IMPLICIT BOUNDARY CONDITIONS IN TRANSFORMATION-OPTICS CLOAKING FOR ELECTROMAGNETIC WAVES

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Abstract—The paper studies boundary conditions in transformation-optics cloaking for two and three dimensional electromagnetic waves. Implicit boundary conditions for these two cases are derived, no matter if the source is placed in the interior or exterior of the cloak layer. More importantly, the two implicit boundary conditions are derived without solving Mie scattering problems, and these conditions are characteristics of the clock-air interface. In particular, the implicit boundary condition for two-dimensional electromagnetic wave case is reported for the first time. In addition, a sensor can be cloaked in two-dimensional electromagnetic waves, i.e., waves can penetrate into the interior of the cloak layer without exterior scattering.

1. INTRODUCTION

Invisibility cloak via transformation optics has received much attention in the past few years. Pendry et al. proposed the transformation optics approach to control electromagnetic (EM) fields, by which a space consisting of the normal free space can be squeezed into a new space with different volumes and inhomogeneous constitutive parameters. The fundamental idea is the invariance of Maxwell's equations under a space-deforming transformation if the material properties are altered accordingly [1–10]. When the source is outside the cloak, usually the Mie scattering problem is solved to obtain the field distributions in different layers [11, 12]. The case when source is in the interior of the cloak, which is much less studied in comparison, is usually tackled by a way similar to the Mie scattering approach, i.e., expressing the source

field as multipole fields and then matching boundary conditions [6]. For convenience, we refer to both methods as the Mie scattering approach (or the method of solving the Mie scattering problem), no matter if the source is placed in the exterior or interior of the cloak layer. Boundary conditions at the inner boundary of the cloak have been of great interest, especially when the source is in the interior of the cloak. To the best knowledge of the author, all existing publications arrive at boundary conditions by first solving the Mie scattering problem and then observing the fields at the boundary.

In this paper, for two-dimensional (2D) and three-dimensional (3D) EM waves, the boundary conditions are derived from constitutive parameters of cloak's material, without resource to the Mie scattering approach, regardless of whether the source is placed in the interior or exterior of the cloak layer. In particular, the implicit boundary condition for 2D EM wave case is reported for the first time. In the analysis, some small quantities, which have been ignored in previous publications, are taken into account with the aid of Taylor's expansions, which leads to several interesting results that have never been reported before. The ideal cloaking is accompanied by shielding: There is a decoupling of the fields inside and outside of the cloaked region, so that external observations do not provide any indication of the presence of a cloaked object, nor is any information about the fields outside detectable inside the cloaked region. In many real-world applications, however, there are needs for effectively cloaking sensors and detectors so that their presence may be less disturbing to the surrounding environment [13, 14]. This paper shows, by using Taylor's expansions, that a sensor can be cloaked in 2D EM waves, i.e., waves can penetrate into the interior of the cloak layer without exterior scattering.

2. TWO-DIMENSIONAL ELECTROMAGNETIC WAVES

For a 2D EM cloak [2, 12, 15], the radial linear transform yields the permittivity and permeability tensors, $\epsilon_\rho/\epsilon_0 = \mu_\rho/\mu_0 = (\rho - R_1)/\rho$, $\epsilon_\phi/\epsilon_0 = \mu_\phi/\mu_0 = \rho/(\rho - R_1)$, and $\epsilon_z/\epsilon_0 = \mu_z/\mu_0 = [R_2/(R_2 - R_1)]^2(\rho - R_1)/\rho$, where R_1 and R_2 are the inner and outer radii of the cloak, respectively. To avoid singularities, the inner boundary of cloak layer is shifted to $\rho = R_1 + \delta$, where δ is a small positive number. The ideal cloaking can be obtained as δ approaches zero. We consider the TM^z mode with respect to the longitudinal direction $\hat{z}$ (Alternatively, it is referred to as the TE^{xy} mode with respect to the xy plane).

