. Level set function.

The evolution of the implicit function φ can be described by the following partial differential equation, known as Hamilton-Jacobi equation [17]

$$
\frac{\partial \varphi}{\partial t} + \|\nabla \varphi\| \, F = 0, \quad \varphi(\vec{r}, 0) = \varphi_0
\tag{15}
$$

where, $\frac{\partial}{\partial t}$ denotes a partial derivative to the temporal variable t and ∇ denotes the gradient operator. The function φ_0 embeds the initial position of the moving surface S.

3.2. The Level Set Dictionary

Once the level set function φ is defined, most of the geometrical quantities of the surface S can be represented in terms of the function φ [16]. The normal vector is given by:

$$
\vec{n} = \frac{\nabla \varphi}{\|\nabla \varphi\|}
\tag{16}
$$

The mean curvature:

$$\kappa = -\nabla \cdot \vec{n} = -\nabla \cdot \frac{\nabla \varphi}{\|\nabla \varphi\|} \tag{17}$$

The Area of S

$$L(\varphi) = \int_\Omega \delta(\varphi) \|\nabla \varphi\| \, dS \tag{18}$$

where $\delta(\varphi)$ denotes the Dirac function

$$\delta(\varphi) = \begin{cases} 1 & \text{if} \quad \varphi = 0 \\ 0 & \text{if} \quad \varphi \neq 0 \end{cases} \tag{19}$$

Moreover, we have that, the surface integral of a function f along S can be writing in function of φ

$$\int_S f(\vec{r}) dS = \int_\Omega f(\vec{r}) \delta(\varphi) \|\nabla \varphi\| \, dr \tag{20}$$

4. METHODOLOGY

The algebraic system (12) can be writing in the following form

$$\int_S A(\vec{r'}) \vec{G} \left(\left| \vec{r'} - \vec{x}_p \right| \right) dS = \vec{M}^i(\vec{x}_p) \quad 1 \leq p \leq m \tag{21}$$

where

$$A(\vec{r'}) = \sum_{k=1}^n a_k \delta(\vec{r}_k - \vec{r'}) \tag{22}$$

$$\delta(\vec{r}_k - \vec{r'}) = \begin{cases} 1 & \text{if} \quad \vec{r'} = \vec{r}_k \\ 0 & \text{if} \quad \vec{r'} \neq \vec{r}_k \end{cases} \tag{23}$$

Let consider D an open subset of Ω enclosed by the auxiliary surface S as shown in Figure 4.

We define φ as a level set function of S by

$$\varphi(\vec{r}, t) = \begin{cases} -\text{distance}(\vec{r}, S) & \text{if} \quad \vec{r} \in D \\ \text{distance}(\vec{r}, S) & \text{if} \quad \vec{r} \notin D \end{cases} \tag{24}$$

S divides the domain D into two parts, and then the level set function φ is negative inside and positive outside.

$$S = \left\{ \vec{r} \in \mathbb{R}^3, \ \varphi(\vec{r}, t) = 0 \right\} \tag{25}$$

By using the property (20) of the level set method, the Equation (21) can be writing as

$$\int_\Omega A(\vec{r'}) \vec{G} \left(\left| \vec{r'} - \vec{x}_p \right| \right) \delta(\varphi) \|\nabla \varphi\| \, dr' = \vec{M}^i(\vec{x}_p) \quad 1 \leq p \leq m \tag{26}$$
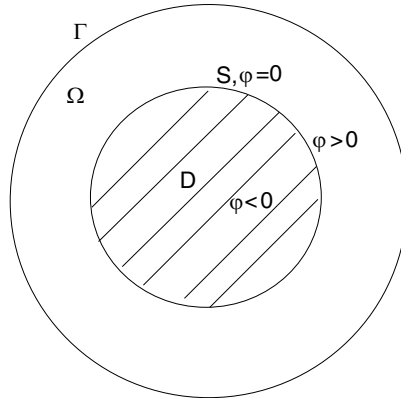
Figure 4. Level-set representation of the auxiliary surface.

