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# AN EFFICIENT HYBRID-SCHEME COMBINING THE CHARACTERISTIC BASIS FUNCTION METHOD AND THE MULTILEVEL FAST MULTIPOLE ALGORITHM FOR SOLVING BISTATIC RCS AND RADIATION PROBLEMS

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**Abstract**—A numerically efficient approach for the rigorous computation of bi-static scattering and radiation problems is presented. The approach is based on an improvement of a previous method scheme that combines the Characteristic Basis Function Method (CBFM) and the Multilevel Fast Multipole Algorithm (MLFMA). The approach combines Characteristic Basis Functions (CBFS) and subdomains functions for reducing the CPU time in the pre-process and in the solving iterative process for simple or multiple excitations. It is intended for use in very large cases where an iterative solution process cannot be avoided, even considering the matrix size reduction achieved by the CBFM. This reduction is particularly important for solving radiation or bistatic problems in which an integral equation is solved once.

## 1. INTRODUCTION

In recent years, CBFM [1, 2] has been developed for solving large electromagnetic scattering or radiation problems. In this method, the unknown currents on the scatterer are expressed in terms of a relatively small set of pre-computed CBFS that extend over relatively large surfaces called "blocks". The CBFS can be considered as macro basis functions because the size of the blocks is on the order of or larger than the wavelength. The use of these functions leads to a

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reduced matrix whose size is much smaller than that obtained in the conventional Method of Moments. As a consequence, direct solvers can be applied to solve problem of moderate electrical size, which could only be addressed previously by relying upon iterative techniques.

However, when the size of the problem increases and the number of CBFS used to model the currents becomes very large, the CBFM system matrix can become so large that it precludes its solution without resorting to iteration, despite a significant reduction in the matrix size realized via the CBFM. Additionally, the memory needed to store the reduced matrix can be a problem as well. One of the most common approaches to ease the burden on the computational resources entails storing only the near-field terms of the Method of Moments (MoM) coupling matrix and computing the far-field interactions via the MLFMA, [3–7]. With MLFMA, the whole geometry is compartmentalized into several first-level cubical groups, which, in turn, generate higher-order cubes as they are grouped. For the first level, the cubes contain a few basis functions, and the coupling between basis functions associated with geometrically close cubes is calculated in a rigorous way and stored for later use. The application of MLFMA-CBFM entails the storage of only the near-field terms of the coupling matrix and the efficient computation of the far-field interactions in the iterative process.

MLFMA-CBFM presents several computational advantages compared with the MoM or MLFMA-MoM techniques in terms of the memory need and CPU time. If the memory requirements of the technique are analyzed, it can be concluded that the reduced matrix and the aggregation and disaggregation terms have a huge influence on the total storage requirements. Thus, there are two main reasons for the reduction in the memory requirements. First, CBFM reduces the number of unknowns of the problem and therefore the size of the matrix coupling impedance and the multipole aggregation and disaggregation terms. The second advantage of this method is the reduction in the CPU time. There are several reasons. First, given the reduction of the unknowns achieved by CBFM, the equation system size will be smaller, and therefore, it will be solved faster. Second, when using SVD, an orthogonal macro-basis functions set is obtained, which improves the conditionality of the matrix equation system. As a consequence, the CPU time employed to solve iteratively the problem is significantly diminished. The third reason is that the iterative process computes the matrix-vector products in an efficient way, as it is described in the MLFMA method.

Figure 1 shows the number of unknowns of the MLFMA versus the number of low-level and high-level basis functions for a plate of

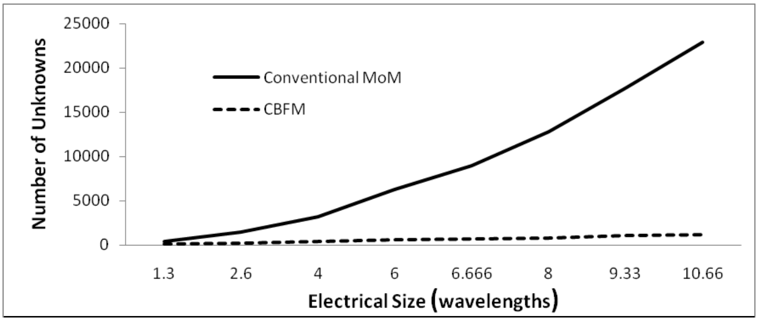


**Figure 1.** Comparison between the number of unknowns using CBFM and MoM.

four meters when the frequency increases. It can be noticed that the ratio between the number of CBFS and the number of low-level basis functions decreases. A more detailed study of this behavior is presented in [2]. Thus, large blocks shall be used.

