Progress In Electromagnetics Research M, Vol. 22, 191–203, 2012

EFFICIENT AND ACCURATE APPROXIMATION OF INFINITE SERIES SUMMATION USING ASYMPTOTIC APPROXIMATION AND FAST CONVERGENT SERIES

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Abstract—We present an approach for very quick and accurate approximation of infinite series summation arising in electromagnetic problems. This approach is based on using asymptotic expansions of the arguments and the use of fast convergent series to accelerate the convergence of each term. It has been validated by obtaining very accurate solution for propagation constant for shielded microstrip lines using spectral domain approach (SDA). In the spectral domain analysis of shielded microstrip lines, the elements of the Galerkin matrix are summations of infinite series of product of Bessel functions and Green's function. The infinite summation is accelerated by leading term extraction using asymptotic expansions for the Bessel function and the Green's function, and the summation of the leading terms is carried out using the fast convergent series.

1. INTRODUCTION

In many electromagnetic problems, electric fields [1], magnetic fields, potentials [2], etc. are expressed in terms of infinite series that have a very slow convergence. For instance, in many multilayered planar shielded circuits in the evaluation of Green's function and in SDA [3], which is used for solving many problems relating to layered media, we often come across infinite series summations. Also when analyzing shielded planar structures, cavity backed antennas, devices inside photonic crystals, time constant in superconductors [4], etc., the calculation of infinite summations is computationally intensive and prevents the development of efficient programs [5]. Tounsi et al. have proposed an extension of the spectral domain immitance approach

Received 8 October 2011, Accepted 30 November 2011, Scheduled 20 December 2011

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(SDIA) for layered multiple conductor shielded microstrip but it needs to be accelerated [6].

In order to accelerate the spectral series summation in application of SDA to shielded microstrip lines, several techniques have been used [7–9]. In the SDA, using appropriate basis functions and adding the asymptotic tails of the series, a drastic improvement in accuracy and speed can be obtained in the evaluation of the elements of K matrix [10]. Cano et al. [8] have proposed an asymptotic approximation technique for acceleration, but it involves the computation of Green's function in the space domain followed by a double integral to find each element of the Galerkin matrix. Medina and Horno [11] have used an asymptotic approximation technique including a partial leading term extraction of the Bessel's function with two terms for the analysis of cylindrical and elliptical microstrip. Tsalamengas and Fikioris [9] have proposed a technique based on the asymptotic approximation in the space domain followed by rapidly convergent series [12] to accelerate the summation of series. Some other good works can be found in [8, 9]. Also these techniques are specific for shielded microstrip using Chebyshev polynomial as basis functions. The mid-point summation (MPS) [13] has also been used for accelerating the SDA, but it converges slowly compared to the techniques mentioned in this paper for small values of the argument.

In this paper, we propose a very efficient and accurate technique to accelerate the convergence of infinite series summation using fast convergent series. The fast convergent series is obtained using the Maclaurin series and integral by parts. The approach has been applied to the numerical acceleration of the SDA for shielded microstrip to obtain accurate results for the propagation constant. This is done by accelerating the convergence of series summation appearing in the elements of the Galerkin matrix using an asymptotic approximation to the Bessel and Green's functions, followed by an approximation of the infinite summation of each leading term with the fast convergent series. Similar fast convergent series have been used recently for the evaluation of periodic Green's function [14]. Thus, this technique involving asymptotic approximation followed by use of fast convergent series is very simple to understand and apply to different problems in electromagnetics.

