

FIELDS IN FRACTIONAL PARALLEL PLATE DB WAVEGUIDES

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Abstract—Time harmonic electric and magnetic fields inside a parallel plate DB boundary waveguide are derived and fractional curl operator is utilized to study the fractional parallel plate DB waveguides. The DB boundary conditions are incorporated by assuming the behavior of boundary as perfect electric conductor (PEC) for transverse electric mode and perfect magnetic conductor (PMC) for transverse magnetic mode. For this purpose a general wave propagating inside the parallel plate wave waveguide is assumed and decomposed into TE and TM modes. Behavior of the fields and transverse impedances of the walls of guide are studied with respect to the fractional parameter describing the order of the fractional curl operator. The results are compared with the corresponding results for fractional waveguides with PEC walls.

1. INTRODUCTION

Fractional derivatives and integrals are mathematical operators involving differentiation and integration of arbitrary (non-integer) real or complex orders. Fractional calculus is a branch of mathematical analysis that studies the properties of taking arbitrary real or complex order differentiation and integration operators [1]. It generalizes the classical calculus, so in this respect traditional calculus may be taken as a special case of the fractional calculus. Using fractional calculus, scientists and engineers have been interested in exploring the potential

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utilities and possible physical implications of mathematical machinery of the subject of fractional calculus, i.e., fractional derivatives and fractional integrals [2]. It has been demonstrated that these mathematical operators are interesting and useful tools in various disciplines of science and engineering [3–5]. Electromagnetic theory has an important role in the modern world of science and engineering. Maxwell equations encapsulate all this revolutionary discipline, whose solutions have a great importance in current research and development activities and fractionalization of solutions to the Maxwell equations has been part of research interests since many years.

Fifteen years ago, Engheta particularly focused on finding out what possible applications and/or physical role, the mathematical operators of fractional calculus can have in electromagnetic theory [6–11]. He applied the concept of fractional derivatives/integrals to certain electromagnetic problems, and obtained interesting results and ideas showing that these mathematical operators are interesting and useful mathematical tools in electromagnetic theory. Some of these ideas include the mathematical link between the electrostatic image methods for the conducting sphere and the dielectric sphere [6], fractional solutions for the scalar Helmholtz equation [7, 10], electrostatic fractional image methods for perfectly conducting wedges and cones [8], and the novel concept of fractional multipoles [9].

Tarasov proved that the electromagnetic fields in dielectric media, whose susceptibility follows a fractional power-law dependence in a wide frequency range, can be described by differential equations with time derivatives of noninteger order [12, 13]. Fractional dimensional space represents an effective physical description of confinement in low-dimensional systems. The concept of fractional calculus to obtain the solution of electrostatic problem in fractional dimensional space, for the fractional order multipoles, is utilized by Muslih and Baleanu [14]. They also introduced the form of fractional scalar potential by finding the solutions of Laplace's equation in fractional dimensional space [15]. They derived potential of charge distribution in fractional space using Gegenbauer polynomials. According to Zubair et al., solutions of Helmholtz equation in fractional space can describe the complex phenomenon of wave propagation in fractal media. With this view, they established a generalized Helmholtz equation for wave propagation in fractional space and found its analytical solution [16–21].

Fractionalization of ordinary derivative and integral operators motivated the researchers to find out the possible fractionalization of other operators and their use in electromagnetics. Engheta fractionalized the kernel of an integral transform and studied

the paradigm of intermediate zones in electromagnetism [22, 23]. Fractionalization of the curl operator, a well known operator in vector calculus, was also introduced by him [24]. He used the fractional curl operator to find the new solutions, which may be regarded as intermediate step between the two given solutions, to the Maxwell equations.

For the sake of completeness, it is decided first to reproduce the meanings of the following concepts given by Engheta [24]: what is meant by fractionalization of a linear operator? How to fractionalize a linear operator? That is, what is the mathematical recipe to fractionalize a linear operator? Fractionalization of the curl operator using this recipe and utilization of the fractional curl operator to find intermediate/fractional solutions to the Maxwell equations for ordinary medium has been discussed [24] and just results of their work are presented here. Time harmonic dependency $\exp(-i\omega t)$ has been considered throughout the paper.

1.1. Fractional Linear Operator in Electromagnetics

1.1.1. Conditions Required to Verify Fractionalization

Fractionalization of any mathematical problem requires two canonical solutions of the problem under consideration and an operator that can transform one canonical solution into the other. Fractionalization of the connecting operator can reveal intermediate solutions between the two canonical solutions. The conditions and recipe for fractionalization of a linear operator $\underline{\underline{L}}$ are reproduced below [24]. It must be mentioned that this recipe has also been used to fractionalize the Fourier transform [25]. The new fractionalized operator $\underline{\underline{L}}^\alpha$ with fractional parameter α , under certain conditions, can be used to obtain the intermediate cases between the canonical case 1 and case 2.

A linear operator $\underline{\underline{L}}$ may become a fractional operator (i.e., $\underline{\underline{L}}^\alpha$) that provides the intermediate solutions to the original problems, if it satisfies the following properties [11, 24].

I. For $\alpha = 1$, the fractional operator $\underline{\underline{L}}^\alpha$ should become the original operator $\underline{\underline{L}}$, which provides us with case 2 when it is applied to case 1. It may be noted that situation for $\alpha = 1$ is the initially given situation.

II. For $\alpha = 0$, the operator $\underline{\underline{L}}^\alpha$ should become the identity operator $\underline{\underline{I}}$ and thus the case 1 can be mapped into itself.