The problem can be formulated as an optimization one

$$(a^*_{1\leq k\leq n},\varphi^*) = \underset{a_k,\varphi}{argmin}J(A,\varphi) \tag{27}$$

Find the amplitudes $a^*_{1\leq k\leq n}$ and the level set function φ^* which minimize the cost functional J.

$$J(A,\varphi)=\frac{1}{m}\sum_{p=1}^{m}\left\|\int_{\Omega}A(\vec{r'})\vec{G}\left(\left|\vec{r'}-\vec{x}_p\right|\right)\delta(\varphi)\left\|\triangledown\varphi\right\|dr'-\vec{M^i}(\vec{x}_p)\right\|^2 \tag{28}$$

The optimal distribution of the radiation centers $\vec{r}_{1\leq k\leq n}$ strongly depends on the area of the auxiliary surface. By shifting the sources into the conducting body the scattered field function becomes more smooth on the surface of the body and the fulfillment of the boundary conditions in the region between collocation points is improved [7, 8]. However, the shift of the auxiliary surface is restricted by the location of the scattered field singularities [10], ignoring this point leads to the divergence of the solution. So, the area of the auxiliary surface should be added to the cost functional J as regularisation term. Therefore, we force the algorithm to search the best-suited auxiliary surface that encloses the singularities.

$$\begin{aligned}J(A,\varphi) &= \frac{1}{m}\sum_{p=1}^{m}\left\|\int_{\Omega}A(\vec{r'})\vec{G}\left(\left|\vec{r'}-\vec{x}_p\right|\right)\delta(\varphi)\left\|\triangledown\varphi\right\|dr'-\vec{M^i}(\vec{x}_p)\right\|^2 \\ &\quad+\beta L(\varphi)\end{aligned} \tag{29}$$

where β is a real-valued regularization coefficient.

4.1. Calculation of the Auxiliary Surface

The evolution of φ is described by the following Hamilton-Jacobi equation.

$$\frac{\partial \varphi}{\partial t} + \|\nabla \varphi\| F = 0, \quad \varphi(\vec{r}, 0) = \varphi_0 \tag{30}$$

We want to choose an evolution law F such that $\frac{\partial J}{\partial t} < 0$, J will decrease in the artificial time evolution during sufficiently small time interval $[0, \tau]$. By applying the chain rule, we get

$$\frac{\partial J}{\partial t} = \frac{\partial J}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial t} \tag{31}$$

from Equation (30) we have

$$\frac{\partial \varphi}{\partial t} = - \|\nabla \varphi\| F \tag{32}$$

So,

$$\frac{\partial J}{\partial t} = - \|\nabla \varphi\| F \cdot \frac{\partial J}{\partial \varphi} \tag{33}$$

An obvious selection for F is

$$F = \rho \frac{\partial J}{\partial \varphi} \quad \text{with} \quad \rho > 0 \tag{34}$$

Refer to Appendix A. for the calculation of $\frac{\partial J}{\partial \varphi}$.

By dynamically updating the level set function, the zero level set of the function is also changed. Thus, to find the optimal auxiliary surface S^* we just need to find the corresponding function φ^*, by resolving (30) with $F = \rho \frac{\partial J}{\partial \varphi}$.

4.2. Calculation of the Radiation Center Positions

Suppose φ is perturbed by a small variation $\delta \varphi$ as shown in Figure 5. Let δr be the resulting variation of the point $\vec{r}$. By taking the variations of the Equation (30) between $t = 0$ and $t = \tau$, we get

$$\delta \varphi + F \tau \|\nabla \varphi\| = 0 \tag{35}$$

We have

$$F \tau = \delta r \tag{36}$$

We find the relation between δr and $\delta \varphi$.

$$\delta r = - \frac{\delta \varphi}{\|\nabla \varphi\|} \tag{37}$$

So, the radiations center positions $\vec{r}_{1 \leq k \leq n}$ are updated as follows.

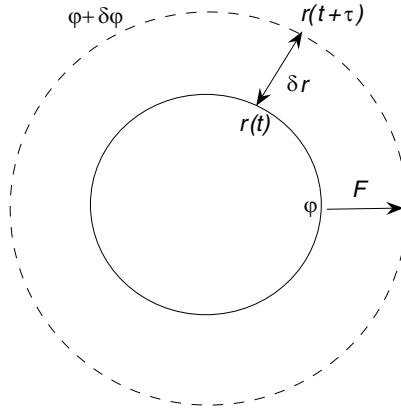$$\vec{r}_k(t + \tau) = \vec{r}_k(t) - \frac{\delta \varphi}{\|\nabla \varphi\|} \tag{38}$$

Figure 5. Deformation of shapes by the level set formulation.