2.1. Source Is inside the Cloak

When a time-harmonic source, with $\exp(-i\omega t)$ dependence, is placed in the interior of the cloak, the z component of electric field in each region can be expressed as

$$E_z^{\text{ext}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} A_n H_n^{(1)}(k_0\rho)e^{in\phi} \quad \text{for} \ \rho > R_2 \tag{1}$$

$$E_z^{\text{clo}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} \left[B_n J_n(k_c(\rho-R_1)) + C_n H_n^{(1)}(k_c(\rho-R_1)) \right] e^{in\phi}$$

$$\text{for} \ R_1+\delta < \rho < R_2 \tag{2}$$

$$E_z^{\text{int}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} \left[D_n J_n(k_0\rho) + K_n H_n^{(1)}(k_0\rho) \right] e^{in\phi},$$

$$\text{for} \ 0 < \rho < R_1 + \delta, \tag{3}$$

where $k_c = k_0 R_2/(R_2 - R_1)$. The magnetic field can be derived as

$$H_\rho = \frac{1}{i\omega\mu_\rho\rho} \frac{\partial E_z}{\partial \phi} \tag{4}$$

$$H_\phi = -\frac{1}{i\omega\mu_\phi} \frac{\partial E_z}{\partial \rho}. \tag{5}$$

Since the source is given, K_n is uniquely determined. For each multipole (of order n), we will solve for four unknowns A_n, B_n, C_n, and D_n. Thus, by matching boundary conditions, i.e., E_z and H_ϕ are continuous at $\rho = R_1 + \delta$ and $\rho = R_2$, we obtain

$$A_n/K_n = C_n/K_n = \frac{2i}{\pi k_0(R_1 + \delta)}/F_n \tag{6}$$

$$B_n/K_n = 0 \tag{7}$$

$$D_n/K_n = G_n/F_n, \tag{8}$$

where

$$F_n = \frac{k_c\delta}{k_0(R_1+\delta)} H_n^{(1)\prime}(k_c\delta) J_n(k_0(R_1+\delta)) - H_n^{(1)}(k_c\delta) J_n'(k_0(R_1+\delta)) \tag{9}$$

$$G_n = -\frac{k_c\delta}{k_0(R_1+\delta)} H_n^{(1)\prime}(k_c\delta) H_n^{(1)}(k_0(R_1+\delta))$$

$$+ H_n^{(1)}(k_c\delta) H_n^{(1)\prime}(k_0(R_1+\delta)) \tag{10}$$

and the Wronskian $J_n(z)H_n^{(1)\prime}(z) - J_n'(z)H_n^{(1)}(z) = 2i/(\pi z)$ is used.

From here onwards, we will use the fact that δ is a small parameter to simplify the obtained values of C_n and D_n.

2.1.1. For $n = 0$

For an infinitesimal parameter z, we have the asymptotics $H_0^{(1)}(z) \approx q_0 \ln z$ and $J_0(z) \approx 1 - z^2/4$ [16]. Using Taylor's expansion, we obtain

$$F_0 \approx \frac{q_0}{k_0 R_1} J_0(k_0 R_1) - q_0 \ln(k_c \delta) J_0'(k_0 R_1) \tag{11}$$

$$G_0 \approx -\frac{q_0}{k_0 R_1} H_0^{(1)}(k_0 R_1) + q_0 \ln(k_c \delta) H_0^{(1)\prime}(k_0 R_1) \tag{12}$$

When $J_0'(k_0 R_1) \neq 0$, we obtain $D_0/K_0 = -H_0^{(1)\prime}(k_0 R_1)/J_0'(k_0 R_1)$ and $C_0/K_0 = O(1/\ln(\delta)))$, where the Landau big-o notation $O(\cdot)$ denotes terms of the same order, i.e., neglecting constants multipliers and higher-order terms. At the boundary $\rho = R_1 + \delta$, we see that following quantities are continuous across the boundary

$$B_\rho = 0 \tag{13}$$

$$H_\phi = O\left(\frac{1}{(R_1 + \delta)/\delta} C_0 H_n^{(1)\prime}(k_c \delta)\right) = O(1/\ln(\delta)) \tag{14}$$

As δ approaches zero, both B_ρ and H_ϕ approach zero.

When $J_0'(k_0 R_1) = 0$, it is easy to find that $D_0 = O(\ln(\delta))$, which is infinite as δ approaches zero. This phenomenon is due to inserting an active source inside a resonator. In practice, loss must be taken into account, which avoids the appearance of infinity. The analysis of such case is beyond the scope of this paper.