III. For any two values α_1 and α_2 of fractional parameter, $\underline{\underline{L}}^\alpha$ have the additive property in α , i.e.,

$$\underline{\underline{L}}^{\alpha_1} \cdot \underline{\underline{L}}^{\alpha_2} = \underline{\underline{L}}^{\alpha_2} \cdot \underline{\underline{L}}^{\alpha_1} = \underline{\underline{L}}^{\alpha_1 + \alpha_2} \quad (1)$$

IV. In order to fractionalize given problem, the operator $\underline{\underline{L}}^\alpha$

should commute with the operator involved in the mathematical description of the problem under consideration. For example: $\underline{\underline{\mathbf{L}}}^\alpha (\nabla \times \mathbf{F}) = \nabla \times (\underline{\underline{\mathbf{L}}}^\alpha \mathbf{F})$.

1.1.2. Recipe for Fractionalization

It is assumed that, present discussion is about a class of linear operators (or mappings) where the domain and range of any linear operator of this class are similar to each other and have the same dimensions. That is, $\underline{\underline{\mathbf{L}}}^\alpha : C^n \rightarrow C^n$ where C^n is a n dimensional vector space over the field of complex numbers. Once a linear operator such as $\underline{\underline{\mathbf{L}}}$ is given, the recipe for constructing the fractional operator $\underline{\underline{\mathbf{L}}}^\alpha$ can be described as follows [24].

1. One finds the eigenvectors and eigenvalues of the operator $\underline{\underline{\mathbf{L}}}$ in the space C^n so that $\underline{\underline{\mathbf{L}}} \cdot \mathbf{A}_m = a_m \mathbf{A}_m$, where \mathbf{A}_m and a_m for $m = 1, 2, 3, \dots, n$, are the eigenvectors and eigenvalues of the operator $\underline{\underline{\mathbf{L}}}$ in space C^n respectively. In present analysis we are dealing with space C^3 .
2. Provided \mathbf{A}_m s form a complete orthonormal basis in the space C^n , any vector in this space can be expressed in terms of linear combination of \mathbf{A}_m . Thus an arbitrary vector \mathbf{G} in space C^n can be written as

$$\mathbf{G} = \sum_{m=1}^n g_m \mathbf{A}_m \quad (2)$$

where g_m s are co-efficients of expansion of \mathbf{G} in terms of \mathbf{A}_m s.

3. Having obtained the eigenvectors and eigenvalues of the operator $\underline{\underline{\mathbf{L}}}$, the fractional operator $\underline{\underline{\mathbf{L}}}^\alpha$ can be seen to have the same eigenvectors \mathbf{A}_m s but with the eigenvalues as $(a_m)^\alpha$, i.e.,

$$\underline{\underline{\mathbf{L}}}^\alpha \cdot \mathbf{A}_m = (a_m)^\alpha \mathbf{A}_m \quad (3)$$

When this fractional operator $\underline{\underline{\mathbf{L}}}^\alpha$ operates on an arbitrary vector \mathbf{G} in the space C^n , one gets

$$\begin{aligned} \underline{\underline{\mathbf{L}}}^\alpha \cdot \mathbf{G} &= \underline{\underline{\mathbf{L}}}^\alpha \sum_{m=1}^n g_m \mathbf{A}_m = \sum_{m=1}^n g_m \underline{\underline{\mathbf{L}}}^\alpha \cdot \mathbf{A}_m \\ &= \sum_{m=1}^n g_m (a_m)^\alpha \mathbf{A}_m \end{aligned} \quad (4)$$

The above equation essentially defines the fractional operator $\underline{\underline{\mathbf{L}}}^\alpha$ from the knowledge of operator $\underline{\underline{\mathbf{L}}}$ and its eigenvectors and eigenvalues. In the next section, above recipe has been applied to fractionalize the curl operator.

1.1.3. Fractional Curl Operator and Maxwell Equations

Consider a three-dimensional vector field \mathbf{F} as a function of three cartesian space coordinates (x, y, z) . Curl of this vector can be written as

$$\text{curl}\mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}} \quad (5)$$

where F_x, F_y, F_z are the cartesian components of vector \mathbf{F} and $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are the unit vectors in the spatial domain. Assuming that spatial Fourier transforms of both the vector functions (\mathbf{F} and $\text{curl}\mathbf{F}$) exist, the Fourier transform of these two vectors can be written as

$$\begin{aligned} & \mathcal{F}_k \{\mathbf{F}(x, y, z)\} \\ &= \tilde{\mathbf{F}}(k_x, k_y, k_z) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{F}(x, y, z) \exp(-ik_x x - ik_y y - ik_z z) dx dy dz \quad (6) \\ & \mathcal{F}_k \{\text{curl}\mathbf{F}(x, y, z)\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{curl}\mathbf{F}(x, y, z) \exp(-ik_x x - ik_y y - ik_z z) dx dy dz \\ &= i\mathbf{k} \times \tilde{\mathbf{F}}(k_x, k_y, k_z) \quad (7) \end{aligned}$$

where a tilde over the vector $\tilde{\mathbf{F}}$ denotes the Fourier transform of vector \mathbf{F} . Hence in the k -domain (k_x, k_y, k_z) , the curl operator can be written as a cross product of vector $i\mathbf{k}$ with the vector $\tilde{\mathbf{F}}$. The fractionalization of curl operator is equivalent to the fractionalization of this cross product operator. With the recipe described in the previous section, fractionalization of the cross product operator as $(i\mathbf{k} \times)^\alpha$ can be obtained in the k -domain.

