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MOLECULAR EM FIELDS AND DYNAMICAL RESPONSES IN SOLIDS WITH MAGNETIC CHARGES

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Abstract—The monopoles are theoretically defined as charges which produce fields whose divergence is, obviously, different from zero. However, the entities which have been experimentally detected in the spin-ices, with mimetic behavior to that of the magnetic monopoles, generate magneti c fields which seem to be compatible with $\nabla \cdot \mathbf{B} = 0$. This apparent contradiction can create confusion and therefore it requires explanation. In this paper we have carried out an analysis of the different electromagnetic fields in the spin-ices materials. We clarify the differences between the average fields of standard Maxwell equations with zero divergence even in spin-ices and the non macroscopic fields when there are magnetic monopoles in these materials. We give the molecular or local fields which allow us to determine the molecular polarizability. We combine the extended Clausius-Mossotti equations with the Lorentz-Drude model for obtaining the extended susceptibility and the optical conductivity which can be used for explaining the action of the electromagnetic fields in spin-ices.

1. INTRODUCTION

Since Dirac argued [1] in 1931 that a possibly explanation of the electric charge quantization requires the existence of magnetic charges, the magnetic monopoles are a permanent and recurrent problem from either theoretical or experimental points of view. A contribution to the interpretation of these monopoles was carried out in 1938 by Jordan [2] and in 1974 Hooft [3] and Polyakov [4] introduced the magnetic monopole idea as a contribution to the second great unification of the three more intense forces, electromagnetic, weak nuclear and

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strong nuclear interactions. These and other theoretical analysis have provoked that the experimental pursuit of the magnetic monopole manifestation is an issue that is both liminal and subliminally present since 1982 [5] and now is receiving a payment of attention in both high energy physics [6] and solid state (SS) physics [6–15].

Other lines of research of magnetic charges and magnetic currents have been developed for the calculation of the fields in the wave scattering of diffraction in the aperture antennas [16], in the studies on the memristors [17] and in the study of equivalent sources in the reconstruction of 3D surfaces [18]. These equivalent field sources are constituted of superficial electric and magnetic currents located in the closed surface. The consideration of the currents of magnetic charges can be an excellent and effective mathematical procedure for calculating complex electromagnetic problems [16–18]. However, these procedures are developed in order to obtain more accuracy in the solutions of the Maxwell equations in very complex electromagnetic systems and do not try to assign independent phenomenology to these magnetic charges and currents.

In any case, the intuitive idea of Dirac and the other pioneers in the magnetic charge theories is being experimentally materialized and appearing now, maybe in a different form what they thought. But some key clues of these theories are serving of inspiration for understanding some experimental electromagnetic phenomenology raised on magnetic materials such as the spin-ices [11, 12]. The presence of magnetic charges detected within the condensed matter does not correspond to elemental particles, but they are due to the spin flip occurrence in low energy excitation states of magnetic structures, such as the spin-ices, or by means of image states of external electrons in topological insulators [9]. These magnetic charges have an effective behavior which is mimetic to that of the magnetic monopoles and their phenomenology is due to the interplay of the interaction with the material medium and the application of an external field. Therefore, in SS physics the magnetic monopoles are effective models for explaining the phenomenological behavior of determined crystals. These models are submitted to the diatribe and speculation being the unique validation the experimental evidences, which fortunately are easier to attain than those concerning the high energy physics. In addition, in SS physics the definition of magnetic monopoles is accepted with the apparent contradictory condition of being $\nabla \cdot \mathbf{B} = 0$, and some researches consider that the existence of these magnetic monopoles are coherent with the standard Maxwell equations, which can induce the question: what kind of magnetic monopole is it?

The central idea of the spin-ice magnetic monopoles is the existence of low energy excitations of the spin field chain when a spin flip is produced between two contiguous crystalline tetrahedral basis. Then the configuration suffers a change in such a way that the magnetic field sources are similar to those due to the existence of micromagnets with two poles, the denominated dumbbells [7, 8, 10–12, 19]. Then, each micromagnet can be considered as a pole-antipole pair which can be broken when the length of the string which form these pairs increases. The propagation of these spin flips can be extended in the crystal, generating a system whose physical image can be represented as similar to Dirac-like strings in which the corresponding poles do not practically suffer any mutual interaction. However, the strings in these spin-ices are theoretically observable [20] and experimentally observed [21], and this point may represent a difference with respect to those of the Dirac theory [20]. In addition, the monopole Dirac charge is quantized in contrast to the effective magnetic charge of the spin-ice strings which can have any value.

The pole-antipole micromagnet breaks are favored when the strength of their inter-pole interaction is less than that existing with the eventual application of an external magnetic field. Then, the existence of two component poles coming from the micromagnet can be possible and they can travel independently over the crystal. The proliferation of the deconfined pole-antipole pairs produces a quasiparticle gas of magnetic charges whose behavior can be visualized as a magnetic plasma [22]. The result is a density of dissociated pairs immersed in a system of non broken micromagnets. When the density of deconfined magnetic charges is sufficiently large, the system can be studied as an interacting gas of magnetic charges whose dynamics is controlled by the Coulomb-like interaction [7, 8, 11, 12, 19, 22] $V_{ij} = \frac{K}{4\pi}\frac{g_i g_j}{r_{ij}}$, where K is a constant which defines the unit of magnetic charge (g_i).

The substitution of the spin system configuration by these interacting magnetic charges implies that the magnetic structural system can be understood as magnetic particle gas within the vacuum background dominated by dielectric and magnetic responses under the electromagnetic external interactions. However, in a real state of this crystal different from that extremal gas situation, deconfined single monopoles of different charge sign can coexist with pole-antipole coupled micromagnets [11, 12]. In this intermediate situation the classical field concepts should be considered in order to establish the equations that can be applied within the Classical Electrodynamics. This analysis is necessary for a future construction of the classical Lagrangian function with magnetic monopoles in solid which have to be used for obtaining, by the correspondence principle, the quantum Schroedinger equation and the corresponding quantum field theory for these systems. Then, in the limit of constant density of deconfined magnetic charges, the corresponding dual electric charge system could be the jellium model of an electronic gas.