2. SPECTRAL DOMAIN APPROACH (SDA)

Figure 1 shows a cross section of a shielded microstrip. Region 2 is air, and region 1 consists of a dielectric material with relative permittivity and permeability ϵ_r and μ_r, respectively. The structure is uniform and

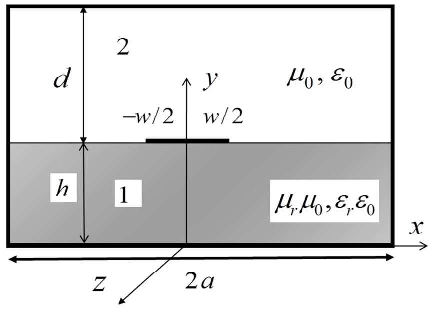


Figure 1. Shielded microstrip.

infinite along the z axis. The thin metal casing and thin metal strip are assumed to be perfect electric conductors (PECs). The solutions for microstrip lines are hybrid modes that can be expressed in terms of superposition of infinite transverse electric (TE) and transverse magnetic (TM) modes which can in turn be expressed in terms of the scalar vector potentials [15]. Then by taking the Fourier transform of all the field components, applying the boundary conditions in the Fourier domain and assuming E_y to be an even function of x for the dominant quasi-transverse electromagnetic (TEM) wave, we obtain [15]:

$$\tilde{E}_{x2}(\alpha_n, h) = G_{xx}(\alpha_n, \beta)\tilde{J}_x(\alpha_n) + G_{xz}(\alpha_n, \beta)\tilde{J}_z(\alpha_n), \qquad (1)$$

$$\tilde{E}_{z2}(\alpha_n, h) = G_{zx}(\alpha_n, \beta)\tilde{J}_x(\alpha_n) + G_{zz}(\alpha_n, \beta)\tilde{J}_z(\alpha_n), \qquad (2)$$

where $\alpha_n = (n - 1/2)\pi/a$ and β is the propagation constant. $\tilde{J}_x(\alpha_n)$ and $\tilde{J}_z(\alpha_n)$ are the Fourier transforms of the transverse current density $J_x(x)$ and the longitudinal current density $J_z(x)$, respectively. The z dependency of the electric and magnetic field has the form of $e^{-j\beta z}$. The expressions for the Green's functions are reported in [15].

The basis currents are chosen such that they are nonzero only on the strip $|x| < w/2$. $J_{xi}(x)$ is a real odd function, and $J_{zi}(x)$ is a real even function. So from the properties of Fourier transforms, $\tilde{J}_{xi}(\alpha_n)$ is a purely imaginary and odd function, and $\tilde{J}_{zi}(\alpha_n)$ is a purely real and even function. The subscript i refers to the ith basis function. The $\tilde{}$ denotes that we are looking at the quantities in the spectral domain. Chebyshev polynomials of the first and second kinds are chosen as the basis for $J_z(x)$ and $J_x(x)$, respectively [16].

$$J_z(x) = (1/\sqrt{1 - (2x/w)^2}) \sum_{i=0}^{M_z - 1} I_{zi} T_{2i}(2x/w) \qquad (3)$$

$$J_x(x) = \sqrt{1-(2x/w)^2} \sum_{i=1}^{M_x} I_{xi} U_{2i-1}(2x/w) k_0 w \tag{4}$$

T_i and U_i are the Chebyshev polynomials of the first and second kinds, respectively. The transverse current $J_x(x)$ is proportional to ω, so as frequency decreases it will become very small compared to $J_z(x)$. Therefore, it has been normalized with $k_0 w$ [17].

The Fourier transforms of the basis functions are given by:

$$\tilde{J}_x(\alpha_n) = \frac{w\pi}{\delta_n} \sum_{i=1}^{M_x} I_{xi} i (-1)^i J_{2i}(\delta_n) k_0 w, \tag{5}$$

$$\tilde{J}_z(\alpha_n) = j\frac{w\pi}{2} \sum_{i=1}^{M_z} I_{z(i-1)} (-1)^i J_{2(i-1)}(\delta_n), \tag{6}$$

where $\delta_n = \alpha_n w/2$ and $J_n(z)$ is the nth order Bessel's function. Further, using the Galerkin method, followed by the Parseval's theorem and the fact that the product of the tangential field component and the surface current is always zero on the line, we obtain the following matrix equation:

$$\begin{bmatrix} K^{xx} & K^{xz} \\ K^{zx} & K^{zz} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where A and B are vectors related to the coefficients I_{xi} and I_{zi}, respectively and

$$K_{ij}^{pq} = \sum_{n=1}^{\infty} F_{ij}^{pq} \quad p = x, z \text{ and } q = x, z, \tag{7}$$

where $F_{ij}^{pq} = \tilde{J}_{pi}(\alpha_n) G_{pq}(\alpha_n, \beta) \tilde{J}_{qj}(\alpha_n)$. Finally, the propagation constant β can be obtained by solving $\det[K] = 0$.