Engheta utilized the fractional curl operator to fractionalize the principle of duality in electromagnetics [24]. Fractionalization of the principle of duality yields new solutions to the Maxwell equations, which may be regarded as intermediate step between the original solution and dual to the original solution and have been termed as fractional dual solutions. In an isotropic, homogeneous, and source free medium described by wavenumber k and impedance η , new set of solutions to the source-free Maxwell equations may be obtained using the following relations [24]

$$\tilde{\mathbf{E}}_{fd} = \left[\frac{1}{(ik)^\alpha} (i\mathbf{k} \times)^\alpha \tilde{\mathbf{E}} \right] \quad (8a)$$

$$\eta \tilde{\mathbf{H}}_{fd} = \left[\frac{1}{(ik)^\alpha} (i\mathbf{k} \times)^\alpha \eta \tilde{\mathbf{H}} \right] \quad (8b)$$

where subscript fd stands for the fractional dual. Inverse Fourier transforming these back into the (x, y, z) -domain, the new set of solutions are obtained as

$$\mathbf{E}_{fd} = \left[\frac{1}{(ik)^\alpha} \text{curl}^\alpha \mathbf{E} \right] \quad (9a)$$

$$\eta \mathbf{H}_{fd} = \left[\frac{1}{(ik)^\alpha} \text{curl}^\alpha (\eta \mathbf{H}) \right] \quad (9b)$$

From Eqs. (9), it can be seen that for $\alpha = 0$, $(\mathbf{E}_{fd}, \eta \mathbf{H}_{fd})$ gives the original solutions whereas $(\mathbf{E}_{fd}, \eta \mathbf{H}_{fd})$ gives dual to the original solution to the Maxwell equations for $\alpha = 1$. Therefore for all values of α between zero and unity, $(\mathbf{E}_{fd}, \eta \mathbf{H}_{fd})$ provides the new set of solutions which can effectively be regarded as intermediate solutions between the the original solution and dual to original solution. These solutions are also called the fractional dual fields as expressed with the subscript fd .

1.2. Previous Contributions

As it has already been stated that concept of the fractional curl operator and its utilization in electromagnetics was given by Engheta [24]. Naqvi and Rizvi extended Engheta's work on fractional curl operator by determining sources corresponding the fractional dual solutions to the Maxwell equations. Results of their valuable work show that surface impedance of the planar reflector, which is intermediate step between the PEC and PMC, is anisotropic in nature [26]. Naqvi et al. further extended work on this topic by finding fractional dual solutions to the Maxwell equations for reciprocal, homogenous, and lossless chiral medium [27]. Lakhtakia pointed out that any fractional operator that commutes with curl operator may yield fractional solutions [23]. Naqvi and Abbas studied the role of complex and higher order fractional curl operators in electromagnetic wave propagation [28]. They also studied the fractional dual solutions in double negative (DNG) medium [29]. Veliev further extended the work on the fractional curl operator by finding the reflection coefficients and surface impedance corresponding to fractional dual planar surfaces with planar impedance surface as original problem [30]. The work on this topic entered into new era when concept of fractional transmission lines, fractional waveguides, and fractional resonator in electromagnetics were introduced [31–40] and nature of the modes supported by fractional dual waveguides and impedance of the walls were addressed. Modelling of transmission of electromagnetic plane wave through a chiral slab using fractional curl operator and fractional dual solutions in bi-isotropic medium are also available [41, 42].

After the introduction of nihility concept by Lakhtakia [43], Tretyakov et al. incorporated the nihility conditions to chiral medium and proposed another metamaterial termed as chiral nihility metamaterial [44, 45]. Chiral nihility is a metamaterial with following properties of constitutive parameters at certain frequency [45].

$$\epsilon \rightarrow 0, \quad \mu \rightarrow 0, \quad \kappa \neq 0$$

Thus the resulting constitutive relations for isotropic chiral nihility metamaterial reduce to

$$\mathbf{D} = i\kappa\sqrt{\epsilon_0\mu_0}\mathbf{H} \quad (10a)$$

$$\mathbf{B} = -i\kappa\sqrt{\epsilon_0\mu_0}\mathbf{E} \quad (10b)$$

Tellegen nihility [46] states that

$$\epsilon \rightarrow 0, \quad \mu \rightarrow 0, \quad \kappa \rightarrow 0, \quad \chi \neq 0$$

and corresponding expressions for constitutive relations for Tellegen nihility metamaterial are

$$\mathbf{D} = \chi\sqrt{\epsilon_0\mu_0}\mathbf{H} \quad (11a)$$

$$\mathbf{B} = \chi\sqrt{\epsilon_0\mu_0}\mathbf{E} \quad (11b)$$

Study of nihility/chiral nihility metamaterials is a topic of current research by several researchers [47–57]. Naqvi contributed many research articles on chiral nihility and fractional dual solutions in chiral nihility metamaterial [51–57].

1.3. DB Boundary Conditions

Before the advent of the idea ‘DB boundary interface’ as proposed by Lindell and Sihvola [58, 59], all the known interfaces dealt with tangential components of electric and magnetic fields. But the DB boundary is analyzed on the basis of normal components of flux densities \mathbf{D} and \mathbf{B} [59, 60]. Waves polarized transverse electric (TE) and transverse magnetic (TM) with respect to the normal of the boundary are reflected as from perfect electric conductor (PEC) and perfect magnetic conductor (PMC) planes (i.e., DB interface behaves like PEC and PMC) [61]. It is worth mentioning here that, all of the previous boundary conditions in electromagnetics are associated to the electromagnetic field vectors \mathbf{E} and \mathbf{H} . The boundary conditions for DB interface may be written as [59–65]

$$\hat{\mathbf{n}} \cdot \mathbf{D} = 0 \quad (12a)$$

$$\hat{\mathbf{n}} \cdot \mathbf{B} = 0 \quad (12b)$$

where $\hat{\mathbf{n}}$ is normal vector to the interface.