2. THE DUMBBELL MODEL

The spin-spin interaction Hamiltonian which is generally accepted in the spin-ice crystal structures [7, 8, 19, 23] contains a sum of nearest-neighbor exchange and long range dipolar interactions,

$$\mathcal{H} = J\sum\nolimits_{\langle ij\rangle} S_i S_j + D\sum\nolimits_{i,j}\left[\frac{3\left(\mathbf{e}_i\cdot\mathbf{r}_{ij}\right)\left(\mathbf{e}_j\cdot\mathbf{r}_{ij}\right)}{r_{ij}^5} - \frac{\mathbf{e}_i\cdot\mathbf{e}_j}{r_{ij}^3}\right] S_i S_j, \quad (1)$$

where J and D are coupling constants, the distance between spins (S_i, S_j) is r_{ij}, and $\mathbf{e}_i$ is the unitary vector en in the i-spin direction. In this Hamiltonian the classical nature given by the authors to the S_i and S_j spin variables is clear, such as they are in the Ising-like models.

The standard magnetic dipole can be thought as a current I in a circuit (*kinetic dipole*) whose surface is S, then the magnetic moment m_e is $m_e = IS$. However, another image of these magnetic moments consists of the substitution of these magnetic dipoles by the so-called dumbbell model. Actually, the magnetic dipoles can also be thought as pairs of equal magnitude and opposite sign magnetic charges $\pm g$, split by a distance d (*split-charge dipole*), and $p_m = gd$. This simple expression can be used for the determination of the magnetic charge g, since p_m is given by the magnetic moment of the corresponding ions of the crystal and d is given by the lattice parameter whose value can be any real number. This dumbbell model is basic for constructing the effective equivalent systems of micromagnets whose elongation generates the spin-ice strings [7, 8]. A typical image of a spin-ice would be a "soup" of strings of different sizes inextricably mixed with dissociated magnetic charges coming from the broken micromagnet due to the excessive length of the corresponding strings. Castelnovo et al. in 2008 [7] showed that the energy in the spin-ice ground state can be accounted for by the magnetic Coulomb energy of the dumbbell model whose expression is

$$\mathcal{E} = \sum_{i,j\ (r_{ij}\neq 0)} \frac{K}{4\pi}\frac{g_i g_j}{r_{ij}} + \sum_{i,j\ (r_{ij}=0)} v_0 g_i g_j, \quad (2)$$

where the g_i and g_j are the magnetic charges of each pole of the dumbbell micromagnet and v_0 is the self-energy of each g charge. The legitimation of the dumbbell model becomes effective since the energy of Equation (2) is equivalent to the dipolar energy of

Equation (1), up to corrections which are small everywhere and vanish with distance at least as fast as $1/r^5$ [7]. Therefore, the Ising-like model of Hamiltonian of Equation (1) can be substituted by an effective Coulomb Hamiltonian. If one considers the kinetic energy of the effective magnetic charges within the magnetic spin-ice monopole gas, the effective Hamiltonian is:

$$\mathcal{H}_{ef} = \sum_i \frac{p_i^2}{2m_i} + \sum_{i,j\,(r_{ij}\neq 0)} \frac{K}{4\pi}\frac{g_i g_j}{r_{ij}} + \sum_{i,j\,(r_{ij}=0)} v_0 g_i g_j, \tag{3}$$

where p_i and m_i are the linear moment and effective mass of the corresponding g_i magnetic charge. In the limit of large density of deconfined magnetic charges, this dumbbell model presents similar ingredients to a magnetic plasma with certain similarities to its dual electronic metallic system [11, 12]. The central point of the study of these cases of generic plasmas is based on the frequency response of the system and the corresponding plasma frequency [22, 24] which depends on the density of charged particles. In the cases of electric charge plasmas, the medium response is the dynamical dielectric function [24] and in the magnetic plasma case, the corresponding frequency responses of the medium is that coming from the magnetic charges [22].

The expression of Equation (3) may suggest a quantum analysis, however this is not our objective in the present work since, in the realistic and intermediate cases, the image is different. In these cases, confined dumbbell pairs coexist with spin-ice strings of different sizes and deconfined (independent) magnetic charges coming from the broken pairs [7, 11, 12]. Then, the knowledge of the frequency responses in order to obtain clues about the dielectric and transport properties seems to be a next logical step [22] in the spin-ice study. These frequency responses are our goal of the next sections.

Equation (2) implies an effective Coulomb-like potential $\phi_m(\mathbf{r}) = \frac{K}{4\pi}\frac{g}{r}$, which produces a microscopic magnetic field (magnetic field in a space point $\mathbf{r}$) around the charge g:

$$\mathbf{B}(\mathbf{r}) = \frac{K}{4\pi} g \frac{\mathbf{r}}{r^3}, \quad \nabla\cdot\mathbf{B} = K g \delta(\mathbf{r}). \tag{4}$$

At great distances, the field of the two models, kinetic dipoles and split-charge dipoles, will be the same when

$$K\mathbf{p}_m = \mu_0 \mathbf{m}_e. \tag{5}$$

We want to emphasize that one should distinguish the field $\mathbf{B}$ from the average field $\langle\mathbf{B}\rangle$ of the standard Maxwell equations whose divergence is null even in the spin-ices [22]. The question is the utility of this average magnetic field concept in the dynamic of the mimetic magnetic charges which have molecular dimensions. In any case, in order to establish the Classical and Quantum Electrodynamics in these materials, the different kinds of the electric and magnetic fields should be analyzed in order to clarify the possible apparent contradictions. This implies a certain revision of the concepts of different microscopic, molecular and average fields, adjusting their determination to the new conditions and properties of the magnetic monopoles which are present in the spin-ices, being this one of the first objectives of this paper.