On the other hand, smaller groups of the MLFMA should be used because, for a small group size, the number of subdomains that belong to adjacent groups is small, and therefore only a short number of rigorous low-level coupling terms must be computed when calculating the reduced matrix. In conclusion, the most convenient approach is to use large blocks and small groups in the MLFMA-CBFM approach. However, there is a limit. If the block size is large, a large number of coupling terms between CBFS (the terms of the reduced matrix that are needed to compute the coupling between CBFS in the same blocks or in contiguous blocks) must be computed rigorously because there will be a large number of CBFS in each block.

The MLFMA-CBFM reduces the CPU time per iteration in the iterative process, but it can require a long time in the pre-process step. When the amount of time employed in solving the problem is large compared to the pre-process time, the use of high-size blocks is recommended. The total time is reduced compared with MLFMA-MoM because the time spend in the pre-process is highly compensated due to the reduction of the time employed in the iterations, which is the case of monostatic RCS problems, in which several excitations shall be solved using the same reduced matrix [8, 9]. However, if the weight of the iterative process is low, such as in antenna radiation pattern computation or bistatic RCS problems, the pre-processing time can be excessive and the CBFM-MLFMA less efficient.

Different sizes for blocks and MLFMA groups can be considered to overcome this burden for these types of problems. Block sizes greater than the group size can be used. In this case, there will be $N$ groups inside each block, which can be a solution, but it has a limitation. If a large block size is used, a large number of CBFS ($P$) will be defined over the entire block, and, of course, defined over all the $N$ groups. If the block is sufficiently large, the number of CBFS over each group can be greater than $M$, the number of low-level basis functions required to represent the current on the group. For every group inside it, aggregation and disaggregation terms must be computed for every CBF, so $N * P$ terms must be computed, compared with only $N * M$ terms using MLFMA with a low-level basis. This method can lead to large memory requirements. It was found by numerical experimentation [10] that the best choice is to consider the block and group size with the same size, where one wavelength or one half of the wavelength are good choices for the size. In [9], the benefits of using the same block size and MLFMA group size were demonstrated, where, even for only one excitation, it is more efficient than a MLFMA based on subdomains.

A new efficient alternative in the MLFMA-CBFM approach for reducing the pre-process time is proposed here. The goal is to use a large block size to reduce the number of CBFS (unknowns) but a low MLFMA group size to avoid the rigorous computation of a high number of low-level coupling terms. To do so, a new hybrid approach that combines CBFS with subdomains will be presented. With this combination, the problem with memory requirements described in the previous paragraph disappears.

The organization of the paper is as follows. Section 2 presents this hybrid approach to overcome the MLFMA-CBFM technique when solving for antenna radiation in bistatic RCS problems. In Section 3, some representative results obtained by using the numerical technique described in this paper are presented. Some conclusions are given in Section 4.

## 2. ALTERNATIVE APPROACH

An efficient alternative in the MLFMA-CBFM approach for reducing the pre-process time of this type of problem is proposed. The goal is to use a large block size to reduce the memory needs of the problem but to use a low MLFMA group size to avoid the rigorous computation of a high number of low-level coupling terms. This hybrid approach must solve the memory problems associated with using different block and group sizes described in the previous section. Using this strategy, an efficient approach can be developed to solve bistatic RCS and antenna radiation problems compared with MLFMA-CBFM and MLFMA-

MoM.

This approach is based on the use of both the low-level basis functions and the macro-basis functions in the solution of the problem. First, the whole geometry is compartmentalized in regions as defined in MLFMA using the first-level box size (group size). In the multilevel scheme, the first-level boxes are grouped, and higher order regions are created. To find an efficient way to define CBFS, the called block-level is defined as the level at which blocks are defined. Thus, every block will be all the geometry contained in any box of this block level. It is assumed that the relation between the block size $B$ and region size $R$ can be expressed as

$$B = NR \tag{1}$$

where $N$ is the number of regions (MLFMA first level box) in each block.

Thus, every block will contain several groups inside it, and every one will have a set of low-level basis functions that will be part of the expansion of the CBFS contained in the block.

To solve the problem, the MLFMA-CBFM formulation is used.

$$\left[Z^R\right] \cdot [J] = \left[V^R\right] \tag{2}$$

where $[V^R]$ represents the excitation vector over every high-level basis function and $[J]$ is the vector of unknowns. $[Z^R]$ represents the operator of the MLFMA-CBFM.