Although PEC and PMC boundaries become special cases of DB boundary for TE and TM excitation respectively but using analysis of fractional DB waveguides it is highlighted that DB boundary has different characteristics from PEC and PMC. For this purpose, fractional dual DB waveguides are studied. The aim of present discussion is to find the fields and impedances of the walls for fractional dual parallel plate DB waveguides and to highlight its differences with fractional dual parallel plate PEC waveguides.

2. GENERAL WAVE BEHAVIOR ALONG A GUIDING STRUCTURE

Consider a waveguide consisting of two parallel plates separated by a dielectric medium with constitutive parameters ϵ and μ . One plate is located at $y = 0$, while other plate is located at $y = b$. The plates are assumed to be of infinite extent and the direction of propagation is considered as positive z -axis. Electric and magnetic fields in the source free dielectric region must satisfy the following homogeneous vector Helmholtz equations

$$\nabla^2 \mathbf{E}(x, y, z) + k^2 \mathbf{E}(x, y, z) = 0 \quad (13a)$$

$$\nabla^2 \mathbf{H}(x, y, z) + k^2 \mathbf{H}(x, y, z) = 0 \quad (13b)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator and $k = \omega\sqrt{\mu\epsilon}$ is the wave number. Taking z -dependance as $\exp(i\beta z)$, Equation (13) can be reduced to two dimensional vector Helmholtz equation as

$$\nabla_{xy}^2 \mathbf{E}(x, y) + h^2 \mathbf{E}(x, y) = 0 \quad (14a)$$

$$\nabla_{xy}^2 \mathbf{H}(x, y) + h^2 \mathbf{H}(x, y) = 0 \quad (14b)$$

where $h^2 = k^2 - \beta^2$, β is the propagation constant.

Since propagation is along z -direction and the waveguide dimensions are considered infinite in xz -plane, so x -dependence can be ignored in the above equations. Under this condition, Equation (14) becomes ordinary second order differential equation as

$$\frac{d^2 \mathbf{E}(y)}{dy^2} + h^2 \mathbf{E}(y) = 0 \quad (15a)$$

$$\frac{d^2 \mathbf{H}(y)}{dy^2} + h^2 \mathbf{H}(y) = 0 \quad (15b)$$

As a general procedure to solve waveguide problems, the Helmholtz equation is solved for the axial field components only. The transverse field components may be obtained using the axial components of the

fields and Maxwell equations. So scalar Helmholtz equations for the axial components can be written as

$$\frac{d^2 E_z}{dy^2} + h^2 E_z = 0 \quad (15c)$$

$$\frac{d^2 H_z}{dy^2} + h^2 H_z = 0 \quad (15d)$$

General solution of the above equations is

$$E_z = a_n \cos(hy) + b_n \sin(hy) \quad (15e)$$

$$H_z = c_n \cos(hy) + d_n \sin(hy) \quad (15f)$$

where a_n, b_n, c_n , and d_n are constants and can be found from the boundary conditions.

Using Maxwell curl equations, the transverse components can be expressed in terms of longitudinal components (E_z, H_z), i.e.,

$$E_x = \frac{1}{h^2} \left(i\beta \frac{\partial E_z}{\partial x} + ik \frac{\partial \eta H_z}{\partial y} \right) \quad (16a)$$

$$E_y = \frac{1}{h^2} \left(i\beta \frac{\partial E_z}{\partial y} - ik \frac{\partial \eta H_z}{\partial x} \right) \quad (16b)$$

$$H_x = \frac{1}{h^2} \left(i\beta \frac{\partial H_z}{\partial x} - \frac{ik}{\eta} \frac{\partial E_z}{\partial y} \right) \quad (16c)$$

$$H_y = \frac{1}{h^2} \left(i\beta \frac{\partial H_z}{\partial y} + \frac{ik}{\eta} \frac{\partial E_z}{\partial x} \right) \quad (16d)$$

where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ is impedance of the medium inside the guide. In the preceding part of this paper, parallel plate waveguide with DB boundary walls has been considered and the fractional dual solutions have been determined and analyzed.

3. PARALLEL PLATE DB BOUNDARY WALLS WAVEGUIDE

A wave of general polarization propagating in positive z-direction through a parallel plate waveguide can be written as a linear sum of the transverse electric (TE^z) modes and transverse magnetic (TM^z) modes. A DB boundary can be simulated as the boundary which behaves like perfect electric conductor (PEC) boundary for (TE^z) modes and perfect magnetic conductor (PMC) boundary for (TM^z) modes. Therefore fields inside a parallel plate DB waveguide may

be obtained by linear superposition of two canonical solutions which are transverse electric (TE^z) mode solution for PEC waveguide and transverse magnetic (TM^z) mode solution for PMC waveguide. Both the cases have been discussed separately.