3. MAGNETIC FIELDS WITHIN A SPIN-ICE

The magnetic monopole behavior of determined entities within the spin-ices, as well as the concurrence of being $\nabla \cdot \langle \mathbf{B} \rangle = 0$ for the average Maxwell field induce us to make an exhaustive analysis of the classical fields which have incidence in the phenomenology of these systems since these two points are apparently contradictory. Therefore, in this section we carry out an analysis around the different magnetic fields associated to the two different models, the spin configuration of the spin-ices and its equivalent dumbbell model. In the classical Electromagnetism, a magnetic potential vector can be associated to each magnetic entity with magnetic moment $\mathbf{m}_i$. This magnetic potential vector generates a magnetic field at a point, usually called microscopic magnetic field [25–27],

$$\mathbf{B}'(\mathbf{r}) = \nabla \times \frac{\mu_0}{4\pi} \sum_i \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r', \tag{6}$$

where $\mathbf{J}_i(\mathbf{r}')$ is the assigned electric current to the particle (or magnetic structural entity) which presents the effective magnetic moment $\mathbf{m}_i = \frac{1}{2} \int \mathbf{r} \times \mathbf{J}_i(\mathbf{r}) d^3 r$. The named macroscopic or average magnetic field within the matter, which appears in the standard Maxwell equations [28], can be obtained by means of the following process [25, 26, 28, 29]. In a first step, one determines the contribution of all magnetic moments in the matter excluding a small volume, ΔV, which contains the point where the magnetic field should be calculated. The dimensions of this ΔV volume are small enough to consider that the Maxwell fields are constant. But are large enough to consider only the long range term in the dipole field of the molecules outside this volume. A second contribution should be added to the first one. This contribution is the average value of the magnetic field created by the magnetic moments of the molecules within this ΔV volume. These two terms constitute the so-called macroscopic or average electromagnetic field in the magnetic material, and it is that of standard Maxwell equations.

The average value of the magnetic field in the volume (ΔV) created by the magnetic moments within this volume is (see appendix A)

$$\left\langle \mathbf{b}' \right\rangle_{\Delta V} = \frac{2}{3}\mu_0 \mathbf{M}_e, \tag{7}$$

where $\mathbf{M}_e$ is the magnetization due to the electric currents.

When one considers the dumbbell model with magnetic charges g_i, the average value of the field created by all g_i-charges located in the small spherical volume ΔV can be written as (see appendix A)

$$\left\langle \mathbf{b} \right\rangle_{\Delta V} = -\frac{K}{3}\mathbf{P}_m, \tag{8}$$

where $\mathbf{P}_m$ is the polarization of magnetic charge dipoles formed by a split of monopole-antimonopole confined pair.

For the validity of the dumbbell model the Equation (5) must be fulfilled, i.e., $K\mathbf{P}_m = \mu_0 \mathbf{M}_e$. Then if the contribution to the magnetic field of the molecules outside the volume ΔV is $\mathbf{B}_0$, the average field of the spin configuration is

$$\left\langle \mathbf{B}' \right\rangle = \mathbf{B}_0 + \left\langle \mathbf{b}' \right\rangle_{\Delta V} = \mathbf{B}_0 + \frac{2}{3}\mu_0 \mathbf{M}_e, \tag{9}$$

and for the dumbbell model

$$\left\langle \mathbf{B} \right\rangle = \mathbf{B}_0 - \frac{1}{3}K\mathbf{P}_m = \mathbf{B}_0 - \frac{1}{3}\mu_0 \mathbf{M}_e. \tag{10}$$

Therefore, the difference between the average magnetic field corresponding to the classical physics interpretation of the spin configuration, and that corresponding to the dumbbell model is

$$\left\langle \mathbf{B}' \right\rangle = \left\langle \mathbf{B} \right\rangle + \mu_0 \mathbf{M}_e. \tag{11}$$

As a consequence, one has to distinguish two different magnetic fields: the average magnetic field $\langle \mathbf{B} \rangle$ of the dumbbell model and the average magnetic field of the spin configuration $\langle \mathbf{B}' \rangle$ which is the only field of these two whose divergence is null and that can be defined as the field of standard Maxwell equations. Therefore, $\nabla \cdot \langle \mathbf{B} \rangle = K\rho_m$, where $\rho_m = -\nabla \cdot \mathbf{P}_m$ is the density of magnetic "dumbbell" charges.

In the spin-ice in three dimensions there is no long-range ordering, they are disordered magnetic systems [30, 31], therefore the macroscopic magnetization $\mathbf{M}_e = 0$ and $\langle \mathbf{B} \rangle = \langle \mathbf{B}' \rangle$. In addition, this point agrees with those recent interpretations about the magnetic field in the spin-ices that claim over magnetic monopole structures with the apparent contradictory condition of $\nabla \cdot \langle \mathbf{B} \rangle = 0$, $\langle \mathbf{B} \rangle$ being the average field of the Maxwell equations.

4. EXTENDED MAXWELL EQUATIONS FOR THE DUMBBELL MODEL

The above magnetic field, generated by each pole charge of the micromagnet, competes with the external magnetic field in the interaction with the magnetic charges. From these competitive actions, the dynamic of the monopole system is such that each pole (or antipole) can be independently moved and then it generates an electric field of the type $\mathbf{E} = \mathbf{v} \times \mathbf{B}$. Therefore, the monopole current $\mathbf{J}_m$ is a source of electric field in an identical way as electric charges in movement generate a magnetic field. In fact, the capability of independent movement of each pole of the micromagnet under magnetic field, which originates the magnetricity and the subsequent formation of the magnetic plasma, is the true phenomenological novelty occurred in the spin-ices. These novel phenomena justifies the denomination of the components of the micromagnet as magnetic charges, since both the magnetic field created by them and their behavior under an external magnetic field is mimetic to that of magnetic monopoles. If there is no independent movement of the different poles of the molecular micromagnet, any phenomenological novelty is discarded. The electromagnetic fields produced by monopoles are governed by the equations

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{E} = -K\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{B} = K\rho_m, \quad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{12}$$

where ρ_m is the monopole density and $\mathbf{J}_m$ the current density of monopoles. Obviously, if the constant K is zero, the above equation are the Maxwell equations in "strictu sensu" without presence of electric charges. On the other hand, the existence of the magnetic charges implies a Lorentz force

$$\mathbf{F} = \frac{K}{\mu_0} g\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) = \kappa g\left(c\mathbf{B} - \frac{1}{c}\mathbf{v} \times \mathbf{E}\right), \tag{13}$$

where $\kappa \equiv K/(c\mu_0)$ is a constant that can also be used to establish the unit of magnetic charge.