4.3. Calculation of the Auxiliary Sources' Amplitudes

To find the optimal auxiliary sources' amplitudes, we should update $a_{1 \le k \le n}$ by following the descent direction of the cost function J. According to the gradient type method the descent direction is given by the negative derivative of J with respect to $a_{1 \le k \le n}$. So, we just need to compute $\frac{\partial J}{\partial a_k}$ for $1 \le k \le n$ and updating a_k^i as follows.

Choose the step size α

$$a_k(t + \tau) = a_k(t) - \alpha \frac{\partial J}{\partial a_k} \tag{39}$$

Refer to Appendix B. for the calculation of $\frac{\partial J}{\partial a_k}$.

4.4. Numerical Scheme

The following numerical scheme is implemented to determine the optimal MAS parameters.

1- Choose the initial level set function φ^0, represents the initial auxiliary surface S^0

2- Choose the initial positions and amplitudes of the radiation centers $(\vec{r}_{1 \le k \le n}^{0}, a_{1 \le k \le n}^{0})$

3- For $i \ge 1$
   - Choose the step size ρ^i, the regularization coefficient β and calculate F

$$F = \rho^i \frac{\partial J}{\partial \varphi} \tag{40}$$

- Determine the level set function φ^i by resolving the Hmilton-Jaccobi equation in the time interval $[0, \tau]$ with the initial condition $\varphi = \varphi^{i-1}$

$$\frac{\partial \varphi}{\partial t} + \|\nabla \varphi\| \, F = 0 \tag{41}$$

- update $\vec{r}_k^i$

$$\vec{r}_k^i = \vec{r}_k^{i-1} - \frac{\delta \varphi}{\|\nabla \varphi\|} \tag{42}$$

- Choose the step size α^i and update a_k^i

$$a_k^i = a_k^{i-1} - \alpha^i \frac{\partial J}{\partial a_k} \tag{43}$$

- Go to the next iteration if not converged.

The error on the boundary condition is used for convergence criterion

$$e_{p,1 \le p \le m} = \frac{\left\| \int_\Omega A(\vec{r'}) \vec{G} \left(\left| \vec{r'} - \vec{x}_p \right| \right) \delta(\varphi) \|\nabla \varphi\| \, dr' - \vec{M}^i(\vec{x}_p) \right\|}{\left\| \vec{M}^i(\vec{x}_p) \right\|} \tag{44}$$

This iterative process will continue until a stop criterion is satisfied, typically when the errors $e_{p,1 \le p \le m}$ exhibit, between two iterations, becomes smaller than a predefined threshold.

The proposed numerical scheme provides three degrees of freedom (auxiliary surface, positions and amplitudes of the radiation centers) to achieve any predesigned boundary condition error, which is a great advantage over classical MAS implementations. Indeed, in the classical MAS implementations, the auxiliary surface and the radiation center positions are fixed beforehand. So, the accuracy is not automatically adjustable, the only degree of freedom is the auxiliary sources' amplitudes. Generally, these techniques are limited to the case when the distance d between the physical surface Γ and the auxiliary surface S satisfies the condition $d < R_{\min}$, where $R_{\min}$ is the minimal radius of positive curvature of the surface Γ, see [10]. Several numerical methods have been proposed to overcome these constraints for the case of two-dimensional scattering problems, such as [18, 19]. By using level set technique. The proposed method shows a great potential to determine the optimal MAS parameters that satisfy a predesigned accuracy for three-dimensional scattering problems.

5. NUMERICAL EXAMPLE

In order to show the feasibility of this method, we present two numerical examples of construction of optimal MAS parameters. The

aim of the first example is to show the ability of the proposed scheme to locate scattered field singularities. The second example compares the accuracy and the computational cost obtained by the proposed scheme and those obtained by the standard MAS implementation.