3.1. Fractional Dual Solutions of Canonical Cases

Case 1: Transverse electric (TE^z) mode propagation through a PEC waveguide

Let us first consider that (TE^z) mode is propagating through a PEC waveguide described in Section 2. For this mode, axial component of the electric field is zero while for the magnetic fields it is given as in Equation (15f). Using Equations (16), the corresponding transverse components can be written as

$$E_x = \left(\frac{ik}{h}\right) [-c_n \sin(hy) + d_n \cos(hy)] \quad (17a)$$

$$H_y = \left(\frac{i\beta}{h}\right) [-c_n \sin(hy) + d_n \cos(hy)] \quad (17b)$$

$$E_y = 0 \quad (17c)$$

$$H_x = 0 \quad (17d)$$

Using boundary conditions for PEC boundary that is $E_{x,z} = 0|_{y=0,b}$, we get the particular solutions as

$$E_x = \left(\frac{ik}{h}\right) [-C_n \sin(hy)] \quad (18a)$$

$$\eta H_y = \left(\frac{i\beta}{h}\right) [-C_n \sin(hy)] \quad (18b)$$

$$\eta H_z = C_n \cos(hy) \quad (18c)$$

$$E_y = 0 \quad (18d)$$

$$H_x = 0 \quad (18e)$$

$$\text{where } C_n = c_n \eta \quad h = \frac{n\pi}{b} \quad n = 1, 2, 3 \dots$$

Re-introducing the z-dependance $\exp(i\beta z)$ and writing Equations (18) in exponential form we can obtain electric and magnetic fields inside the dielectric region as sum of two plane waves as

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \quad (19a)$$

$$\eta \mathbf{H} = \eta \mathbf{E}_1 + \eta \mathbf{H}_2 \quad (19b)$$

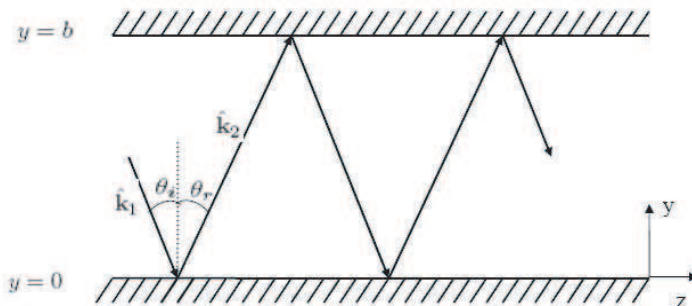


Figure 1. Plane wave representation of the fields inside the waveguide.

where $(\mathbf{E}_1, \mathbf{H}_1)$ are the electric and magnetic fields associated with one plane wave. These fields are given below

$$\mathbf{E}_1 = \left(\frac{C_n}{2} \right) \left(\frac{k}{h} \hat{\mathbf{x}} \right) \exp(-ihy + i\beta z) \quad (20a)$$

$$\eta \mathbf{H}_1 = \left(\frac{C_n}{2} \right) \left(\frac{\beta}{h} \hat{\mathbf{y}} + \hat{\mathbf{z}} \right) \exp(-ihy + i\beta z) \quad (20b)$$

Quantities $(\mathbf{E}_2, \eta \mathbf{H}_2)$ are the electric and magnetic fields associated with the second plane wave and are given below

$$\mathbf{E}_2 = \left(\frac{C_n}{2} \right) \left(-\frac{k}{h} \hat{\mathbf{x}} \right) \exp(ihy + i\beta z) \quad (21a)$$

$$\eta \mathbf{H}_2 = \left(\frac{C_n}{2} \right) \left(-\frac{\beta}{h} \hat{\mathbf{y}} + \hat{\mathbf{z}} \right) \exp(ihy + i\beta z) \quad (21b)$$

This situation can be shown as in Figure 1.

Once we have obtained electric and magnetic fields inside the dielectric region in terms of two plane waves, recipe for fractionalization [24, 32] can be applied to get the fractional dual solutions as

$$\begin{aligned} \mathbf{E}_{PECfd}^{TE} = C_n \left(\frac{k}{h} \right) & \left[-iC_\alpha \sin \left(hy + \frac{\alpha\pi}{2} \right) \hat{\mathbf{x}} + \frac{\beta}{k} S_\alpha \cos \left(hy + \frac{\alpha\pi}{2} \right) \hat{\mathbf{y}} \right. \\ & \left. - i\frac{h}{k} S_\alpha \sin \left(hy + \frac{\alpha\pi}{2} \right) \hat{\mathbf{z}} \right] \exp \left[i \left(\beta z + \frac{\alpha\pi}{2} \right) \right] \end{aligned} \quad (22a)$$

$$\begin{aligned} \eta \mathbf{H}_{PECfd}^{TE} = C_n \left(\frac{k}{h} \right) & \left[-S_\alpha \cos \left(hy + \frac{\alpha\pi}{2} \right) \hat{\mathbf{x}} - i\frac{\beta}{k} C_\alpha \sin \left(hy + \frac{\alpha\pi}{2} \right) \hat{\mathbf{y}} \right. \\ & \left. + \frac{h}{k} C_\alpha \cos \left(hy + \frac{\alpha\pi}{2} \right) \hat{\mathbf{z}} \right] \exp \left[i \left(\beta z + \frac{\alpha\pi}{2} \right) \right] \end{aligned} \quad (22b)$$

where

$$\begin{aligned} C_\alpha &= \cos\left(\frac{\alpha\pi}{2}\right) \\ S_\alpha &= \sin\left(\frac{\alpha\pi}{2}\right) \end{aligned}$$