The excitation vectors are calculated by multiplying the coefficients for each CBF by the inner product between the impressed field $W$ and the low-level testing functions:

$$V_l = \sum_{i=1}^{N_n} \alpha_{l,n}(i) \langle W, R_i(u, v)\rangle \tag{3}$$

where $\alpha_{l,n}(i)$ denotes the expansion of CBF $l$ in terms of subdomains, $W$ is the impressed field over the surface, $R_i(u, v)$ is the testing function $i$-th and $V_l$ represents the $l$-th element of the excitation vector $[V^R]$.

As concluded in the introduction section, the terms of $[Z^R]$ that correspond to near field coupling are the main problem in terms of the pre-process time when using large blocks in MLFMA-CBFM, and the main goal in the present approach is to avoid this problem.

It is common to use iterative solvers when the size of the problem is large to avoid the computational cost of inverting the matrix. In these methods, an initial solution is proposed, and some operations are performed to reduce its initial error. Some of them are matrix-vector products, which are the most computational expensive operations.

$$[Y] = \left[Z^R\right] \cdot [X] \tag{4}$$

To avoid the use of the reduced matrix in these products, the proposed technique uses MLFMA-MoM.

$$[y] = [Z] \cdot [x] \tag{5}$$

The main advantage of using this formulation is that the conventional MLFMA-MoM matrix, which takes the information of the coupling between the subdomains of the problem, is used. Its computation uses small regions, and thus a small number of low-level basis functions must be computed. As a consequence, the pre-process CPU time is reduced compared to MLFMA-CBFM using the same block and group size, and no memory problems appear with different block and group sizes.

To obtain the $x$ vector, a low-level basis function expansion of every term in the $[X]$ is applied:

$$x_i = \sum_{i=1}^{N_n} \beta_{l,n}(l) X_l \tag{6}$$

where $x_i$ is the amplitude of the $l$ CBFS in the $i$ low-level basis functions. Analogously, the elements of the vector $[Y]$ are computed once the MLFMA-MoM matrix-vector product is found.

Consider a set of $P$ high-level basis functions defined over a block and $N$ groups inside it. These functions are, of course, defined over all the $N$ groups, so every group will have the definition of a part of every CBF. If there are $S$ low-level basis functions on that block, it can be assumed that there will be a mean of $M = S/N$ low-level basis functions per group.

MLFMA-MoM products for a block have a complexity that is proportional to $S * \log(S)$. This approach adds only the expansion operations of the high-level vectors into the low-level vectors before the product and the low-level to the high level once the result of the product with a complexity proportional to $S * P$ is obtained. Thus, the final complexity of the approach is proportional to $S * (2 * P + \log(S))$.

Now a computational comparison between the proposed method and MLFMA-MoM and MLFMA-CBFM is presented using the same block and group size. It is important to note that this technique will be efficient when the iteration process time does not dominate the total time, as is typical for bistatic RCS and antenna radiation pattern problems.

In terms of memory, the proposed approach requires a slightly lower amount of memory than the MLFMA-MoM method due to the unknown reduction achieved by CBFM, which reduces the vectors used in the iterative process. In terms of the CPU time, the pre-process step will require somewhat more time due to the CBF computation process;

however, the CPU time required to achieve the solution in the iterative process will be lower because of the lower number of unknowns and the best conditioning of the problems due to the orthogonalization of the CBFS.

If the proposed approach is compared to MLFMA-CBFM, the memory required is moderately higher because the aggregation and disaggregation of every unknown must be stored in terms of subdomains instead of CBFS (the total number of subdomains in the problem will be always higher than the total number of high-level basis functions). However, the pre-process time is drastically reduced because the low-level rigorous coupling computations are only made between elements that are in or adjacent to a group of size $M$ instead of elements that are in or adjacent to its block, which are $P/S$ times larger.

In conclusion, using this technique, a reduction in the pre-process time is achieved compared to traditional MLFMA-CBFM, but it is slightly larger than the pre-process time for MLFMA-MoM. However, in low-weight iteration-process problems, solver CPU time is reduced compared to MLFMA-MoM, so the total time could be lower.