Note that with the exception of the minus sign of $\mathbf{J}_m$, in the curl equation of the electric field, the above equations are symmetric with respect to the two fields $\mathbf{E}$ and $\mathbf{B}$. The duality transformation consists of changing $\mathbf{E} \to c\mathbf{B}$, $c\mathbf{B} \to -\mathbf{E}$, $q \to \kappa g$, and $\kappa g \to -q$. Then the above equations (Maxwell and Lorentz equations) are the standard ones. This implies the existence of a dual phenomenology with magnetic charges with respect to that existing with electric charges where the first experimental manifestation is the magnetricity measured by Bramwell et al. [11, 12]. These authors give account of the movement of the magnetic monopoles in spin-ices when the interaction of these charges with the external magnetic field surpasses the Coulomb interaction among themselves. However, this is not an isolated phenomenon since many symmetrical physical phenomena can appear when the entities with magnetic charge behavior are present within the matter.

A simple example, deduced form the Lorentz force, is the braking effect of a sample with free magnetic charges under an electric field whose direction is perpendicular to the sample velocity. This phenomenon is a symmetric dualism of that suffered by a conductor under a magnetic field, in similar direction conditions. Actually, if one considers a conductor moving at a velocity $\mathbf{u}$, it slows down when entering in a space region with magnetic field $\mathbf{B}$ perpendicular to its velocity, since electrical charges will move at $\delta\mathbf{v} = \frac{q}{m}\mathbf{u} \times \mathbf{B}\delta t$. Then there will be a Lorentz force

$$\mathbf{F} = q\delta\mathbf{v} \times \mathbf{B} = \frac{q^2}{m}(\mathbf{u} \times \mathbf{B}) \times \mathbf{B}\delta t = -\frac{q^2}{m}\delta t B^2\mathbf{u}. \tag{14}$$

This force slows down the conductor until it stops. it is the well known magnetic braking. A similar effect should happen when a material with free magnetic monopoles enters in a space region where there is an electric field. The speed of monopoles after a time δt will be $\delta\mathbf{v} = \frac{\kappa g}{mc}\mathbf{v} \times \mathbf{E}\delta t$, and the Lorentz's force will be

$$\mathbf{F} = -\frac{\kappa g}{c}\delta\mathbf{v} \times \mathbf{E} = \frac{\kappa^2 g^2}{mc^2}(\mathbf{u} \times \mathbf{E}) \times \mathbf{E}\delta t = -\frac{\kappa^2 g^2}{mc^2}\delta t E^2\mathbf{u}. \tag{15}$$

The result is that the sample with magnetic charges also slows. This electric braking of a material with free magnetic monopoles could be used to verify the existence of such monopoles.

5. UNIFIED FIELD EQUATIONS WITH ELECTRIC AND MAGNETIC CHARGES

The above equations, (12) and (13), are responsible for the classical dynamics of the magnetic charges when there are not any electric charges directly interacting (indirectly it is unavoidable) with the magnetic monopoles due to the insulator nature of the spin-ices. However, this is a particular case of another more general study in which the electric charges and magnetic charges coexist in different particles, and even in the same particles (i.e., we consider the case in which the charge of the particles has electric and magnetic components). If one assumes, in addition to the electric charges and the corresponding current densities (ρ_e, $\mathbf{J}_e$), a certain density of magnetic monopoles (ρ_m) with their current density ($\mathbf{J}_m$), the general modified Maxwell equations are [28, 32–34]

$$\nabla \cdot \mathbf{G} = \frac{1}{\varepsilon_0}\varrho, \quad \nabla \times \mathbf{G} = \frac{1}{\varepsilon_0 c}\Omega \mathbf{J} + \frac{1}{c}\frac{\partial}{\partial t}\Omega \mathbf{G}. \tag{16}$$

In these "Maxwell" equations, the electromagnetic charge and current densities (ϱ, $\mathbf{J}$), and the electromagnetic field $\mathbf{G}$ are defined as

$$\varrho \equiv \begin{pmatrix} \rho_e \\ \kappa\rho_m \end{pmatrix}, \quad \mathbf{J} \equiv \begin{pmatrix} \mathbf{J}_e \\ \kappa\mathbf{J}_m \end{pmatrix}, \quad \mathbf{G} \equiv \begin{pmatrix} \mathbf{E} \\ c\mathbf{B} \end{pmatrix}. \tag{17}$$

Note that ϱ, $\mathbf{J}$ and $\mathbf{G}$ are column matrices with two components: one in a subspace that is electrical and another that is magnetic. Also in Equation (16), we have defined $\Omega \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which is fundamental since it controls the symmetry between the electric and magnetic fields and in addition, it establishes the limits of this symmetry. The existence of the minus sign (-1) in Ω comes from that of $\mathbf{J}_m$ in the curl electric field equation, and it is necessary in order to maintain the energy conservation principle.

The Lorentz force is [28, 32, 33]

$$\begin{aligned} \mathbf{F} &= \frac{K}{\mu_0} g \left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) + q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \\ &= Q^T \left(\mathbb{1} - \frac{\mathbf{v}}{c} \times \Omega\right) \mathbf{G}. \end{aligned} \tag{18}$$

The electromagnetic charge Q [$Q^T \equiv (q, \kappa g)$], sometimes called *dyon* [35], has also two components, an electric, q, and a magnetic, κg.