Case 2: Transverse magnetic (TE^z) mode propagation through a PMC waveguide

Similar to the treatment done in Case 1, using Equation (15e) and Equations (16), we can write the results for transverse magnetic mode propagating through a PMC waveguide as

$$\begin{aligned} \mathbf{E}_{PMCfd}^{TM} &= A_n \left(\frac{k}{h}\right) \left[-S_\alpha \cos\left(hy + \frac{\alpha\pi}{2}\right) \hat{\mathbf{x}} - i\frac{\beta}{k} C_\alpha \sin\left(hy + \frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} \right. \\ &\quad \left. + \frac{h}{k} C_\alpha \cos\left(hy + \frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right] \exp\left[i\left(\beta z + \frac{\alpha\pi}{2}\right)\right] \end{aligned} \quad (23a)$$

$$\begin{aligned} \eta \mathbf{H}_{PMCfd}^{TM} &= A_n \left(\frac{k}{h}\right) \left[iC_\alpha \sin\left(hy + \frac{\alpha\pi}{2}\right) \hat{\mathbf{x}} - \frac{\beta}{k} S_\alpha \cos\left(hy + \frac{\alpha\pi}{2}\right) \hat{\mathbf{y}} \right. \\ &\quad \left. + \frac{ih}{k} S_\alpha \sin\left(hy + \frac{\alpha\pi}{2}\right) \hat{\mathbf{z}} \right] \exp\left[i\left(\beta z + \frac{\alpha\pi}{2}\right)\right] \end{aligned} \quad (23b)$$

3.2. Fractional Dual Parallel Plate DB Waveguide

Fractional dual solutions for the DB waveguide can be written by taking linear sum of the fractional dual fields of the above two cases as

$$\begin{aligned} \mathbf{E}_{fd} &= \mathbf{E}_{PECfd}^{TE} + \mathbf{E}_{PMCfd}^{TM} \\ \eta \mathbf{H}_{fd} &= \eta \mathbf{H}_{PECfd}^{TE} + \eta \mathbf{H}_{PMCfd}^{TM} \end{aligned}$$

which give

$$\begin{aligned} \mathbf{E}_{fd} &= \left(\frac{k}{h}\right) \exp\left[i\left(\beta z + \frac{\alpha\pi}{2}\right)\right] \left\{ -(A_n S_\alpha C_{y+\alpha} + iC_n C_\alpha S_{y+\alpha}) \hat{\mathbf{x}} \right. \\ &\quad + \frac{\beta}{k} (C_n S_\alpha C_{y+\alpha} - iA_n C_\alpha S_{y+\alpha}) \hat{\mathbf{y}} \\ &\quad \left. + \frac{h}{k} (A_n C_\alpha C_{y+\alpha} - iC_n S_\alpha S_{y+\alpha}) \hat{\mathbf{z}} \right\} \end{aligned} \quad (24a)$$

$$\begin{aligned} \eta \mathbf{H}_{fd} &= \left(\frac{k}{h}\right) \exp\left[i\left(\beta z + \frac{\alpha\pi}{2}\right)\right] \left\{ -(C_n S_\alpha C_{y+\alpha} - iA_n C_\alpha S_{y+\alpha}) \hat{\mathbf{x}} \right. \\ &\quad - \frac{\beta}{k} (A_n S_\alpha C_{y+\alpha} + iC_n C_\alpha S_{y+\alpha}) \hat{\mathbf{y}} \\ &\quad \left. + \frac{h}{k} (C_n C_\alpha C_{y+\alpha} + iA_n S_\alpha S_{y+\alpha}) \hat{\mathbf{z}} \right\} \end{aligned} \quad (24b)$$

with

$$\begin{aligned}C_\alpha &= \cos\left(\frac{\alpha\pi}{2}\right) \\ S_\alpha &= \sin\left(\frac{\alpha\pi}{2}\right)\end{aligned}$$

$$\begin{aligned}C_{y+\alpha} &= \cos\left(hy + \frac{\alpha\pi}{2}\right) \\ S_{y+\alpha} &= \sin\left(hy + \frac{\alpha\pi}{2}\right)\end{aligned}$$

A_n, C_n are the constant to be determined from initial conditions. The fields given in Equation (24) have been plotted in Figure 2 for different values of the α at an observation point $(hy, \beta z) = (\pi/4, \pi/4)$.