3. APPROXIMATION OF SUMMATION WITH FAST CONVERGENT SERIES

3.1. Asymptotic Approximation of Green's Function

Using Taylor expansion, for large α_n, the Green's functions are approximated as [18]:

$$G_{xx} \approx G_{xx0} \alpha_n w \left(1 - y_{xx}^2/\alpha_n^2\right), \tag{8}$$

$$G_{xz} \approx G_{xz0} \left(1 - y_{xz}^2/\alpha_n^2\right), \tag{9}$$

$$G_{zz} \approx \frac{G_{zz0}}{\alpha_n w} \left(1 - y_{zz}^2/\alpha_n^2\right), \tag{10}$$

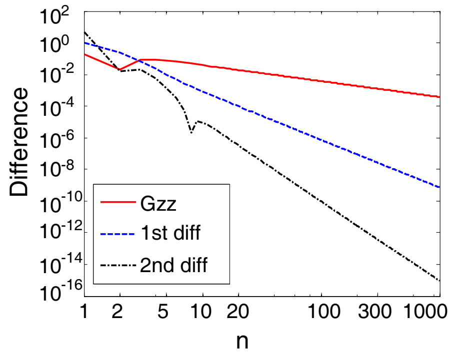


Figure 2. Convergence of G_{zz}, G_{zz} minus the first asymptotic term and G_{zz} minus the first two asymptotic terms as function of n for $\beta = 3k_0$ for the shielded microstrip shown in Figure 1 and parameters $\epsilon_r = 11.7$, $\mu_r = 1$, $f = 4\,\text{GHz}$, $h = 3.17\,\text{mm}$, $w = 3.04\,\text{mm}$, $2a = 34.74\,\text{mm}$, $d = 50\,\text{mm}$.

where

$$G_{xx0} = \frac{1}{1+\epsilon_r},\ G_{xz0} = \frac{\beta}{(1+\epsilon_r)k_0},$$

$$G_{zz0} = \frac{\left(\beta^2 - k_1^2\right) + \mu_r\left(\beta^2 - k_2^2\right)}{k_0^2(1+\epsilon_r)(1+\mu_r)}, \tag{11}$$

$$y_{xx}^2 = \frac{\beta^2}{2} + \frac{\epsilon_r k_1^2 + k_2^2}{2(1+\epsilon_r)}, \tag{12}$$

$$y_{xz}^2 = \frac{\beta^2}{2} + \frac{\left(k_2^2 - k_1^2\right)(1-\mu_r)}{2(1+\mu_r)} - \frac{\epsilon_r k_2^2 + k_1^2}{2(1+\epsilon_r)}, \tag{13}$$

$$y_{zz}^2 = \beta^2 - k_2^2 + \frac{1}{2}\left[\left(k_2^2 - k_1^2\right)\left(\frac{1}{1+\mu_r} + \frac{1}{1+\epsilon_r}\right) - \frac{(\beta^2 - k_1^2)(\beta^2 - k_2^2)(1+\mu_r)}{(\beta^2 - k_1^2) + \mu_r(\beta^2 - k_2^2)}\right]. \tag{14}$$

In Figure 2, Gzz refers to the Green's function G_{zz}, the $1st\ diff$ refers to the difference between Green's function and the first asymptotic term of G_{zz}, and $2nd\ diff$ refers to the difference between Green's function and the first two asymptotic terms of G_{zz}. Also Figure 2 verifies the accuracy of the asymptotic extraction for G_{zz} as using first k leading terms G_{zz} converges as $1/n^{2k+1}$.