## 3. RESULTS

First, a simple test case is considered. It is a $10\lambda$-plate on the $XY$-plane fed with a vertical dipole placed in the position (0.0, 0.0, $1.0\lambda$), see Figure 2. The radiation pattern cut $\varphi = 0°$ was computed. A

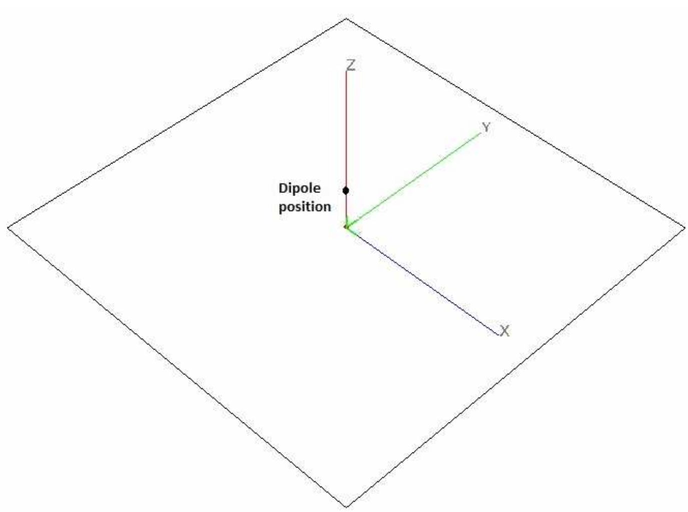


**Figure 2.** $10\lambda$ plate fed by a vertical dipole.

comparison of the results and an analysis of the time execution with all methods are shown in Figure 3 and Table 1, respectively. There is good agreement in the results.

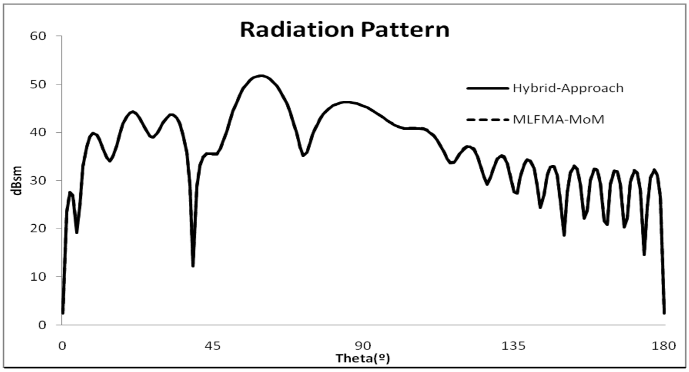


**Figure 3.** Antenna pattern of a vertical dipole located over a $10\lambda$ plate.

**Table 1.** Computational analysis of the $10\lambda$ plate case.

| Method | Block size | MLFMA group size | $N$ of Unknowns | |
|---|---|---|---|---|
| | | | CBFS | Subdom |
| MLFMA MoM | | 0.25 | - | 19800 |
| MLFMA CBFM | 0.5 | 0.5 | 11642 | - |
| MLFMA CBFM | 1 | 1 | 5127 | - |
| MLFMA CBFM | 2 | 2 | 2676 | - |
| Hybrid approach | 1 | 0.25 | 5127 | 19800 |
| Hybrid approach | 2 | 0.25 | 2676 | 19800 |
| Hybrid approach | 1 | 0.5 | 5127 | 19800 |
| Hybrid approach | 2 | 1 | 2676 | 19800 |

| Method | Pre-process time | Solver time | Total time |
|---|---|---|---|
| MLFMA MoM | 1’ 29” | 24’ 44” | 26’ 13” |
| MLFMA CBFM | 4’ 19” | 10’ 1” | 14’ 20” |
| MLFMA CBFM | 10’ 59” | 3’ 26” | 14’ 25” |
| MLFMA CBFM | 2 h 32’ 8” | 1’ 8” | 2 h 33’ 16” |
| Hybrid approach | 2’ 25” | 6’ 31” | 8’ 56” |
| Hybrid approach | 4’ 6” | 2’ 29” | 6’ 35” |
| Hybrid approach | 4’ 37” | 5’ 44” | 10’ 21” |
| Hybrid approach | 12’ 18” | 1’ 44” | 14’ 2” |

Some conclusions can be drawn from the results in Table 1. When the block size is increased in MLFMA-CBFM, the number of unknowns is reduced, and as a consequence, the system with a lower number of unknowns is solved. Therefore, the solver time is also reduced. However, the pre-process time is high for two main reasons. First, SVD must to be applied to a higher number of unknowns, and more low-level basis function rigorous coupling terms must be computed.

Using the hybrid approach, the pre-process time is similar to that of the MLFMA-MoM approach with the same group size, and the solver time is slightly higher than that of the MLFMA-CBFM approach with the same block size. However, in terms of the total time, a better performance is obtained compared to the other methods.

It is important to set properly the block size in the hybrid approach. We should choose a proper block size to minimize the total CPU-time, composed by the pre-process and solving times. As shown in Table 1, the reduction in the number of unknowns is low using a small block size and we do not achieve the best CPU-time reduction in the solution process. But when we increase the block size, we improve this CPU-time in the solution problem, but we spend a slightly higher pre-process time because we must apply SVD to a higher number of low-level subdomains.