5.1. Field Equations in Matter

One of the key objects in the definition of the electrodynamics within the matter is the dipole moment concept, whose extended definition when there are magnetic charges can be formulated by as follows:

$$\mathbf{P} \equiv \begin{pmatrix} \mathbf{P}_e \\ \kappa\mathbf{P}_m \end{pmatrix} \equiv \frac{1}{\Delta\mathcal{V}} \int_{\Delta\mathcal{V}} \mathbf{r} \begin{pmatrix} \rho_e \\ \kappa\rho_m \end{pmatrix} d^3 r, \tag{19}$$

$$\mathbf{M} \equiv \begin{pmatrix} \mathbf{M}_e \\ \kappa\mathbf{M}_m \end{pmatrix} \equiv \frac{1}{2\Delta\mathcal{V}} \int_{\Delta\mathcal{V}} \mathbf{r} \times \begin{pmatrix} \mathbf{J}_e \\ \kappa\mathbf{J}_m \end{pmatrix} d^3 r, \tag{20}$$

where, $\Delta\mathcal{V}$ is a volume that approaches zero, $\mathbf{P}_e$ ($\mathbf{P}_m$) is the electric (magnetic) polarization due to a split in the gravity centers of the electric (magnetic) charges, and $\mathbf{M}_e$ ($\mathbf{M}_m$) is the magnetization due to the electric (magnetic) currents. We call $\mathbf{P}$ *split-charge polarization* and $\mathbf{M}$ *kinetic polarization*.

Following a coherent and parallel way to the standard Classical Electrodynamics the result is

$$\nabla \cdot \mathbf{G} = \frac{1}{\varepsilon_0} \left(\varrho - \nabla \cdot \mathbf{P} \right), \quad \nabla \times \mathbf{G} = \mu_0 c \Omega \left(\mathbf{J} + \nabla \times \mathbf{M} \right). \tag{21}$$

5.2. Linear Response

The Lorentz's force, Equation (18), of an external field $\mathbf{G}$ on a static dyon with $Q^T = (q, \kappa g) \equiv |Q|(\cos\zeta, \sin\zeta)$, will be

$$\mathbf{F}_0 = |Q| \left(\cos\zeta, \sin\zeta \right) \mathbf{G} \tag{22}$$

and the separation between charges is proportional to this force. Thus, it is reasonable that the split charge dipole moment induced in a molecule will be

$$\mathbf{p} = \frac{\alpha}{|Q|^2} |Q| \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix} \mathbf{F}_0 = \alpha \Theta \mathbf{G}, \tag{23}$$

where α is the *polarizability* of the molecule and the Θ-matrix expression is given by

$$\Theta \equiv \frac{1}{|Q|^2} \begin{pmatrix} q \\ \kappa g \end{pmatrix} (q, \kappa g) = \begin{pmatrix} \cos^2\zeta & \cos\zeta \sin\zeta \\ \cos\zeta \sin\zeta & \sin^2\zeta \end{pmatrix}. \tag{24}$$

The procedure is similar for a dyon moving at velocity $\mathbf{v}$. The Lorentz force due to the velocity is

$$\mathbf{F}_v = -|Q| \left(\cos\zeta, \sin\zeta \right) \frac{\mathbf{v}}{c} \times \Omega \mathbf{G}, \tag{25}$$

and the modification of the current will be proportional to this force. Therefore, it is expected that the kinetic dipole moment induced will be

$$\mathbf{m} = \frac{\alpha'}{|Q|^2} |Q| \begin{pmatrix} \cos\zeta \\ \sin\zeta \end{pmatrix} \mathbf{F}_v = \alpha' \Theta' \mathbf{G}, \tag{26}$$

where the constant α' depends on the molecule and the $\Theta' \equiv -\Theta\Omega$ (see Appendix B for a detailed expression).

6. MOLECULAR FIELD AND CLAUSIUS-MOSSOTTI EQUATION

The dumbbell model of spin-ice systems can be conceived as a medium with magnetic charges of different sign traveling in the matter as a

Coulomb gas similar to a plasma [22]. In this image, the medium response before modifications of the magnetic charge density has a significant importance for determining their electric, magnetic, optical and electromagnetic wave propagation properties [24, 33]. These properties depend on the dielectric and magnetic permeability functions of the material media based on the effective fields able to electrically and magnetically polarize the molecules.

Therefore, we have to establish an intermediate field concept (between the average Maxwell field $\langle \mathbf{B} \rangle$ and the point field $\mathbf{B}$) that depends on the environment and determines the magnetic and electric response functions. This field is the so-called molecular or local field $\mathbf{B}_{\text{mol}}$, and it is defined [25] as the average field intensity acting on a given molecule within the material. The molecular field $\mathbf{B}_{\text{mol}}$ is due to all external sources plus every other molecule constituting the material, but excluding the own field of the molecule in question. It may be determined by removing the molecule in question, maintaining all other molecules in their time-averaged states, and calculating the space-averaged magnetic field in the cavity previously occupied by the removed molecule.

If the cavity can be chosen as a spherical volume of radius r_0, then the results of Appendix A give

$$\mathbf{E}_{\text{mol}} = \mathbf{E} + \frac{1}{4\pi\varepsilon_0 r_0^3}\mathbf{p}_e + \frac{2K}{4\pi r_0^3}\mathbf{m}_m, \tag{27}$$

where the second term is due to the split-charge dipolar moment $\mathbf{p}_e$ of the electrical charges, and the third term takes into account the kinetic dipoles $\mathbf{m}_m$ of the magnetic currents, included in the small molecular cavity. For the magnetic field

$$\mathbf{B}_{\text{mol}} = \mathbf{B} + \frac{K}{4\pi r_0^3}\mathbf{p}_m - \frac{2\mu_0}{4\pi r_0^3}\mathbf{m}_e, \tag{28}$$

where we have assumed that there are split-charge dipolar moments $\mathbf{p}_m$ due to the monopoles, and kinetic dipolar moments $\mathbf{m}_e$ due to the electric currents.