2.1.2. For $n > 0$

For an infinitesimal parameter z and an integer $n > 0$, we have the asymptotics $H_n^{(1)}(z) = q_n z^{-n} + O(z^{-n+2})$ and $J_n(z) = p_n z^n + O(z^{n+2})$ [16]. Using Taylor's expansion, we obtain

$$F_n \approx (k_c \delta)^{-n} q_n \left\{ -\left[\frac{n}{k_0 R_1} J_n(k_0 R_1) + J_n'(k_0 R_1)\right] + O(\delta)\right\} \tag{15}$$

$$G_n \approx (k_c \delta)^{-n} q_n \left\{ \left[\frac{n}{k_0 R_1} H_n^{(1)}(k_0 R_1) + H_n^{(1)\prime}(k_0 R_1)\right] + O(\delta)\right\} \tag{16}$$

When $J_n'(k_0 R_1) + \frac{n}{k_0 R_1} J_n(k_0 R_1) \neq 0$, we obtain $D_n/K_n = -[H_n^{(1)\prime}(k_0 R_1) + \frac{n}{k_0 R_1} H_n^{(1)}(k_0 R_1)]/[J_n'(k_0 R_1) + \frac{n}{k_0 R_1} J_n(k_0 R_1)]$ and $C_n/K_n = O(\delta^n)$. At the boundary $\rho = R_1 + \delta$, we see that although B_ρ and H_ϕ do not vanish as δ approaches zero, a linear combination of them,

$$B_\rho - i\mu_0 H_\phi = C_n \frac{e^{in\phi}}{\omega(R_1 + \delta)} \left[n H_n^{(1)}(k_c \delta) + k_c \delta H_n^{(1)\prime}(k_c \delta)\right]$$
$$= C_n O((k_c \delta)^{-n+2}) = O(\delta^2) \tag{17}$$

does vanish as δ approaches zero. As discussed in Section 2.1.1, the case when $J_n'(k_0 R_1) + \frac{n}{k_0 R_1} J_n(k_0 R_1) = 0$ is not discussed in this paper.

2.2. Source Is outside of Cloak

When a source is placed in outside the cloak, E_z in each region can be expressed as

$$E_z^{\text{ext}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} \left[K_n J_n(k_0 \rho) + A_n H_n^{(1)}(k_0 \rho) \right] e^{in\phi} \quad \text{for } \rho > R_2 \quad (18)$$

$$E_z^{\text{clo}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} \left[B_n J_n(k_c(\rho - R_1)) + C_n H_n^{(1)}(k_c(\rho - R_1)) \right] e^{in\phi}$$
$$\text{for } R_1 + \delta < \rho < R_2 \quad (19)$$

$$E_z^{\text{int}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} D_n J_n(k_0 \rho) e^{in\phi}, \quad \text{for } 0 < \rho < R_1 + \delta. \quad (20)$$

Since the source is given, K_n is uniquely determined. By matching E_z and H_ϕ at $\rho = R_1 + \delta$ and $\rho = R_2$, we obtain

$$B_n/K_n = 1 \quad (21)$$

$$A_n/K_n = C_n/K_n = G_n/F_n \quad (22)$$

$$D_n/K_n = \frac{2i}{\pi k_c \delta}/F_n \quad (23)$$

where

$$F_n = J_n(k_0(R_1 + \delta)) H_n^{(1)\prime}(k_c \delta) - \frac{k_0(R_1 + \delta)}{k_c \delta} J_n'(k_0(R_1 + \delta)) H_n^{(1)}(k_c \delta) \quad (24)$$

$$G_n = -J_n(k_0(R_1 + \delta)) J_n'(k_c \delta) + \frac{k_0(R_1 + \delta)}{k_c \delta} J_n'(k_0(R_1 + \delta)) J_n(k_c \delta) \quad (25)$$

2.2.1. For n = 0

Using Taylor's expansion, we obtain

$$F_0 \approx -q_0 \frac{k_0 R_1}{k_c \delta} \ln(k_c \delta) J_0'(k_0 R_1) + \frac{q_0}{k_c \delta} J_0(k_0 R_1) \quad (26)$$

$$G_0 \approx \frac{k_0 R_1}{k_c \delta} J_0'(k_0 R_1) + O(\delta^0) \quad (27)$$

When $J_0'(k_0 R_1) \neq 0$, we obtain $C_0/K_0 = O(1/\ln(\delta))$ and $D_0/K_0 = O(1/\ln(\delta))$, both of which approach zero as δ approaches zero.