In the analysis shown in Table 1, we can see the CPU-Time using different block sizes (directly associated with the block level). In our experience, a block size of about 1 or $2\lambda$ is proper, which is related

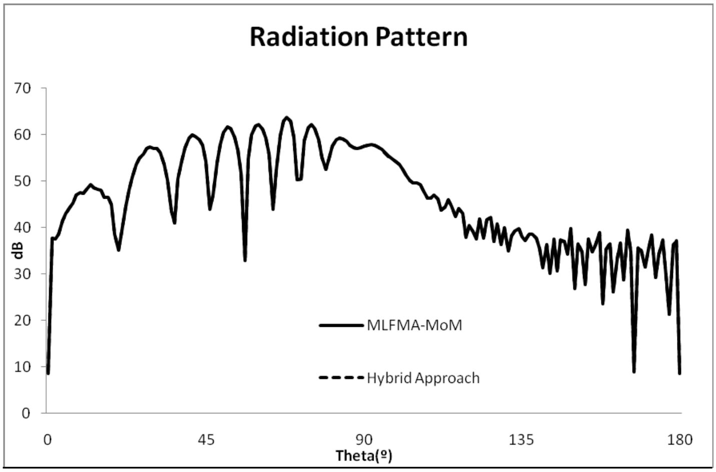


**Figure 4.** Antenna pattern of a horn located in front of the $40\lambda$ plate.

with a block level of 3 or 4, assuming the rule of

$$BS = FS * 2^{(\text{level-1})} \tag{7}$$

where $BS$ is the block size and $FS$ is the level 0 MLFMA group size. It is assumed a group size of a quarter of wavelength.

This difference between the execution times increases for larger electrical size problems, e.g., when the frequency of the simulation is

**Table 2.** Computational analysis of the $40\lambda$ plate case.

| Method | Block size | MLFMA group size | $N$ of Unknowns | |
|---|---|---|---|---|
| | | | CBFS | Subdom |
| MLFMA MoM | | 0.25 | - | 319200 |
| MLFMA CBFM | 0.5 | 0.5 | 190617 | - |
| MLFMA CBFM | 1 | 1 | 82562 | - |
| Hybrid approach | 1 | 0.5 | 82562 | 319200 |
| Hybrid approach | 2 | 0.25 | 45897 | 319200 |

| Method | Pre-process time | Solver time | Total time |
|---|---|---|---|
| MLFMA MoM | 22' 6" | 6 h 10' 1" | 6 h 32' 7" |
| MLFMA CBFM | 28' 51" | 1 h 55' 11" | 2 h 24' 2" |
| MLFMA CBFM | 3 h 04' 1" | 55' 32" | 3 h 59'33" |
| Hybrid approach | 30' 31" | 1 h 14' 2" | 1 h 44' 32" |
| Hybrid approach | 27' 13" | 43' 21" | 1 h 10' 34" |

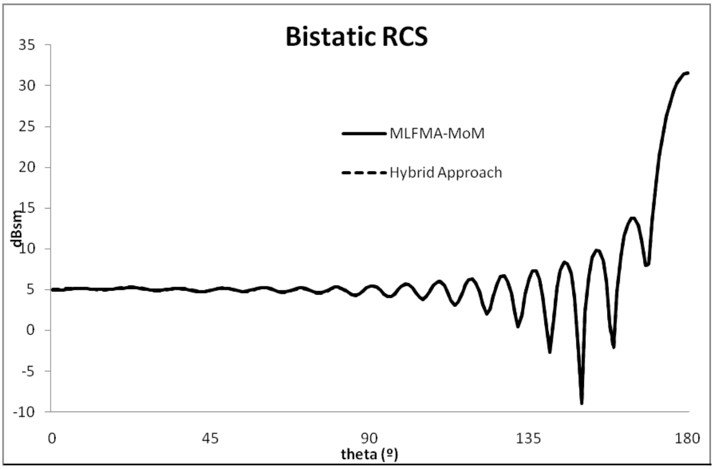


**Figure 5.** Bistatic RCS values for the sphere test case.

increased by a factor of four. Comparisons between both methods and between the computational parameters are shown in Figure 4 and Table 2, respectively.

Another simple case is simulated in order to compare the hybrid approach and MLFMA-MoM. Bistatic RCS values from a 1 meter radius sphere at the frequency of 1 GHz computed for the $\theta = 0°$ and $\varphi = 0$ incident direction and a sweep in $\varphi = 0$ from 0° to 180° for the observation angles with $\theta$-$\theta$ polarization are shown in Figure 5.