3.2. Asymptotic Approximation of the Bessel's Function

The Bessel function [19] can be written in the following form:

$$J_n(z) = \left(\frac{2}{z\pi}\right)^{1/2} \left[\cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\left(1 - \frac{C_n^2}{(8z)^2} + \dots\right) - \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\left(\frac{C_n^1}{8z} - \frac{C_n^3}{(8z)^3} + \dots\right)\right], \tag{15}$$

where $n \in \{0, 1, 2, 3, \dots\}$, $C_n^0 = 1$ and

$$C_n^k = \frac{1}{k!}\prod_{m=1}^{k}[4n^2 - (2m-1)^2] = \frac{4n^2 - (2k-1)^2}{k} C_n^{k-1}. \tag{16}$$

Using the asymptotic forms of the Bessel function and Green's functions for large α_n and considering terms up to $1/\alpha_n^5$

$$\begin{aligned} \tilde{F}_{kl}^{pq}(-1)^{i+j}\pi/2 = \frac{G_{pq0}}{(\alpha_n w)^2}\Bigg\{ & 1 + \sin(\alpha_n w) + \left(C_{2i}^1 + C_{2j}^1\right)\frac{\cos(\alpha_n w)}{4\alpha_n w} \\ & - \Big[16y_{pq}^2 w^2 + C_{2i}^2 + C_{2j}^2 - C_{2i}^1 C_{2j}^1 + \Big(16y_{pq}^2 w^2 + C_{2i}^2 + C_{2j}^2 \\ & + C_{2i}^1 C_{2j}^1\Big)\sin(\alpha_n w)\Big]/(4\alpha_n w)^2 - \Big[C_{2i}^3 + C_{2j}^3 + C_{2i}^2 C_{2j}^1 \\ & + C_{2i}^1 C_{2j}^2 + 16y_{pq}^2 w^2\left(C_{2i}^1 + C_{2j}^1\right)\Big]\frac{\cos(\alpha_n w)}{(4\alpha_n w)^3}\Bigg\}, \end{aligned} \tag{17}$$

where $i = k-1$ for $p = z$, $i = k$ for $p = x$ and $j = l-1$ for $q = z$, $j = l$ for $q = x$.

$$K_{kl}^{pq} \approx \sum_{n=1}^{N_{\max}}\left[F_{kl}^{pq} - \tilde{F}_{kl}^{pq}\right] + \sum_{n=1}^{\infty}\tilde{F}_{kl}^{pq}. \tag{18}$$

Therefore, from (17), one observes that $\tilde{F}_{kl}^{pq}$ only consists of a weighted sum of terms of the form $\sin(nz)/n^k$, $\cos(nz)/n^k$ and $1/n^k$. Thus, in the second term on the right hand side in (18), the terms involving infinite summation of sinusoidal functions divided by α_n^k are approximated using fast convergent series developed for different powers of n in the next section. And those of the form $1/\alpha_n^k$ can be evaluated using the Riemann Zeta function [20].

In order to consider up to the fourth asymptotic term in the expression for $\tilde{F}_{kl}^{pq}$, we need to consider only first two asymptotic terms in the Green's function. Also, as the frequency increases beyond a few tens of GHz for dimensions in the *mm* range, one needs to consider even the third order term in the asymptotic expansion for the Green's function if faster convergence is required in the value of the propagation constant.

4. FAST CONVERGENT SERIES

For any series of the form [16] $\sum_{n=1,3,5\dots}^{\infty} e^{jnz}/n^k$ where $k = 2, 3, 4, \dots,$ we have

$$\sum_{n=1,3\dots}^{\infty} \frac{e^{jnz}}{n^k} = j \sum_{n=1,3\dots}^{\infty} \int_0^z \left(\frac{e^{jnz}}{n^{k-1}}\right) dz + \sum_{n=1,3\dots}^{\infty} \frac{1}{n^k}. \tag{19}$$