When $J_0'(k_0 R_1) = 0$, we obtain $C_0/K_0 = O(\delta)$ and $D_0/K_0 = O(\delta^0)$. Thus, whereas the scattered field ($A_0 = C_0$) approaches zero as δ approaches zero, the interior field (D_0) does not. To the best of author's knowledge, this property has never been reported before. This property can be used to cloak a sensor, which will be discussed in Section 2.2.3.

At the boundary $\rho = R_1 + \delta$, we see that no matter $J_0'(k_0 R_1)$ is equal to zero or not the following quantities are continuous across the boundary

$$B_\rho = 0 \tag{28}$$

$$H_\phi = O\left(D_0 J_0'(k_0(R_1 + \delta))\right) = O(1/\ln(\delta)) \text{ or } O(\delta) \tag{29}$$

As δ approaches zero, both B_ρ and H_ϕ approach zero.

2.2.2. For $n > 0$

Using Taylor's expansion, we obtain

$$F_n \approx (k_c \delta)^{-n-1} q_n \left\{ \left[-n J_n(k_0 R_1) - k_0 R_1 J_n'(k_0 R_1) \right] + O(\delta) \right\} \tag{30}$$

$$G_n \approx (k_c \delta)^{n-1} p_n \left\{ \left[-n J_n(k_0 R_1) + k_0 R_1 J_n'(k_0 R_1) \right] + O(\delta) \right\} \tag{31}$$

When $J_n'(k_0 R_1) + \frac{n}{k_0 R_1} J_n(k_0 R_1) \neq 0$, we obtain $C_n/K_n = O(\delta^{2n})$ and $D_n/K_n = O(\delta^n)$, both of which approach zero as δ approaches zero.

When $J_n'(k_0 R_1) + \frac{n}{k_0 R_1} J_n(k_0 R_1) = 0$, we obtain $C_n/K_n = O(\delta^{2n-1})$ and $D_n/K_n = O(\delta^{n-1})$. For $n > 1$, both terms approach zero as δ approaches zero. For $n = 1$, whereas the scattered field ($A_1 = C_1 = O(\delta)$) approaches zero as δ approaches zero, the interior field ($D_1 = O(\delta^0)$) does not. As mentioned earlier, this property can be used to cloak a sensor.

At the boundary $\rho = R_1 + \delta$, both $B_\rho = D_n n/[\omega(R_1 + \delta)] J_n(k_0(R_1 + \delta))e^{in\phi}$ and $H_\phi = -D_n k_0/(i\omega\mu_0) J_n'(k_0(R_1 + \delta))e^{in\phi}$ are continuous. For the cases (1) $n > 1$ or (2) $n = 1$ and $J_n'(k_0 R_1) + \frac{n}{k_0 R_1} J_n(k_0 R_1) \neq 0$, both B_ρ and H_ϕ approach zero as δ approaches zero. For $n = 1$ and $J_n'(k_0 R_1) + \frac{n}{k_0 R_1} J_n(k_0 R_1) = 0$, although B_ρ and H_ϕ do not vanish as δ approaches zero, a linear combination of them,

$$B_\rho - i\mu_0 H_\phi = \frac{D_n}{\omega} \left[\frac{n}{R_1} J_n(k_0 R_1) + k_0 J_n'(k_0 R_1) + O(\delta) \right] e^{in\phi}$$

$$= D_1 O(\delta) = O(\delta) \tag{32}$$

does vanish as δ approaches zero.

2.2.3. Cloaking a Sensor

In ideal cloaking, there is a decoupling of the fields inside and outside of the cloaked region, i.e., external observations cannot detect the presence of a cloaked object, nor is any field outside detectable inside the cloaked region. In many real-world applications, however, there are needs for effectively cloaking sensors and detectors so that their presence may be less disturbing to the surrounding environment. The interesting phenomenon presented in Section 2.2.1 and Section 2.2.2 can be used to cloak a sensor. We insert a cylinder of radius R_0 inside the cloak, with its center at the origin. The sensor is placed inside the inserted cylinder and is able to measure the electric field at the surface of the cylinder. The inserted cylinder has the surface impedance boundary condition $E_z/H_\phi = (i\omega\mu_0/k_0)/\alpha$. The electric field in the space $R_0 \leq \rho \leq R_1$ is given by

$$E_z^{\text{int}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} \left[D_n J_n(k_0\rho) + E_n H_n^{(1)}(k_0\rho) \right] e^{in\phi}, \qquad (33)$$

and the fields in other regions are still the same as Eqs. (18) and (19). The impedance boundary condition at $\rho = R_0$ yields

$$\left[D_n J_n'(k_0R_0) + E_n H_n^{(1)\prime}(k_0R_0) \right] + \alpha \left[D_n J_n(k_0R_0) + E_n H_n^{(1)}(k_0R_0) \right] = 0. \tag{34}$$