5.1. Perfect Electric Conductor Ellipsoid

We consider a PEC ellipsoid illuminated by an incident plane wave at 100 MHz. The considered ellipsoid is defined by the following equation

$$\frac{x^2}{4} + \frac{y^2}{2} + z^2 = 0.1 \qquad (45)$$

The Figure 6 shows contour plots at 0, 23, 50 and 70 iterations on a slice through the middle of the auxiliary surface. It is well-known that the scattered field from a metallic ellipsoid has phase centers located in the ellipsoid foci's region [10]. Figure 6 shows the construction of the auxiliary surface where the boundary condition error does not exceed 1%. We deduce that the obtained surface encloses the ellipsoid foci's region, which is consistent with the analytic result.
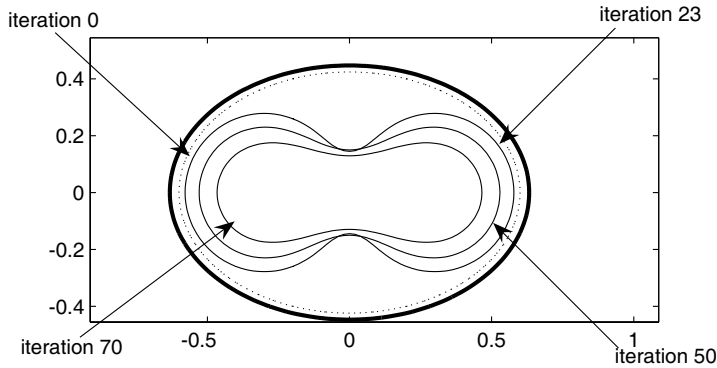

Figure 6. Singularities location of a PEC ellipsoid.

5.2. Perfect Electric Conductor Dumbbell

We consider a PEC smooth dumbbell, consists of two uniform spheres of radius $R = 0.3$ m joined by a cylinder of length $l = 0.5$ m and radius $r = 0.1$ m, as shown in Figure 7. The dumbbell was illuminated by an incident plane wave at $f = 100$ MHz.

As we have noticed in previous sections, to start the numerical scheme one needs to have an initial guess for both, the auxiliary surface and the distribution of the radiation sources. Since we have no prior knowledge on either unknown, we proceed as follows:
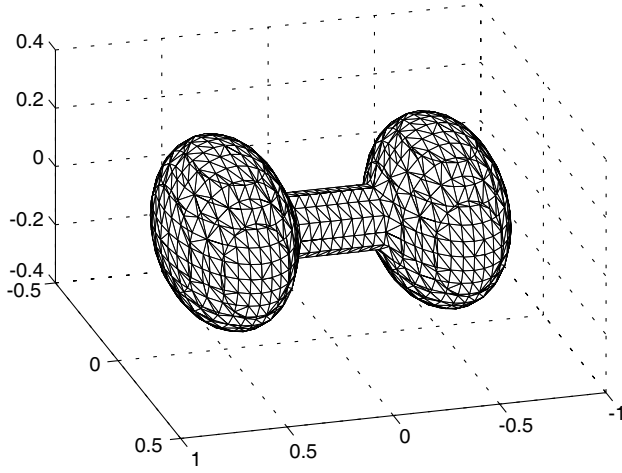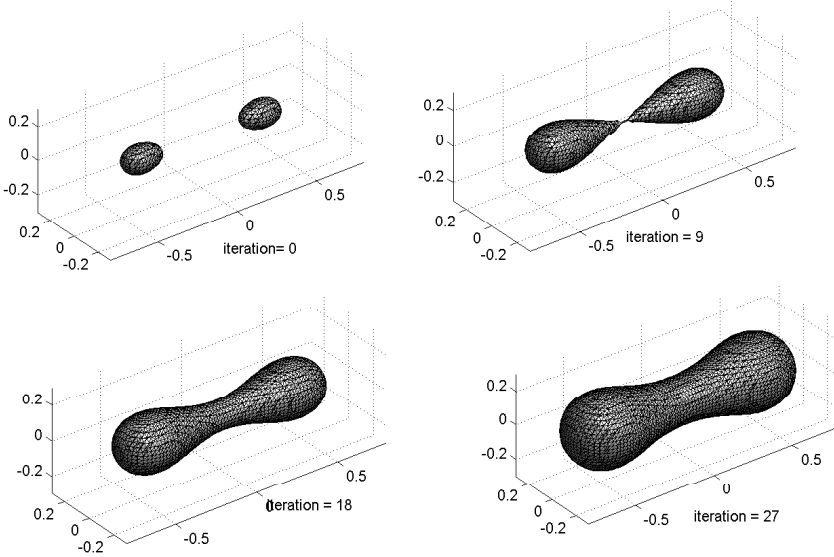
Figure 7. PEC smooth dumbbell.