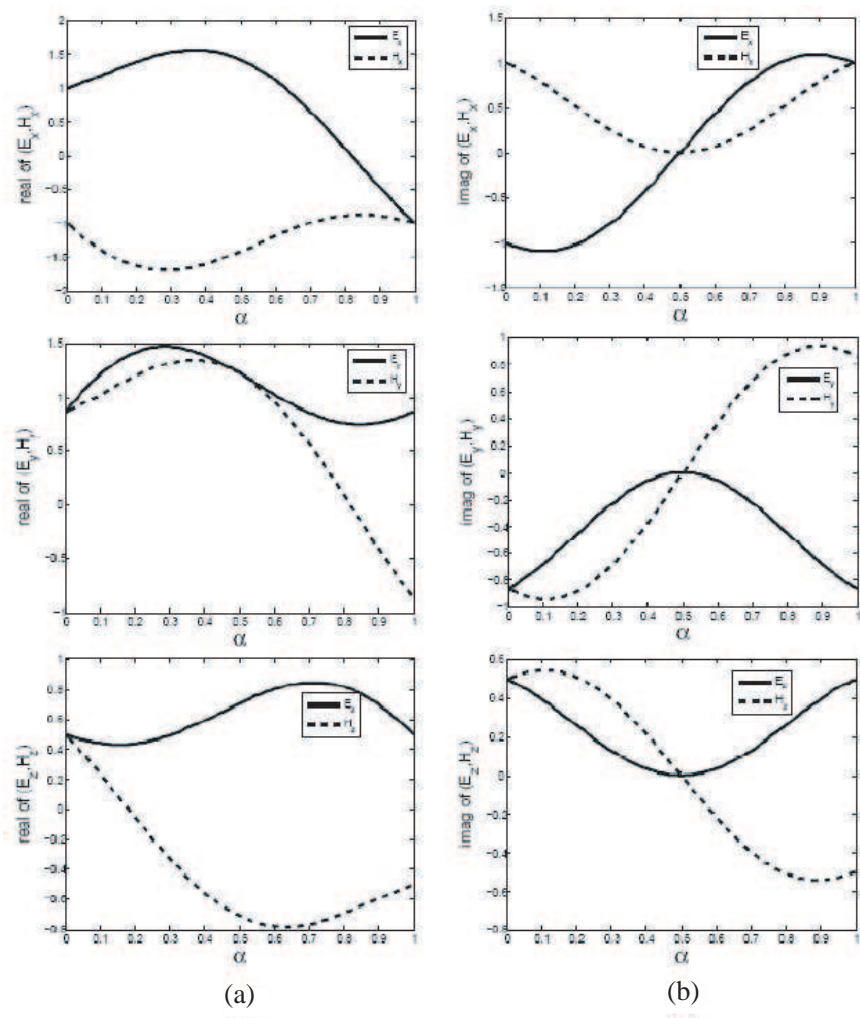


Figure 2. Plots of fractional dual TE^z polarized fields at a point $(hy, \beta z) = (\pi/4, \pi/4)$, (a) real parts, (b) imaginary parts.

From Figure 2, it can be seen that fractional dual fields satisfy the principle of duality, i.e., for $\alpha = 0$

$$\begin{aligned} E_{fdx} &= E_x, & \eta H_{fdx} &= \eta H_x \\ E_{fdy} &= E_y, & \eta H_{fdy} &= \eta H_y \\ E_{fdz} &= E_z, & \eta H_{fdz} &= \eta H_z \end{aligned}$$

and for $\alpha = 1$

$$\begin{aligned} E_{fdx} &= \eta H_x, & \eta H_{fdx} &= -E_x \\ E_{fdy} &= \eta H_y, & \eta H_{fdy} &= -E_y \\ E_{fdz} &= \eta H_z, & \eta H_{fdz} &= -E_z \end{aligned}$$

4. RESULTS AND DISCUSSION

4.1. Behavior of Fields inside the Fractional Parallel Plate DB Waveguide

In order to study the behavior of fields inside the fractional parallel plate DB waveguide, electric and magnetic field lines are plotted in the yz -plane and are shown in Figure 3. We have selected yz -plane as an observation plane. The instantaneous field expressions are obtained by multiplying the phasor vector expressions (24) with $\exp(j\omega t)$ and taking the real part of the product. Equation describing the behavior of electric field lines at a given time t can be found from the following relation.

$$\frac{dy}{E_{fdy}} = \frac{dz}{E_{fdz}}$$

Finally integration of the equation gives us the field lines behavior.

These plots are for the mode propagating through the guide at an angle $\pi/6$ so that $\beta/k = \cos(\pi/6)$, $h/k = \sin(\pi/6)$. Initial conditions for both the modes are taken as same. Solid lines show the electric as well as magnetic field plots for DB waveguide while the fields of PEC waveguide are shown by dashed lines as a reference. From the figure we see that there is no normal component of the electric as well as magnetic field for $\alpha = 0$. This is because the plates of the guide behave as perfect electric conductors for transverse electric components while they behave as perfect magnetic conductor for transverse magnetic modes. For the reference PEC results, there is no tangential component of the electric field at the guide surface while magnetic field has no normal component. As value of α increases from 0, normal components of both the fields in DB guide start appearing and become maximum at $\alpha = 0.5$. After this value normal components start decreasing and again become zero at $\alpha = 1$.