A straightforward calculation shows that when we choose $\alpha = -J_n'(k_0R_0)/J_n(k_0R_0)$, the coefficient E_n has be to zero. In this case, the linear equation system is exactly the same as those in Section 2.2.1 and Section 2.2.2. Thus, when $J_0'(k_0R_1) = 0$ or $J_1'(k_0R_1) + \frac{1}{k_0R_1}J_1(k_0R_1) = 0$ is satisfied, the electromagnetic fields in the region $R_0 \leq \rho \leq R_1$ do not approach zero and at the same time the scattered fields outside the cloak approach zero. This is to say it achieves cloaking a sensor. It is worth stressing that the advantage of the proposed cloaking-sensor method over those published ones [13, 14] is that it is much simpler.

2.3. Implicit Boundary Condition

Section 2.1 and Section 2.2 discuss several boundary conditions at $\rho = R_1 + \delta$. A careful observation shows that the boundary condition $B_\rho - i\mu_0 H_\phi = 0$ (as δ approaches zero) is always satisfied at $\rho = R_1 + \delta$, regardless of whether the source is in the interior or exterior of the cloak. In the aforementioned two Sections, we first obtain electromagnetic fields via solving the Mie scattering problem, and consequently obtain the aforementioned boundary condition. In fact,

this boundary condition is characteristic of the cloak-air interface, and it can be obtained without solving the Mie scattering problem. Now we only use the permittivity and permeability of the cloak to derive the boundary condition.

For $n = 0$, it is obvious that $B_\rho = 0$ and thus it is sufficient to show $H_\phi = 0$. Due to the asymptotics $H_0^{(1)}(z) \approx q_0 \ln z$ and $J_0(z) \approx 1 - z^2/4$ for a small parameter z [16], we see that the electric field at the inner boundary of cloak is dominated by $E_z(R_1 + \delta) \approx C_0 H_0^{(1)}(k_c\delta)$. Since the other side of the boundary is air and the electric field in air is finite, the continuity of E_z across the boundary yields $C_0 H_0^{(1)}(k_c\delta) = O(\delta^0)$. Thus, $C_0 = O(1/\ln(\delta))$. Consequently, we see from Eq. (4) that $H_\phi(R_1 + \delta) = O(\frac{1}{\mu_\phi} C_0 H_0^{(1)\prime}(k_c\delta)) = O(1/\ln(\delta))$, which vanishes as δ approaches zero.

For $n > 0$, we have the asymptotics $H_n^{(1)}(z) = q_n z^{-n} + O(z^{-n+2})$ and $J_n(z) = p_n z^n + O(z^{n+2})$ for an infinitesimal parameter z [16]. At the inner boundary of the cloak $\rho = R_1 + \delta$, the electric field at the inner boundary of cloak is dominated by $E_z(R_1 + \delta) \approx C_n H_n^{(1)}(k_c\delta)$. Since the other side of the boundary is air and the electric field in air is finite, the continuity of E_z across the boundary yields $C_n H_n^{(1)}(k_c\delta) = O(\delta^0)$. Thus, $C_n = O(\delta^n)$. We straightforwardly calculate from Eqs. (4) and (5) that $B_\rho - i\mu_0 H_\phi = e^{in\phi}/(\omega(R_1 + \delta))C_n \left[nH_n^{(1)}(k_c\delta) + k_c\delta H_n^{(1)\prime}(k_c\delta) \right] = O(C_n\delta^{-n+2}) = O(\delta^2)$, vanishes as δ approaches zero.

Thus, we have finished the proof that the condition $B_\rho - i\mu_0 H_\phi = 0$ (as δ approaches zero) is always satisfied at $\rho = R_1 + \delta$. To the best of our's knowledge, this boundary condition has never been reported before. In addition, this boundary condition can be proven without solving the Mie scattering problem.