Also from [16], we know

$$\sum_{n=1,3\dots}^{\infty} \frac{e^{jnz}}{n} = -\frac{1}{2} \ln\left[\tan\left(\frac{z}{2}\right)\right] + j\frac{\pi}{4}, \quad 0 < z < \pi, \tag{20}$$

Using Mathematica, the Maclaurin series expansion for $\ln(\tan z)$ can be written as:

$$\begin{aligned} \ln(\tan z) = &\ln z + z^2/3 + 7z^4/90 + 62z^6/2835 \\ &+ 127z^8/18900 + 146z^{10}/66825 + O[z]^{12}, \quad |z| < \pi. \end{aligned} \tag{21}$$

The first term on the right hand side of (19) is evaluated by substituting the value of $\ln[\tan(z/2)]$ from (21) into (20) and then substituting the expression for $\sum_{n=1,3..}^{\infty} \frac{e^{jnz}}{n}$, thus we obtain (19). The second term on the right hand side of (19) is calculated using the Riemann Zeta function [21]. Then by comparing the real and imaginary parts on the right and left hand sides, one obtains:

$$\begin{aligned} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin(nz)}{n^2} = &- \Big[(z/2)\ln(z/2) - z/2 + z^3/72 + 7z^5/14400 \\ &+ 31z^7/1270080 + 127z^9/87091200 + \dots\Big] \end{aligned} \tag{22}$$

$$\sum_{n=1,3,5,\dots}^{\infty} \frac{\cos(nz)}{n^2} = \pi^2/8 - \pi z/4 \tag{23}$$

Thus, following the same procedure by substituting the expression for $\sum_{n=1,3\dots}^{\infty} \frac{e^{jnz}}{n^{k-1}}$ in (19), sequentially fast convergent series for values of k up to 5 have been obtained. There will exist a closed form expression for the infinite summation over odd integers for $\sin(nz)$ divided odd powers of n and also for $\cos(nz)$ divided even powers of n. The expressions for these have been given in the appendix.

Following the same procedure, a series for summation of n over $n = 1, 2, 3\dots$ can also be obtained.

5. NUMERICAL RESULTS

The approach was numerically validated using a shielded microstrip with parameters $\epsilon_r = 11.7$, $\mu_r = 1$, $f = 4\,\mathrm{GHz}$, $h = 3.17\,\mathrm{mm}$, $w = 3.04\,\mathrm{mm}$, $2a = 34.74\,\mathrm{mm}$, $d = 50\,\mathrm{mm}$ [8].

The kth difference in Figure 3 refers to considering up to kth asymptotic term for $\tilde{F}_{11}^{zz}$ in the difference $F_{11}^{zz} - \tilde{F}_{11}^{zz}$. It shows that if we use up to k leading terms, the difference converges as $1/n^{k+1}$ which confirms the accuracy of the asymptotic expansion. From Figures 4(a) and 4(b), it is observed that the convergence of K_{11}^{zz} and determinant of K matrix changes from $1/N_{\max}$ using the direct summation to $1/N_{\max}^3$

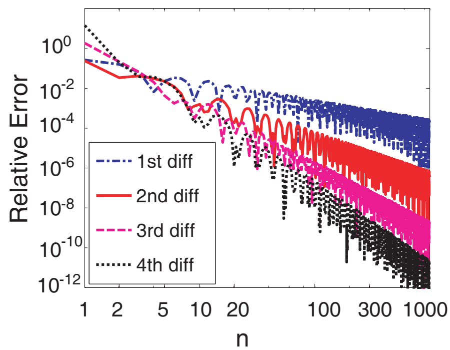
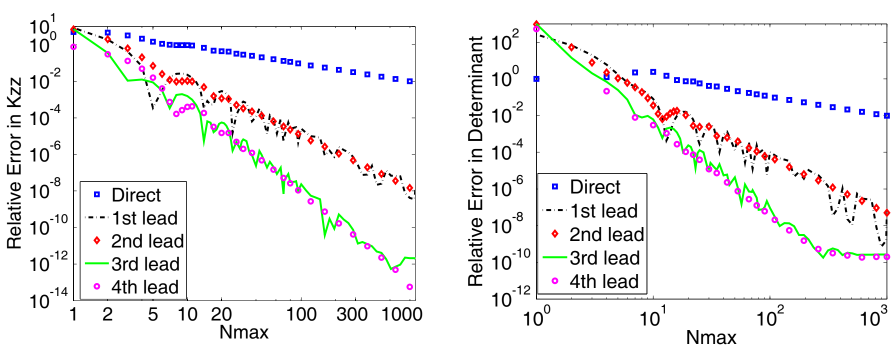