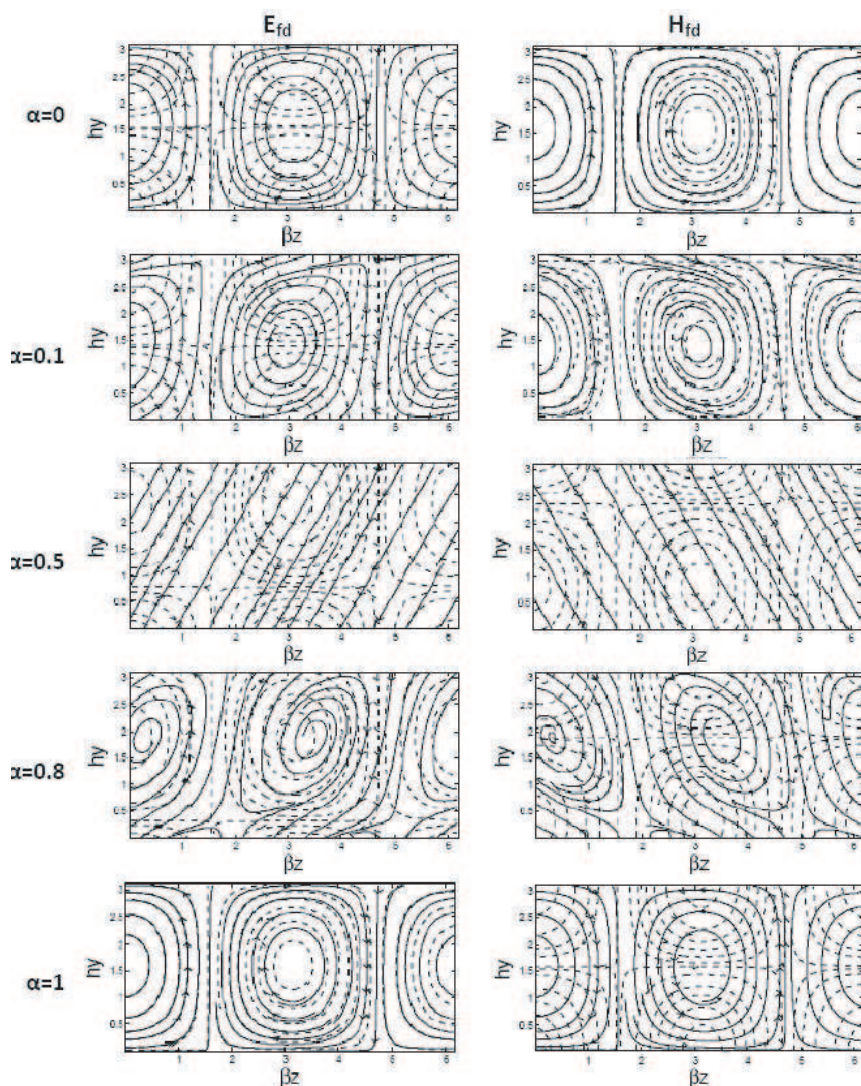


Figure 3. Field lines in yz -plane at different values of α ; solid lines are for the fractional DB waveguides while dashed lines are for the fractional PEC waveguides.

It may be noted that electric field distribution in the DB waveguide is same as the magnetic field distribution for the limiting values of α while it is different for the intermediate values. Further it may be noted that field behavior for the original and dual situation is similar.

4.2. Transverse Impedances of Wall

Wave impedance is defined by ratio of the transverse components of the electric and magnetic fields as

$$Z_{fdxz} = -\frac{E_{fdx}}{H_{fdz}} = \eta \frac{k}{h} \frac{A_n S_\alpha C_{y+\alpha} + i C_n C_\alpha S_{y+\alpha}}{C_n C_\alpha C_{y+\alpha} + i A_n S_\alpha S_{y+\alpha}}$$

$$Z_{fdzx} = \frac{E_{fdz}}{H_{fdx}} = \eta \frac{h}{k} \frac{A_n C_\alpha C_{y+\alpha} - i C_n S_\alpha S_{y+\alpha}}{C_n S_\alpha C_{y+\alpha} - i A_n C_\alpha S_{y+\alpha}}$$

At $y = 0$, these impedances become impedance of the new reflecting boundary called the fractional dual boundary. The normalized impedance matrix of the DB boundary wall taking $A_n = C_n$ can be written as

$$\underline{\underline{z}}_{fd} = \left\{ \frac{k}{h} z_{fdxz} \hat{\mathbf{x}} \hat{\mathbf{z}} + \frac{h}{k} z_{fdzx} \hat{\mathbf{z}} \hat{\mathbf{x}} \right\}, \quad 0 \leq \alpha \leq 1$$

where

$$z_{fdxz} = \frac{S_\alpha C_\alpha + i C_\alpha S_\alpha}{C_\alpha C_\alpha + i S_\alpha S_\alpha}$$

$$z_{fdzx} = \frac{C_\alpha C_\alpha - i S_\alpha S_\alpha}{S_\alpha C_\alpha - i C_\alpha S_\alpha}$$

These impedance components have been plotted for whole range of α as in Figure 4.

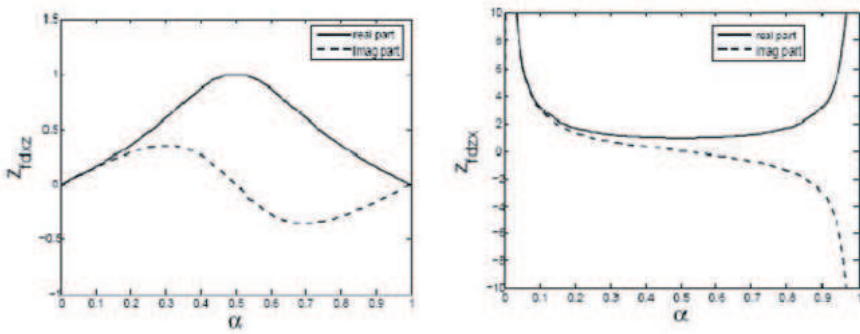


Figure 4. Transverse impedance of walls of fractional DB waveguides vs. fractional parameter α .

It may be noted that z_{fdxz} corresponds to the impedance for TE^z modes and z_{fdzx} corresponds to the impedance for TM^z modes. Since DB boundary behaves as PEC for the TE^z modes so it is zero at $\alpha = 0$ and $\alpha = 1$ while it is complex for the intermediate range of α . Similarly

Since DB boundary behaves as PMC for the TM^z modes so impedance is infinitely high at $\alpha = 0$ and $\alpha = 1$, while it is finite complex for the intermediate range of α .

5. CONCLUSIONS

Fractional dual solutions to the Maxwell equations for fields inside a parallel plate DB waveguide are derived using the fractional curl operator. The waveguides described by such fields are termed as fractional dual parallel plate DB waveguides. Electric field distribution in the fractional DB waveguides is same as the magnetic field distribution for the limiting integer values of the fractional parameter ($\alpha = 0, 1$) while it is different for the intermediate values, i.e., $0 < \alpha < 1$. For limiting cases, transverse impedance of DB wall is zero for transverse electric mode and infinitely high for transverse magnetic mode while it is non zero complex value for the intermediate situations. Fractional dual waveguide explains the situation which is an intermediate step of DB boundary waveguide and its dual situation obtained through duality principle.

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