3. THREE-DIMENSIONAL ELECTROMAGNETIC WAVES

For a 3D EM cloak [1, 2, 5, 6, 11, 17], the radial linear transform yields the permittivity and permeability tensors, $\epsilon_t/\epsilon_0 = \mu_t/\mu_0 = R_2/(R_2 - R_1)$, $\epsilon_r/\epsilon_t = \mu_r/\mu_t = (r - R_1)^2/r^2$, where the subscript t denotes the tangential components (i.e., components in the $\hat{\theta}$ and $\hat{\phi}$ directions). To avoid singularities, we let the inner boundary of the cloaking material be at $R_1 + \delta$, where δ is a small positive number.

Electromagnetic fields can be decomposed into two independent modes, TE and TM modes, which are dual to each other. For the TM

mode, the **B** field can be expressed as

$$\mathbf{B} = \nabla \times (\hat{r}\Phi_M).$$
(35)

Now the magnetic field **H** and the electric field **E** can be expressed as

$$\mathbf{H} = \mu_t^{-1} \left(\frac{1}{r\sin\theta} \frac{\partial \Phi_M}{\partial \phi} \hat{\theta} - \frac{1}{r} \frac{\partial \Phi_M}{\partial \theta} \hat{\phi} \right)$$
(36)

$$\mathbf{E} = \frac{\overline{\overline{\epsilon}}^{-1} \cdot \mu_t^{-1}}{-i\omega} \left[-\frac{1}{r^2\sin\theta} \left(\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial \Phi_M}{\partial\theta}) + \frac{1}{\sin\theta} \frac{\partial^2 \Phi_M}{\partial \phi^2} \right) \hat{r} \right.$$

$$\left. + \frac{1}{r} \frac{\partial^2 \Phi_M}{\partial r \partial\theta} \hat{\theta} + \frac{1}{r\sin\theta} \frac{\partial^2 \Phi_M}{\partial r \partial\phi} \hat{\phi} \right].$$
(37)

3.1. Source Is inside the Cloak

When a source is placed in the interior of the cloak, the potential Φ_M in the three regions is written as

$$\Phi_M^{\text{ext}}(r,\theta,\phi) = \sum_n \sum_m A_{nm} \hat{H}_n^{(1)}(k_0 r) Y_n^m(\theta,\phi) \quad \text{for } r > R_2$$
(38)

$$\Phi_M^{\text{clo}}(r,\theta,\phi) = \sum_n \sum_m \left[B_{nm} \hat{J}_n(k_c(r-R_1)) + C_{nm} \hat{H}_n^{(1)}(k_c(r-R_1)) \right]$$

$$Y_n^m(\theta,\phi) \qquad \text{for } R_1 + \delta < r < R_2$$
(39)

$$\Phi_M^{\text{int}}(r,\theta,\phi) = \sum_n \sum_m \left[D_{nm} \hat{J}_n(k_0 r) + K_{nm} \hat{H}_n^{(1)}(k_0 r) \right] Y_n^m(\theta,\phi)$$

$$\text{for } 0 < r < R_1 + \delta, \quad (40)$$

where $k_c = \omega\sqrt{\mu_t\epsilon_t}$, $\hat{J}_n(z) = zj_n(z)$ and $\hat{H}_n^{(1)}(z) = zh_n^{(1)}(z)$ are the Riccati-Bessel functions of the first and third kinds respectively, and Y_n^m are spherical harmonics.

Since the source is given, K_{nm} is uniquely determined. Eqs. (36) and (37) indicate that the continuities of tangential components of **H** and **E** across the boundaries $r = R_1 + \delta$ and $r = R_2$ amount to the continuities of Φ_M/μ_t and $(1/\mu_t\epsilon_t)\partial\Phi_M/\partial r$, respectively. We solve the linear equation system to obtain

$$A_{nm}/K_{nm} = i/F_n$$
(41)

$$B_{nm} = 0$$
(42)

$$C_{nm} = A_{nm}R_2/(R_2 - R_1)$$
(43)

$$D_{nm}/K_{nm} = G_n/F_n,$$
(44)

where

$$F_n = \hat{H}_n^{(1)\prime}(k_c\delta)\hat{J}_n(k_0(R_1+\delta)) - \hat{H}_n^{(1)}(k_c\delta)\hat{J}_n'(k_0(R_1+\delta)) \qquad (45)$$

$$G_n = -\hat{H}_n^{(1)\prime}(k_c\delta)\hat{H}_n^{(1)}(k_0(R_1+\delta)) + \hat{H}_n^{(1)}(k_c\delta)\hat{H}_n^{(1)\prime}(k_0(R_1+\delta)) \ (46)$$

and the Wronskian $\hat{J}_n(z)\hat{H}_n^{(1)\prime}(z) - \hat{J}_n'(z)\hat{H}_n^{(1)}(z) = i$ is used.