Figure 3. Convergence of $(F_{11}^{zz} - \tilde{F}_{11}^{zz})\alpha_n^2$ as function of n for $\beta = k_0$ for different number of leading terms for a shielded microstrip with parameters as given in Figure 2.



(a) Convergence of K_{11}^{zz} (b) Determinant of K matrix

Figure 4. Convergence of (a) K_{11}^{zz} and (b) determinant of K matrix for $M_z = 1$, $M_x = 1$ and $\beta = 3k_0$ using different number of leading terms for a shielded microstrip with parameters same as in Figure 2.

using up to second leading term and finally to $1/N_{\max}^5$ using up to fourth leading term. Here, the result using four leading terms with $N_{\max} = 10^6$ is used as reference. Also Figures 4(a), 4(b), 5(a) and 5(b) show that the results are similar if we use odd number of leading terms or the next even number of leading terms. This can be explained by the fact that the even leading terms do not have a constant term but only sinusoidal functions which converge faster than the constant term.

Table 1 shows the comparison of our results with [9]. We have used MATLAB to write our codes, and the CPU time for getting ϵ_{reff} correct up to 12 digits is less than 0.05 seconds using a 2.66 GHz processor. We have used $c = 299,792,456.2\,\text{m/s}$ which is the same as that used in [9] and was determined by email communication. Using $M_z = 2, M_x = 2$ and $N_{\max} = 52$, an $\epsilon_{reff} = 8.8100416$ is obtained, which is the same as that in [8, 13]. For a given basis, the convergence of the approach can be accelerated by using larger number of asymptotic terms.

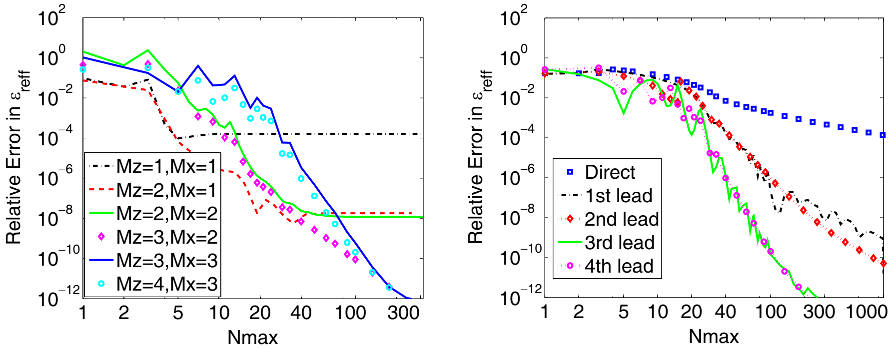
Table 2 shows the comparison of ϵ_{reff} for a different number of basis functions using the proposed approach (FCS) and MPS [13] with $c = 299,792,458\,\text{m/s}$ [22]. It is seen that MPS converges nearly two

Table 1. ϵ_{reff} for different basis functions and $N_{\max} = 40$, 70 for a shielded microstrip with parameters same as in Figure 2.

