

SCATTERING OF GAUSSIAN BEAM BY A SPHEROIDAL PARTICLE

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Abstract—Gaussian beam scattering by a spheroidal particle is studied in detail. A theoretical procedure is given to expand an incident Gaussian beam in terms of spheroidal vector wave functions within the generalized Lorenz-Mie theory framework. Exact analytic solutions are obtained for an arbitrarily oriented spheroid with non-confocal dielectric coating. Normalized differential scattering cross sections are shown and discussed for three different cases of a dielectric spheroid, spheroid with a spherical inclusion and coated spheroid.

1. INTRODUCTION

The generalized Lorenz-Mie theory (GLMT) developed by Gouesbet et al. is effective for describing the interaction of a shaped beam with a spherical particle by relying on the separability of variables [1–3], and has been extended by so many researchers to multilayered spheres [4, 5], spheroids [6] and infinite cylinders [7–9]. Various applications of focused beam scattering include optimizing the rate at which morphology-dependent resonances (MDRs) are excited, laser trapping, particle manipulation, and the analysis of optical particle sizing instruments [10, 11]. Within the GLMT framework for spheroids, one fundamental problem concerns the expansion of the incident shaped beam in terms of spheroidal vector wave functions, in which the expansion or beam shape coefficients (BSCs) are at the core. Due to the complexity of the spheroidal vector wave functions, it is a convenient and effective approach to evaluate the BSCs in spheroidal coordinates by virtue of the known expressions of the BSCs in spherical coordinates.

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By using such an approach, Han et al. have obtained the BSCs in spheroidal coordinates when the propagation direction of the incident Gaussian beam is parallel with the symmetry axis of the spheroid, and then analyzed the scattering of an off-axis Gaussian beam and of a femtosecond pulse by a spheroid [12]. With an appropriate coordinate rotation, a theoretical procedure is presented by Xu et al. to expand an incident shaped beam in terms of the spheroidal vector wave functions with respect to a spheroid, and then a general case is studied of an arbitrarily oriented, located, and shaped beam scattered by a homogeneous spheroidal particle [13, 14]. The method used by Xu et al. is first to have a description of a shaped beam in the particle coordinates through coordinate rotations, and then to calculate the BSCs in the spherical coordinates and in the spheroidal coordinates as usual. Since it has been found that the localization approximation method, an efficient method of evaluating the BSCs, can no longer be applied under coordinate rotations, the BSCs given by Xu et al. are computed by a triple integral or a double integral due to the use of the quadrature method, which lacks flexibility and is often computer-time consuming.

To overcome the difficulty of the inapplicability of the localization principle under coordinate rotations, contrary to Xu's theory, a method is provided by us which uses a rotation of the coordinate system after the localization principle is applied. As a result, the localization principle holds, and the addition theorems for spherical vector wave functions under coordinate rotations are required [15, 16]. The resultant BSCs in the spheroidal coordinates are expressed explicitly in terms of the BSCs in the spherical coordinates evaluated by a localized beam model, proving their evaluations highly efficient. Based on such an expansion, a strong effort has been devoted by us to the study of Gaussian beam scattering by a spheroidal particle [17], and by a spheroidal particle with concentric non-confocal dielectric coating, in which a transformation from spheroidal vector wave functions to spherical ones is necessary [18]. In this paper, we present a detailed discussion of the expansion of Gaussian beam in spheroidal coordinates within the GLMT framework, and of the scattering of Gaussian beam by a dielectric spheroid and coated spheroid.