For an infinitesimal parameter z, we have the asymptotics $\hat{J}_n(z) \approx a_n z^{n+1}$ and $\hat{H}_n^{(1)}(z) \approx b_n z^{-n}$ [16]. Using Taylor's expansion, we have

$$F_n \approx -b_n(k_c\delta)^{-(n+1)}\left[n\hat{J}_n(k_0R_1) + \delta(nk_0+k_c)\hat{J}_n'(k_0R_1)\right] \quad (47)$$

$$G_n \approx b_n(k_c\delta)^{-(n+1)}\left[n\hat{H}_n^{(1)}(k_0R_1) + \delta(nk_0+k_c)\hat{H}_n^{(1)\prime}(k_0R_1)\right]. \ (48)$$

When $\hat{J}_n(k_0R_1) \neq 0$, $A_{nm} = O(\delta^{n+1})$, and $D_{nm}/K_{nm} = -\hat{H}_n^{(1)}(k_0R_1)/\hat{J}_n(k_0R_1)$. Thus, we see from Eq. (40) that at the inner boundary of the cloak $r = R_1 + \delta$, the potential Φ_M approaches zero as δ approaches zero, and consequently Eqs. (36) and (37) yield that H_θ, H_ϕ and D_r all approach zero.

When $\hat{J}_n(k_0R_1) = 0$, $A_{nm} = O(\delta^n)$, and $D_{nm} = O(\delta^{-1})$. In practice, loss must be taken into account to avoid the appearance of infinity. As discussed in Section 2.1.1, such case is beyond the scope of this paper.

3.2. Source Is outside Cloak

When the source is placed outside the cloak, the potential Φ_M in the three regions is written as

$$\Phi_M^{\text{ext}}(r,\theta,\phi) = \sum_n\sum_m\left[K_{nm}\hat{J}_n(k_0r) + A_{nm}\hat{H}_n^{(1)}(k_0r)\right]Y_n^m(\theta,\phi)$$
$$\text{for } r > R_2 \quad (49)$$

$$\Phi_M^{\text{clo}}(r,\theta,\phi) = \sum_n\sum_m\left[B_{nm}\hat{J}_n(k_c(r-R_1)) + C_{nm}\hat{H}_n^{(1)}(k_c(r-R_1))\right]$$
$$Y_n^m(\theta,\phi) \qquad \text{for } R_1+\delta < r < R_2 \quad (50)$$

$$\Phi_M^{\text{int}}(r,\theta,\phi) = \sum_n\sum_m D_{nm}\hat{J}_n(k_0r)Y_n^m(\theta,\phi), \ \text{for } R_0 < r, < R_1+\delta. \ (51)$$

Since the source is given, K_{nm} is uniquely determined. By matching boundary conditions, we obtain

$$A_{nm}/K_{nm} = G_n/F_n \qquad (52)$$

$$B_{nm}/K_{nm} = R_2/(R_2 - R_1) \qquad (53)$$

$$C_{nm} = A_{nm}R_2/(R_2 - R_1) \qquad (54)$$

$$D_{nm}/K_{nm} = i/F_n, \qquad (55)$$

where

$$F_n = \hat{J}_n(k_0(R_1 + \delta))\hat{H}_n^{(1)\prime}(k_c\delta) - \hat{J}_n'(k_0(R_1 + \delta))\hat{H}_n^{(1)}(k_c\delta) \quad (56)$$

$$G_n = -\hat{J}_n(k_0(R_1 + \delta))\hat{J}_n'(k_c\delta) + \hat{J}_n'(k_0(R_1 + \delta))\hat{J}_n(k_c\delta). \quad (57)$$

Using small parameter approximations, we have

$$F_n \approx -b_n(k_c\delta)^{-(n+1)}\left[n\hat{J}_n(k_0R_1) + \delta(nk_0 + k_c)\hat{J}_n'(k_0R_1)\right] \quad (58)$$

$$G_n \approx a_n(k_c\delta)^n\left[-(n+1)\hat{J}_n(k_0R_1) + \delta(-(n+1)k_0 + k_c)\hat{J}_n'(k_0R_1)\right]. (59)$$

When $\hat{J}_n(k_0R_1) \neq 0$, $A_{nm} = O(\delta^{2n+1})$, and $D_{nm} = O(\delta^{n+1})$. At the inner boundary of the cloak $r = R_1 + \delta$, the potential Φ_M approaches zero as δ approaches zero, and consequently Eqs. (36) and (37) yield that H_θ, H_ϕ and D_r all approach zero.