M_z	M_x	$N_{\max} = 40$	$N_{\max} = 70$	$N_{\max} = 40$ **[9]**	$N_{\max} = 70$ **[9]**
1	1	8.8114915	8.8114916	8.8114918	8.8114916
2	1	8.8100414	8.8100414	8.8100416	8.8100414
2	2	8.8100417	8.8100416	8.8100418	8.8100417
3	2	8.8100416	8.8100416	8.8100417	8.8100416
3	3	8.81004157493		$N_{\max} = 230$	
3	3	8.81004157493		$N_{\max} = 260$ [9]	
4	3	8.81004157493		$N_{\max} = 220$	

Table 2. Comparison of the FCS and the MPS ϵ_{reff} for different basis functions for a shielded microstrip with parameters same as in Figure 2.

M_z	M_x	ϵ_{reff}	$N_{\max(FCS)}$	$N_{\max(MPS)}$
1	1	8.81	4	11
2	1	8.810041	18	42
2	2	8.8100416	52	61
3	2	8.810041567	130	140
3	3	8.81004156779	243	255
4	3	8.81004156779	232	243



(a) (b)

Figure 5. The convergence of ϵ_{reff} for different basis and different number of leading terms for a shielded microstrip with parameters same as given in Figure 2. (a) Convergence of ϵ_{reff} using different number of basis. (b) Convergence of ϵ_{reff} using different number of leading terms.

times slower than the FCS especially when only a few digits of accuracy is required which is the case in most practical applications. For very high accuracies, the FCS converges slightly faster than MPS.

Figure 5(a) shows that the rate of convergence decreases as we increase the number of basis functions for small $N_{\max}$, but by using higher order basis functions and larger number of terms we get even more accurate results. From Figure 5(b) it is observed that the rate of convergence increases as we use more number of leading terms. Also it can be seen from Figure 5(b) that in the evaluation of the elements of K matrix, if we are considering odd number of leading terms, the next leading term does not speed up the convergence so much, because even leading terms do not have constant terms but only sinusoidal function, which have a much smaller contribution to the matrix elements than the constant terms and will therefore not significantly affect their convergence.

6. CONCLUSION

The speed of computation and accuracy are major issues for design applications, and very efficient computer programs are required in such applications. By using the present approach, it is possible to write very efficient programs for many applications in electromagnetics for computing the Green's functions, electric fields, magnetic fields, analysis of shielded planar circuits and transmission lines, and problems involving SDA in which the speed is limited by the slow convergence of infinite series. In addition, this approach is very simple to implement and does not involve the overheads of the computation of complex coefficients or numerical integration.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation CAREER Grant ECS-0547161.

APPENDIX A.

The super convergent series for the infinite summation over odd integers for $\sin(nz)$ divided even powers of n and also for $\cos(nz)$ divided odd powers of n thus obtained are:

$$\sum_{n=1,3,5,\ldots}^{\infty} \sin(nz)/n^2 = -z/2\log(z/2) - z/2 + z^3/72 + 7z^5/14400 + 31z^7/1270080 + 127D^9/87091200 \quad \text{(A1)}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \cos(nz)/n^3 = 1.05179979026464499725 + z^2/4\log(z/2) - 3z^2/8 + z^4/288 + 7z^6/86400 \quad \text{(A2)}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \sin(nz)/n^4 = 1.05179979026464499725z + z^3/12\log(z/2) - 11z^3/72 + z^5/1440 + z^7/86400 \quad \text{(A3)}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \cos(nz)/n^5 = 1.0045237627951387 - (0.5258998951323225z^2 + z^4/48\log(z/2) - 25z^4/576 + z^6/8640) \quad \text{(A4)}$$

The closed form expressions for the infinite summation over odd integers for $\sin(nz)$ divided odd powers of n and also for $\cos(nz)$ divided even powers of n are:

$$\sum_{n=1,3,5,\ldots}^{\infty} \sin(nz)/n^3 = \pi^2 z/8 - \pi z^2/8 \quad \text{(A5)}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \cos(nz)/n^4 = \pi^4/96 - \pi^2 z^2/16 + \pi z^3/24 \quad \text{(A6)}$$

$$\sum_{n=1,3,5,\ldots}^{\infty} \sin(nz)/n^5 = \pi^4 z/96 - \pi^2 z^3/48 + \pi z^4/96 \quad \text{(A7)}$$

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