Assuming parallel and equal polarization for all local molecules:

$$\frac{3}{4\pi r_0^3}\mathbf{m}_{e,m} = \mathbf{M}_{e,m}, \quad \frac{3}{4\pi r_0^3}\mathbf{p}_{e,m} = \mathbf{P}_{e,m}, \tag{29}$$

it gives

$$\mathbf{E}_{\text{mol}} = \mathbf{E} + \frac{1}{3\varepsilon_0}\mathbf{P}_e + \frac{2}{3}K\mathbf{M}_m, \quad \mathbf{B}_{\text{mol}} = \mathbf{B} + \frac{K}{3}\mathbf{P}_m - \frac{2}{3}\mu_0\mathbf{M}_e. \tag{30}$$

Also one can calculate the molecular field by splitting the volume of material in two, through a virtual sphere of volume ΔV. The field due to molecules outside the sphere is **E**, **B**. The field due to the molecules on the surface of the sphere is

$$\mathbf{E}_{\text{surf}} = \frac{1}{3\varepsilon_0}\mathbf{P}_e + \frac{2}{3}K\mathbf{M}_m, \quad \mathbf{B}_{\text{surf}} = \frac{1}{3}K\mathbf{P}_m - \frac{2}{3}\mu_0\mathbf{M}_e. \tag{31}$$

In certain circumstances (cubic symmetry, for example), the field due to the molecules within the sphere can be considered zero, and then $\mathbf{E}_{\text{mol}} = \mathbf{E} + \mathbf{E}_{\text{surf}}$ and $\mathbf{B}_{\text{mol}} = \mathbf{B} + \mathbf{B}_{\text{surf}}$, that are the same equations found above, Equation (30).

In both methods, if we consider the existence of electric and magnetic charges, as well as magnetic and electric currents, we have the following expressions for the extended molecular field:

$$\mathbf{G}_{\text{mol}} = \mathbf{G} + \frac{\mathbf{P}}{3\varepsilon_0} - \frac{2}{3}\mu_0 c\Omega\mathbf{M}. \tag{32}$$

This is a particular case of a more general relationships of the extended molecular field

$$\mathbf{G}_{\text{mol}} = \mathbf{G} + \gamma\frac{\mathbf{P}}{\varepsilon_0} + (\gamma' - 1)\mu_0 c\Omega\mathbf{M}, \tag{33}$$

when $\gamma = \gamma' = 1/3$, we obtain the specific case for a spherical volume, ΔV.

The minimal energy principle implies the tendency to orientate the dipole moments **p** and **m** in the direction of the extended field $\mathbf{G}_{\text{mol}}$. Therefore, according to the Equations (23) and (26) if there are N molecules per unit volume

$$\mathbf{P} = N\mathbf{p} = N\alpha\Theta\mathbf{G}_{\text{mol}} \equiv \Lambda\Theta\mathbf{G}_{\text{mol}}, \tag{34}$$

$$\mathbf{M} = N\mathbf{m} = N\alpha'\Theta'\mathbf{G}_{\text{mol}} \equiv \Lambda'\Theta'\mathbf{G}_{\text{mol}}. \tag{35}$$

In the appendix, we give some properties of matrices that fulfill: $\Theta\mathbf{P} = \mathbf{P}$; $\Theta\Omega\mathbf{M} = 0$; $\Theta'\mathbf{P} = 0$; $\Theta'\Omega\mathbf{M} = \mathbf{M}$. As a consequence we have the following expressions for the polarization **P** and **M**:

$$\mathbf{P} = \Lambda\Theta\mathbf{G} + \Lambda\gamma\frac{\mathbf{P}}{\varepsilon_0}; \quad \mathbf{M} = \Lambda'\Theta'\mathbf{G} + \Lambda'(\gamma' - 1)\mu_0 c\mathbf{M}. \tag{36}$$

The split-charge susceptibility is:

$$\mathbf{P} = \frac{\Lambda}{1 - \Lambda\gamma/\varepsilon_0}\Theta\mathbf{G} = \varepsilon_0\chi_s\Theta\mathbf{G} \;\Rightarrow\; \chi_s = \frac{\Lambda/\varepsilon_0}{1 - \Lambda\gamma/\varepsilon_0}. \tag{37}$$

In the case of absence of magnetic charges, one obviously obtains the standard Clausius-Mossotti expression for the electrical susceptibility, if $\Lambda = N\alpha$ and $\gamma = 1/3$

$$\chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0}. \tag{38}$$

The kinetic susceptibility can also be calculated from following equations:

$$\mathbf{M} = \frac{\Lambda'}{1 - \Lambda'(\gamma' - 1)\mu_0 c}\Theta'\mathbf{G} = \frac{\xi}{\mu_0 c}\Theta'\mathbf{G} \tag{39}$$

and therefore

$$\xi = \frac{\Lambda'\mu_0 c}{1 - \Lambda'(\gamma' - 1)\mu_0 c} \;\Rightarrow\; \chi_k \equiv \frac{\xi}{1 - \xi} = \frac{\Lambda'\mu_0 c}{1 - \Lambda'\gamma'\mu_0 c}. \tag{40}$$

This result is also coherent with the corresponding standard result for the magnetic susceptibility when the magnetic charges are absent [25].

7. SUSCEPTIBILITY IN THE LORENTZ-DRUDE MODEL

The electromagnetic dynamical conductivity and absorption can be determined from the frequency dielectric function which in turn can be calculated from the Lorentz-Drude model. This model, which although uses a classical harmonic oscillator for the charged particles, presents features for the corresponding response functions that allow to describe system conductivity properties in concordance with some quantum models. We consider a dyonic charge Q under an extended molecular electromagnetic field $\mathbf{G}_{\text{mol}} = \mathbf{G}_0 e^{-i\omega t}$. Assuming a logical extension of the classical Lorentz-Drude oscillator in the x direction, the equation of motion can be written

$$\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + \omega_0^2 x = \frac{Q^T\mathbf{G}_{loc}}{m}, \tag{41}$$

where ω_0 is the resonant frequency of the classic oscillator corresponding to a particle with a dyonic charge Q, and Γ is the damping constant. The split-charge polarization with a particle density per volume unity N is:

$$\mathbf{P} = NQx = NQ\frac{Q^T\mathbf{G}_0/m}{\omega_0^2 - \omega^2 - i\Gamma\omega}e^{-i\omega t} = \frac{N|Q|^2\Theta\mathbf{G}_0/m}{\omega_0^2 - \omega^2 - i\Gamma\omega}e^{-i\omega t}. \tag{42}$$