Figure 8. Auxiliary surface evolution at 0, 9, 18 and 27 iterations.

- We define the initial auxiliary surface S_0 as two spheres placed within the conductor.
- We take $n = 150$ sources uniformly distributed on S_0.
- We take $m = 150$ collocation points uniformly distributed on the

conductor surface.

In the numerical implementation and in order to have a good accuracy, we have replaced the delta function δ by the following smooth function

$$\delta_h(\varphi) = \frac{h}{\pi(\varphi^2 + h^2)} \tag{46}$$

h is chosen sufficiently small.

The proposed method has been developed and implemented within MATLAB environment, and the level set Toolbox of Ian Mitchell [15]. The Figure 8 shows the auxiliary surface evolution at 0, 9, 18 and 27 iterations.

Once the proposed scheme converges, we calculate the bistatic RCS of the dumbbell. Figure 9 shows a comparison between the bistatic RCS values obtained from the proposed scheme and those obtained by the standard MAS implementation as described in [10]. The result from FEKO electromagnetic simulation software is taken as reference. With FEKO environment, the dumbbell surface is modelled by PEC impedance and the RCS is calculated by the Method of Moments.

The standard MAS is implemented according to the recommendations [10] with the following configuration
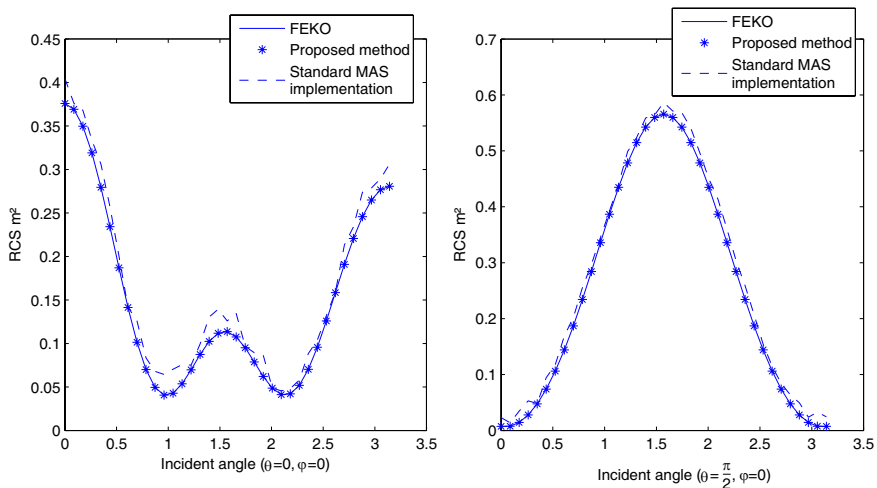

Figure 9. Comparison between the bistatic RCS values obtained from the proposed scheme and those obtained by the standard MAS implementation.

  - We take the distance d between the auxiliary surface and the boundary as $d = \frac{R_{\min}}{2}$, where $R_{\min}$ is the minimal radius of positive curvature of the dumbbell's surface.
  - We take $n = 300$ sources uniformly distributed on the auxiliary surface.
  - We take $m = 300$ collocation points uniformly distributed on the dumbbell's surface.

The above implementation leads to a boundary condition error $e = 5\%$. However, the proposed method leads to a boundary condition error $e = 0.1\%$ with 150 auxiliary sources and 150 collocation points. Which proves that the proposed method is able to achieve high accuracy with less implementation cost than the standard MAS implementation.