When $\hat{J}_n(k_0R_1) = 0$, by Wronskian we see that $\hat{J}_n'(k_0R_1) \neq 0$. Consequently, $A_{nm} = O(\delta^{2n+1})$, and $D_{nm} = O(\delta^n)$. Since the lowest multipole in electromagnetic radiation is the dipole, n starts from one. Thus, D_{nm} approaches zero as δ approaches zero. At the inner boundary of the cloak, the potential Φ_M approaches zero as δ approaches zero, and consequently H_θ, H_ϕ and D_r all approach zero.

3.2.1. Cloaking a Sensor

Since the electromagnetic fields inside the cloak always approach zero as δ approaches zero, no matter $\hat{J}_n(k_0R_1)$ equals zero or not, the interesting sensor effect, as presented in Section 2.2.3 in 2-D case, does not appear in 3-D case. The main reason for such a difference is that the lowest multipole in 2D case is monopole, whereas it is dipole in 3D case. It is worth mentioning that although the radial linear transformation cannot achieve cloaking a sensor for 3D EM waves, there are advanced linear transformations that can achieve it [18].

3.3. Implicit Boundary Condition

For the TM mode, from Eq. (37), we see that the tangential components of electric field (E_θ and E_ϕ) at the inner boundary of cloak $r = R_1 + \delta$ is dominated by $C_{nm}\partial\hat{H}_n^{(1)}(k_c\delta)/\partial r$. Since the other side of the boundary is air and the electric field in air is finite, the continuity of E_θ and E_ϕ requires $C_{nm}\partial\hat{H}_n^{(1)}(k_c\delta)/\partial r = O(\delta^0)$. Thus, $C_{nm} = O(\delta^{n+1})$. We straightforwardly calculate from Eqs. (39) and (50) that $\Phi_M^{clo}(R_1+\delta) = O(C_{nm}\hat{H}_n^{(1)}(k_c\delta)) = O(C_{nm}\delta^{-n}) = O(\delta^1)$, which vanishes as δ approaches zero. Consequently, Eqs. (36) and (37)

yield that H_θ, H_ϕ, D_r all approach zero as δ approaches zero. Thus, the implicit boundary condition for the TM mode is the perfectly magnetic conductor (PMC). For the TE mode, we obtain by duality that the implicit boundary condition is the perfectly electric conductor (PEC). Alternatively, no matter for the TE mode or TM mode, or a linear combination of them, the implicit boundary condition can be summarized as $D_r = B_r = 0$. Once again, the implicit boundary is obtained without solving the Mie scattering problem.

4. CONCLUSION AND DISCUSSION

Transformation-optics cloaking for 2D and 3D electromagnetic waves is discussed in this paper. No matter if the source is placed in the interior or exterior of the cloak layer, there are implicit boundary conditions in the inner boundary of the cloak. More importantly, the two implicit boundary conditions are derived without solving Mie scattering problems. In particular, the implicit boundary condition for 2D electromagnetic wave case is reported for the first time. Taylor's expansion is used to take into account some small quantities, which leads to several interesting results that have never been reported before. One of such results is that a sensor can be cloaked in 2D electromagnetic waves, i.e., waves can penetrate into the interior of the cloak layer without exterior scattering, which has many real-world applications since the presence of sensor that is cloaked is less disturbing to the surrounding environment. The method presented in this paper can be extended to study acoustic wave cloaking as well. For an acoustic cloak that is based on the radial linear transformation, the normal component of velocity approaches zero at the inner boundary of the cloak. In a way similar to that presented in Section 2.2.3, cloaking a sensor can also be achieved for the monopole component of acoustic wave when the condition $j_0'(k_0 R_1) = 0$ is satisfied. Finally, it is worth mentioning that since the cloak layer is made of metamaterials,which are usually periodic composite materials, it may be difficult in practice to accurately control the infinitely small perturbation of the inner boundary of the cloak layer.

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