Then, we can define polarizability α [25] with Equation (23) resulting $\mathbf{P} \equiv N\alpha\Theta\mathbf{G}_{\text{mol}}$, and thus we obtain

$$\frac{N\alpha}{\varepsilon_0} = \omega_p^2\frac{1}{\omega_0^2 - \omega^2 - i\Gamma\omega}, \quad \omega_p^2 \equiv \frac{N|Q|^2}{m\varepsilon_0}. \tag{43}$$

If we consider the above Clausius-Mossotti equation

$$\chi_s = \frac{3N\alpha/\varepsilon_0}{3 - N\alpha/\varepsilon_0} = \frac{\omega_p^2}{\omega_0^2 - \omega_p^2/3 - \omega^2 - i\Gamma\omega}. \tag{44}$$

This results implies an effective frequency defined by $\omega_i^2 \equiv \omega_0^2 - \omega_p^2/3$. If one considers several classes of particles with ω_i oscillator frequencies, then the split-charge susceptibility can be obtained considering analogical arguments to those of the standard Lorentz-Drude model, and the results are

$$\chi_s = \sum_i \frac{\omega_p^2}{\omega_i^2 - \omega^2 - i\Gamma\omega}, \quad \mathbf{P} = \varepsilon_0 \chi_s \Theta \mathbf{G}. \tag{45}$$

These expressions are valid for determined approximations within Quantum Physics of the Solid, fundamentally in the random phase approximation for the dielectric functions. The only difference is the physics meaning of the ω_i frequencies which in the quantum theory correspond to the possible transitions among the different quantum eigenstates which are a discrete number in the atomic level. These summations are converted in a continuum spectrum if one considers the solid and their determination requires the formulation of the band structures for the magnetic charge entities which may be formulated from the effective Hamiltonian of Equation (3). From these band structures, the spectral absorptions corresponding to the systems whose spectral transitions are fixed by the ω_i-energies can be determined as well as a valid approximation for the optical conductivity in function of the frequencies of the external electromagnetic field which interact with this system. As the optical conductivity is given by $\overleftrightarrow{\sigma}(\omega) = \varepsilon_0 \omega \mathrm{Im} \chi_s \Theta$, when the dissipative force coming from the Γ-parameter tends to zero, then the conductivity is given by

$$\overleftrightarrow{\sigma}(\omega) = \varepsilon_0 \frac{\pi}{2} \omega_p^2 \mathcal{N}(\omega) \Theta, \tag{46}$$

where $\mathcal{N}(\omega)$ is the density of states of magnetic charges for an energy $\hbar\omega$. The formalization of the solid state physics of these magnetic charges and the determination of their band structures are objectives of our investigation in progress. The relationships given in Equations (37), (39), (45) and (46) show properties of the lineal responses of dyonic matter under molecular electromagnetic fields seen since the Classical Electrodynamics which can obviously serve as a guide for researching experimental properties of these solids.

8. CONCLUDING REMARKS

One of the main goals of this paper is to analyze the different fields within the spin-ices since there are some apparent contradictions in the interpretation of the nature of these mimic magnetic charges. For instance, recently it was published that the field produced by these monopoles is due to a potential energy $V_m(\mathbf{r}) = \pm\frac{\mu_0}{4\pi}\frac{Q_m^2}{r}$, but however, the divergence of the generated magnetic field due to this magnetic energy is zero, affirming, in addition, that the standard Maxwell equations are valid in these spin-ices materials [22]. We have established a distinction in the above third section between the microscopic fields applicable to the molecular dimension of the dumbbells and the average field concepts coming from the standard Maxwell equations whose $\nabla \cdot \langle \mathbf{B} \rangle = 0$ even in the spin-ice systems. However, the concept of these average field equations (standard Maxwell equations) do not seem to be useful in the molecular dumbbell dimension. This analysis, in our opinion, clarifies this point and destroys these apparent contradictions. The field equations for the dumbbell charges have to be those corresponding to the microscopic field equations such as explained in this third section. This vision allows the connection of the micromagnets generated in the spin-ices with the toy model of pole-antipole pairs whose strong attractive interaction implies an extreme difficulty for detecting poles of a determined sign in freedom space. However, within the matter, the many-body system produces an intermediate interaction provoking a certain weakness of this pole-antipole interaction and the consequent breaking of the micromagnet under the presence of an external magnetic field. This interplay between the interpole interaction and that existing between the pole charges and the magnetic external field in an environment conditioned by the dielectric and magnetic responses of the material produces the magnetic pole gas [22]. Then, the dynamic in this system is governed by the optical conductivity and dielectric/magnetic responses. The classical calculation of these responses is determined in the second part of this paper via the analysis of the Clausius-Mossotti-like and Drude-Lorentz formalism applied to the systems with magnetic charges.

The true novelty of the phenomenologies in spin-ices such as the Coulomb state of magnetic charges [7], magnetricity [11, 12] and other dual properties [33, 34] is a direct consequence of the breaking of the binding among poles and antipoles in the so-called dumbbells. These dumbbells are magnetic dipoles produced by the spin-flip of two contiguous tetrahedron basis of the crystal, and their possible breaks can be favored by the dielectric function of the system. Therefore the dynamic of the spin-ices is conditioned by dielectric and permeability responses which do allow the presence of entities with magnetic charge swimming in the solid. This leads to a dual property whose dynamic is similar to that of the electric charges in electrolytes such as measured in 2009 [11, 12].

The difference existing between the average magnetic field in the spin configuration, $\langle \mathbf{B}' \rangle$, and the average field of the dumbbell model, $\langle \mathbf{B} \rangle$, whose relationship is that of Equation (11), implies that these two models can be considered equivalent in static conditions when $\langle \mathbf{m}_e \rangle \equiv \mathbf{M}_e = 0$. On the contrary, if $\mathbf{M}_e \neq 0$, this difference between the two fields of Equation (11) could imply a different magnetic energy. This difference of magnetic energy would be considered as the origin of the independent and possible deconfined movement of the different magnetic poles in the dumbbell model. This independent movement of the two poles of the micromagnets is not considered in the standard Maxwell equations of the classical interpretation of the magnetic configuration of the spin-ices.