The CPU-Time spent in the simulation is exposed in Table 3, and we can set the same conclusions that previous test case.

Next, consider a complex antenna. The circular two-layer reflectarray with a diameter of 406 mm as described in [11] was also analyzed. The reflectarray was designed to radiate in the direction of $\theta_o = 19°$ and $\varphi_o = 0°$ at 11.95 GHz. Figure 6 shows an image

**Table 3.** Computational analysis of the sphere case.

| Method | Block size | MLFMA group size | $N$ of Unknowns | Pre-process time | Solver time | Total time |
|---|---|---|---|---|---|---|
| MLFMA MoM | | 0.25 | 28812 | 8' 12" | 14' 11" | 22' 23" |
| Hybrid approach | 1 | 0.25 | 28812 | 10' 30" | 5' 23" | 15' 53" |

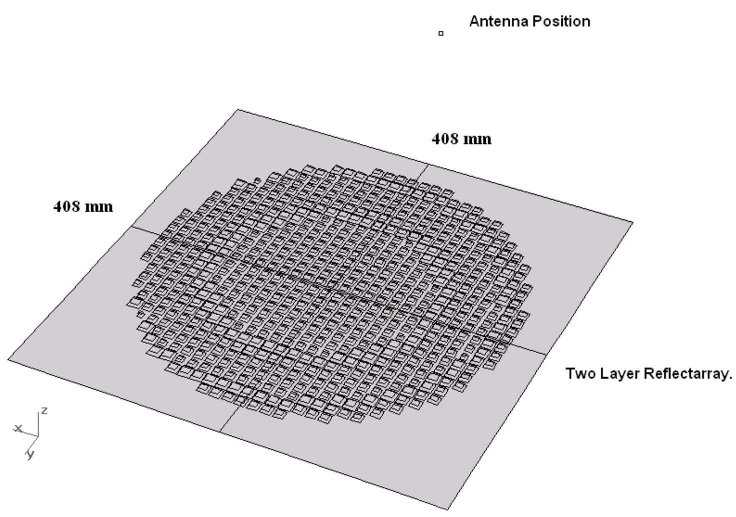


**Figure 6.** Geometrical description of the reflectarray.

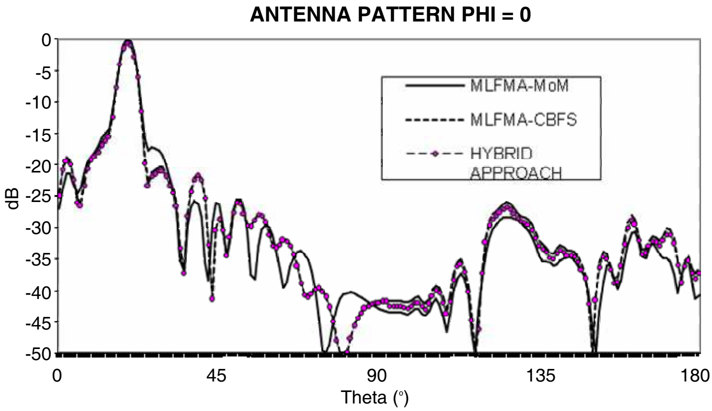


**Figure 7.** Antenna pattern of the reflectarray.

**Table 4.** Computational analysis of the reflectarray case.

| Method | Block size | MLFMA group size | $N$ of Unknowns | Pre-process time | Solver time | Total time |
|---|---|---|---|---|---|---|
| MLFMA MoM | | 0.25 | 122475 | 1 h 17' 15" | 9 h 22' 41" | 10 h 39' 56" |
| MLFMA CBFM | 0.5 | 0.5 | 73725 | 2 h 51' 03" | 5 h 22' 52" | 8 h 13' 55" |
| MLFMA CBFM | 1 | 1 | 34677 | 17 h 48' 25" | 1 h 55' 13" | 19 h 43' 38" |
| Hybrid approach | 1 | 0.25 | 34677 | 2 h 06' 11" | 2 h 29' 14" | 4 h 35' 25" |

of the geometrical model of the reflectarray and the antenna position at coordinates (−0.16, 0.0, 0.340) in units of meters. Results for the main cut of the normalized radiation pattern of the structure obtained using several approaches are shown in Figure 7. Table 4 presents the CPU-time spent for the reflectarray analysis by these approaches. The maximum in the radiation pattern appears in the direction that was specified in the design of the reflectarray.

Another case is defined by a cylinder with a flat back ended cavity, as shown in Figure 8 [12]. The cylinder is 300 mm long, and it has a radius of 150 mm. The cavity is in the center of the curved surface of the cylinder, its aperture is 150 × 75 mm in size, and the maximum depth is 2 mm.