6. CONCLUSIONS

We have reported a numerical scheme to determine the optimal MAS parameters for three-dimensional scattering problems, by using the level set technique. Comparison between the RCS values obtained from the proposed scheme and those obtained from the standard MAS implementation, showing that the proposed method can achieve high accuracy with less implementation cost. We have limited our theoretical study to perfect electric conductors, but the proposed method can be easily extended to study dielectric objects.

APPENDIX A.

Let V a space of functions defined over Ω. The Gateaux differential is defined in the sense of distributions as:

For a given $F : V \mapsto \mathbb{R}$ which maps elements from a space V to real numbers, we say that $G(\varphi)$ is the Gateaux differential of $F(\varphi)$ if

$$G(\varphi) = \lim_{\mu \to 0} \frac{d}{d\mu} F(\varphi + \mu h) \qquad (A1)$$

Normally, we write $\frac{\partial F}{\partial \varphi} = G(\varphi)$.

The cost functional J is witting as

$$J(A, \varphi) = g(A, \delta(\varphi) \left\| \nabla \varphi \right\|) + \beta L(\varphi) \qquad (A2)$$

where

$$g(A, X) = \frac{1}{m} \sum_{p=1}^{m} \left\| \int_{\Omega} A(\vec{r'}) \vec{G} \left(\left| \vec{r'} - \vec{x}_p \right| \right) X dr' - \vec{M}^i(\vec{x}_p) \right\|^2 \qquad (A3)$$

and

$$L(\varphi) = \int_{\Omega} \delta(\varphi) \left\| \nabla \varphi \right\| dS \tag{A4}$$

So,

$$\frac{\partial J}{\partial \varphi} = \frac{\partial (\delta(\varphi) \left\| \nabla \varphi \right\|)}{\partial \varphi} \frac{\partial g}{\partial X} \big|_{X = \delta(\varphi) \left\| \nabla \varphi \right\|} + \beta \frac{\partial L(\varphi)}{\partial \varphi} \tag{A5}$$

we have from [16]

$$\frac{\partial L}{\partial \varphi} = -\delta(\varphi) \nabla \cdot \frac{\nabla \varphi}{\left\| \nabla \varphi \right\|} = \delta(\varphi) \kappa(\varphi) \tag{A6}$$

and

$$\begin{aligned}
\frac{\partial (\delta(\varphi) \left\| \nabla \varphi \right\|)}{\partial \varphi} &= \delta'(\varphi) \left\| \nabla \varphi \right\| - \delta(\varphi) \nabla \cdot \frac{\nabla \varphi}{\left\| \nabla \varphi \right\|} \\
&= \delta'(\varphi) \left\| \nabla \varphi \right\| + \delta(\varphi) \kappa(\varphi)
\end{aligned} \tag{A7}$$

By applying the definition of the Gateaux differential we get,

$$\frac{\partial g}{\partial X} = \frac{2}{m} \sum_{p=1}^{m} A(\vec{r}) \vec{G}(|\vec{r} - \vec{x}_p|) \cdot \int_{\Omega} \left[A(r') \vec{G} \left(\left| \vec{r'} - \vec{x}_p \right| \right) X - \vec{M}^i(\vec{x}_p) \right] dr' \tag{A8}$$

APPENDIX B.

By applying the chain's rule, we get

$$\frac{\partial J}{\partial a_k} = \frac{\partial J}{\partial A} \frac{\partial A}{\partial a_k} \tag{B1}$$

We have

$$A(r) = \sum_{k=1}^{n} a_k \delta(\vec{r}_k - \vec{r}) \tag{B2}$$

So, it is easy to see that

$$\frac{\partial A}{\partial a_k} = \delta(\vec{r}_k - \vec{r}) \tag{B3}$$

By applying the definition of the Gateaux differential we get,

$$\begin{aligned}
\frac{\partial J}{\partial A} &= \frac{2}{m} \sum_{p=1}^{m} \delta(\varphi) \left\| \nabla \varphi \right\| \vec{G}(|\vec{r} - \vec{x}_p|) \\
&\quad \cdot \int_{\Omega} \left[A(\vec{r'}) \vec{G} \left| \vec{r'} - \vec{x}_p \right| \right) \delta(\varphi) \left\| \nabla \varphi \right\| - \vec{M}^i(\vec{x}_p) \right] dr' \tag{B4}
\end{aligned}$$

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