Concerning the question posed in the introduction of this paper: what type of monopoles are these which are present in spin-ices? We have explained in previous sections that these monopoles are entities which mimic the action of the magnetic charges to generate microscopic and molecular fields, but the resulting macroscopic averaged magnetic field has null divergence and, in addition, the magnitude of its magnetic charge is not quantized since $g = p_m/d$, where the lattice parameter (d) can have as value any real number.

APPENDIX A. AVERAGE VALUES

When the magnetic field is generated by magnetic charges, the average magnetic field generated by all g_i-charges located in the small spherical volume ($\Delta V = \frac{4}{3}\pi R^3$) is

$$\begin{aligned} \langle \mathbf{b} \rangle_{\Delta V} &= -\frac{3}{4\pi R^3}\frac{K}{4\pi}\sum_i \int_{\Delta V} \nabla \frac{g_i}{|\mathbf{r}-\mathbf{r}_i|} d^3 r \\ &= -\frac{3}{4\pi R^3}\frac{K}{4\pi}\sum_i g_i \oint_{\Delta S} \frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}_i|} dS \\ &= -\frac{K}{4\pi R^3}\sum_i g_i \mathbf{r}_i = -\frac{K}{3}\mathbf{P}_m, \end{aligned} \tag{A1}$$

where $\mathbf{P}_m$ is the polarization of magnetic charge dipoles formed by a split of monopole-antimonopole confined pair. Analogously,

$$\langle \mathbf{e}' \rangle_{\Delta V} = -\frac{1}{4\pi\varepsilon_0 R^3}\sum_i q_i \mathbf{r}_i = -\frac{1}{3\varepsilon_0}\mathbf{P}_e. \tag{A2}$$

On the other hand, the average value of the magnetic field created by an electric current density $\mathbf{J}_e$, within this volume, is

$$\begin{aligned}\langle \mathbf{b}' \rangle_{\Delta V} &= \frac{3}{4\pi R^3}\frac{\mu_0}{4\pi}\int_{\Delta V}\int_{\Delta V} \nabla\times\frac{\mathbf{J}_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'd^3r \\ &= \frac{3}{4\pi R^3}\frac{\mu_0}{4\pi}\int_{\Delta V}\oint_{\Delta S}\mathbf{n}\times\frac{\mathbf{J}_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dSd^3r' \\ &= \frac{3}{4\pi R^3}\frac{\mu_0}{4\pi}\int_{\Delta V}\left(\oint_{\Delta S}\frac{\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}dS\right)\times\mathbf{J}_e(\mathbf{r}')d^3r' \\ &= \frac{\mu_0}{4\pi R^3}\int_{\Delta V}\mathbf{r}'\times\mathbf{J}_e(\mathbf{r}')d^3r' = \frac{2\mu_0}{4\pi R^3}\mathbf{m}_e = \frac{2}{3}\mu_0\mathbf{M}_e, \end{aligned} \tag{A3}$$

where $\mathbf{M}_e$ is the macroscopic magnetization due to the electric currents. Similarly, the average value of the electric field created by a magnetic current density $\mathbf{J}_m$ within this volume is

$$\langle \mathbf{e} \rangle_{\Delta V} = -\frac{K}{4\pi R^3}\int_{\Delta V}\mathbf{r}'\times\mathbf{J}_m(\mathbf{r}')d^3r' = -\frac{2K}{4\pi R^3}\mathbf{m}_m = -\frac{2K}{3}\mathbf{M}_m, \tag{A4}$$

where $\mathbf{M}_m$ is the macroscopic magnetization due to the magnetic currents.

APPENDIX B. SOME PROPERTIES OF THE Θ-MATRIX

From the definition of Θ matrix

$$\Theta \equiv \begin{pmatrix} \cos^2\zeta & \cos\zeta\sin\zeta \\ \cos\zeta\sin\zeta & \sin^2\zeta \end{pmatrix} \Rightarrow \Theta^2 = \Theta, \quad \det|\Theta| = 0. \tag{B1}$$

It is also convenient to define another matrix Θ'

$$\Theta' \equiv -\Theta\Omega = \begin{pmatrix} -\cos\zeta\sin\zeta & \cos^2\zeta \\ -\sin^2\zeta & \cos\zeta\sin\zeta \end{pmatrix}. \tag{B2}$$

If Ξ is a symmetric matrix

$$\Xi \equiv \begin{pmatrix} X & Z \\ Z & Y \end{pmatrix} \Rightarrow \Xi\Omega\Xi = \det|\Xi|\Omega, \tag{B3}$$

since Θ is symmetric and $\Omega^2 = -\mathbb{1}$

$$\begin{aligned} &\Theta'\Theta = -\Theta\Omega\Theta = -\det|\Theta|\Omega = 0 \\ &\Theta'\Theta' = -\Theta'\Theta\Omega = 0 \Rightarrow \Theta\Omega\Theta' = -\Theta'\Theta' = 0. \\ &\Theta\Theta' = -\Theta\Theta\Omega = -\Theta\Omega = \Theta' \Rightarrow \Theta'\Omega\Theta' = -\Theta\Omega\Omega\Theta' = \Theta\Theta' = \Theta'. \end{aligned} \tag{B4}$$

For the split-charge polarization

$$\mathbf{P} = \Lambda\Theta\mathbf{G}_{\text{mol}} \Rightarrow \Theta\mathbf{P} = \mathbf{P}, \quad \Theta'\mathbf{P} = 0, \tag{B5}$$

and for the kinetic polarization

$$\mathbf{M} = \Lambda'\Theta'\mathbf{G}_{\text{mol}} \Rightarrow \Theta'\mathbf{M} = 0, \quad \Theta\Omega\mathbf{M} = 0, \quad \Theta'\Omega\mathbf{M} = \mathbf{M}. \tag{B6}$$

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