Bistatic RCS values at a frequency of 36.0 GHz for the $\theta = 90°$

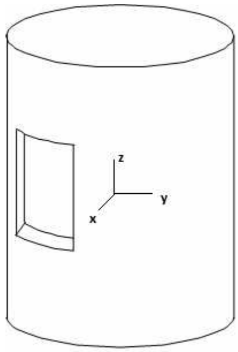


**Figure 8.** Cylinder geometry.

**Table 5.** Parameters employed in the computation of the RCS of the geometry shown in Figure 8 at 36.0 GHz.

| Method | $N$ of Unknowns | Group size | Block size | Total time | Pre-process time | Solver time |
|---|---|---|---|---|---|---|
| MLFMA -MoM | 1,275,477 | 0.25$\lambda$ | | 3 h 26' 18" | 48' 57" | 2 h 37' 21" |
| MLFMA -CBFM | 259,523 | 1.0$\lambda$ | 1.0$\lambda$ | 2 h 56' 44" | 1 h 46' 23" | 10' 21" |
| Hybrid approach | 259,523 | 0.25$\lambda$ | 1.0$\lambda$ | 1 h 26' 20" | 58' 38" | 27' 42" |

and $\varphi = 0$ incident direction and a sweep in $\varphi$ from 0° to 180° for the observation angles with $\theta$-$\theta$ polarization are shown in Figure 9, where the results obtained using MLFMA-CBFM, MLFMA-MoM and the hybrid approach are compared. The number of low-level basis functions obtained is 1,275,477. The results were obtained using a parallel version of the tool over a machine with twelve Intel(R) Xeon(R) 2.0 GHz processor with 3 GB of RAM for each processor. As can be seen in Table 5, the CPU time spent in the computation of the pre-process stage in MLFMA-CBFM with a block and group size of 2.0$\lambda$ was 1 h 46' 23", which is a long time compared to the hybrid approach, which had a CPU time of only 58' 38". Thus, this presented approach is more efficient when the block size grows.

Next, radiation pattern of a vertical dipole feeding an airplane, shown in Figure 10, is computed. The cut $\varphi = 0$ of the pattern of the structure computed with the hybrid approach and compared with MLFMA-MoM is shown in Figure 11. Table 6 shows a computational comparison between both methods.

Finally, another radiation pattern is computed for a ship illuminated by a vertical dipole. Details of the simulation are shown in

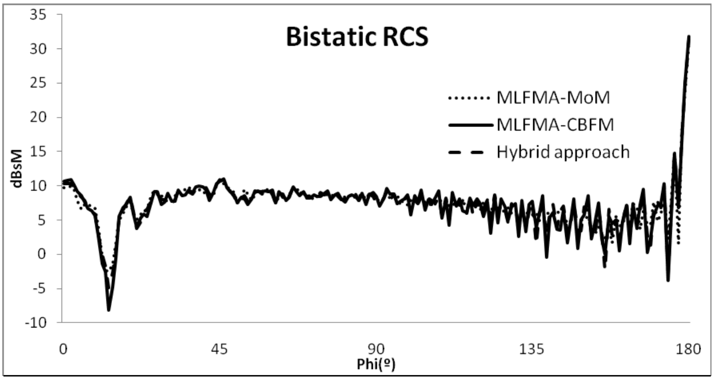


**Figure 9.** Bistatic RCS values at 36.0 GHz for the test case shown in Figure 8.

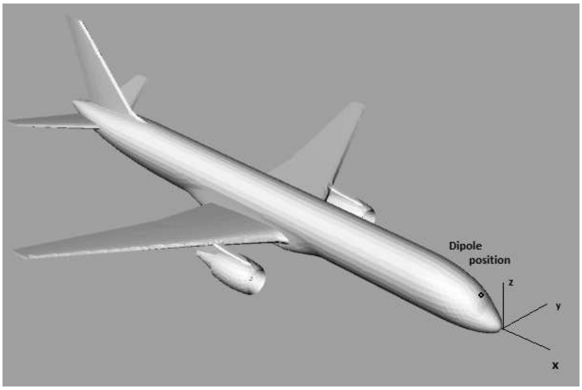


**Figure 10.** Boeing 757 fed with a vertical dipole.

**Table 6.** Computational analysis for the test case shown in Figure 10.

| Method | $N$ of Unknowns | Group size | Block size | Total time | Pre-process time | Solver time |
|---|---|---|---|---|---|---|
| MLFMA -MoM | 2,134,057 | 0.25$\lambda$ | | 29 h 20' 29" | 16 h 1' 2" | 13 h 19' 27" |
| Hybrid approach | 262,376 | 0.25$\lambda$ | 4.0$\lambda$ | 20 h 12' 41" | 16 h 59' 27" | 3 h 13' 14" |

Figure 12. The cut $\varphi = 0$ of the radiation pattern form the structure computed with the hybrid approach and compared with MLFMA-MoM is shown in Figure 13, and a computational analysis is exposed in Table 7. The results were obtained using a parallel version of the tool over a machine with sixteen AMD(R) Opteron(R) 2.2 GHz processor.

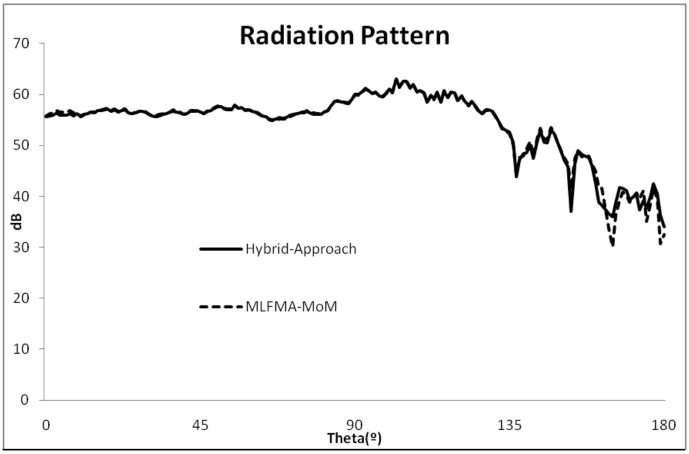


**Figure 11.** Radiation pattern for the test case shown in Figure 10.

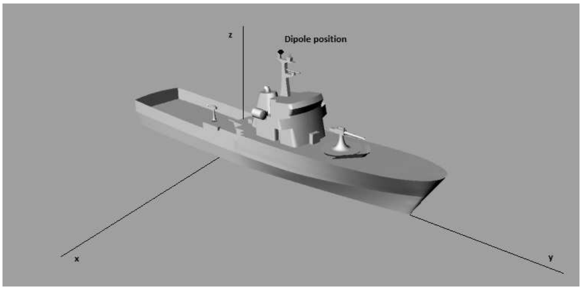


**Figure 12.** Ship fed with a vertical dipole.

**Table 7.** Computational analysis for the test case shown in Figure 12.

| Method | $N$ of Unknowns | Group size | Block size | Total time | Pre-process time | Solver time |
|---|---|---|---|---|---|---|
| MLFMA -MoM | 2,076,267 | $0.25\lambda$ | | 6 h 35' 48" | 1 h 09' 31" | 5 h 26' 17" |
| Hybrid approach | 249,242 | $0.25\lambda$ | $4.0\lambda$ | 3 h 7' 26" | 1 h 57' 23" | 1 h 10' 3" |

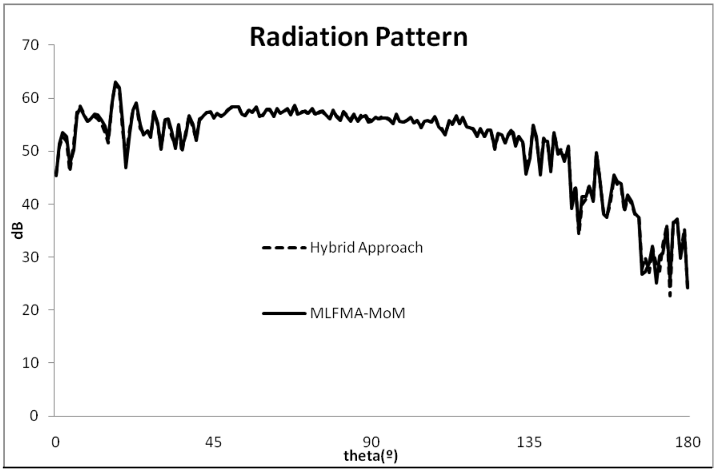


**Figure 13.** Radiation pattern for the test case shown in Figure 12.

## 4. CONCLUSIONS

A novel technique was developed for the rigorous analysis of antenna radiation problems and bistatic scattering problems. It is based on the combination of CBFM and MLFMA and reduces the CPU time during pre-processing in these problems due to a combination of high-level and low-level basis functions in the solution process. Thus, the approach is well suited for the analysis of electrically large complex targets in an efficient way, both in terms of memory requirements and CPU time. Several results were obtained to demonstrate the accuracy and computational benefits of the hybrid approach.

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