2. EXPANSION OF GAUSSIAN BEAM IN SPHEROIDAL COORDINATES

As illustrated in Fig. 1, a Gaussian beam propagates in free space and from the negative z' to the positive z' axis in its own Cartesian coordinate system $O'x'y'z'$, with the beam center located at origin

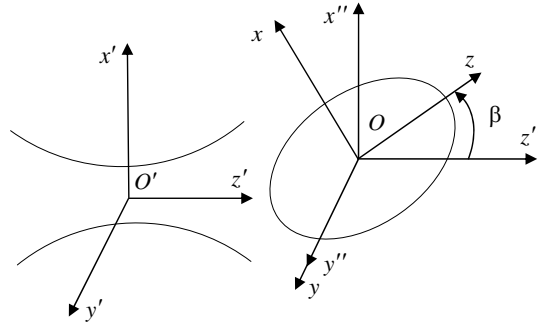


Figure 1. The Cartesian coordinate system $Ox''y''z''$ is parallel to the Gaussian beam coordinate system $O'x'y'z'$, and the Cartesian coordinates of O in $O'x'y'z'$ are (x_0, y_0, z_0) . $Oxyz$ is obtained by a rigid-body rotation of $Ox''y''z''$ through a single Euler angle β . A spheroidal particle is natural to $Oxyz$.

O' and the time-dependent part of the electromagnetic fields assumed to be $\exp(-i\omega t)$. An accessory system $Ox''y''z''$ that is parallel to $O'x'y'z'$ is introduced, and the system $Oxyz$ is obtained by rotating $Ox''y''z''$ through a single Euler angle β [19]. The center of a spheroid is located at origin O and has the Cartesian coordinates (x_0, y_0, z_0) in $O'x'y'z'$, and the major axis is along the z axis of $Oxyz$. The semifocal distance, semimajor and semiminor axes of the spheroid are denoted by f , a and b .

By following Davis's first-order approximation, the electromagnetic fields of the Gaussian beam can be described in its own Cartesian coordinate system $O'x'y'z'$ as [20]

$$E_{x'} = E_0 \psi_0 e^{ikz'} \quad (1a)$$

$$E_{y'} = 0 \quad (1b)$$

$$E_{z'} = \frac{x'}{l} 2Q E_{x'} \quad (1c)$$

$$H_{x'} = 0 \quad (1d)$$

$$H_{y'} = H_0 \psi_0 e^{ikz'} \quad (1e)$$

$$H_{z'} = \frac{y'}{l} 2Q H_{y'} \quad (1f)$$

where

$$\psi_0 = iQ \exp[-iQ(\xi^2 + \eta^2)] \quad (1g)$$

as well as

$$\xi = \frac{x'}{w_0} \quad (1h)$$

$$\eta = \frac{y'}{w_0} \quad (1i)$$

$$Q = \frac{1}{i - \frac{2z'}{l}} \quad (1j)$$

$$l = kw_0^2 \quad (1k)$$

where w_0 is the beam waist radius.

From the scatterer geometry described in Fig. 1, the following equation relating the two systems $O'x'y'z'$ and $Oxyz$ can be obtained as:

$$\begin{bmatrix} x' - x_0 \\ y' - y_0 \\ z' - z_0 \end{bmatrix} = [A] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2)$$

where the transformation matrix is given by

$$[A] = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \quad (3)$$

By virtue of Eq. (2), i.e., to transform the coordinates x' , y' , z' to x , y , z according to Eq. (2), in Eq. (1) the beam descriptions of $(E_{x'}, E_{y'}, E_{z'})$ and $(H_{x'}, H_{y'}, H_{z'})$ in the beam coordinate system $O'x'y'z'$ can be transformed to their counterparts (E_x, E_y, E_z) and (H_x, H_y, H_z) in the particle coordinate system $Oxyz$, in the following form

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = [A] \begin{bmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{bmatrix} \quad (4)$$

where F stands for E or H , and the coordinates x' , y' , z' at the right side are transformed to x , y , z according to Eq. (2).

Within the framework of the GLMT, the incident Gaussian beam can be expanded in terms of the spherical vector wave functions with respect to the system $Oxyz$ as follows [21]

$$\mathbf{E}^i = E_0 \sum_{n=1}^{\infty} \sum_{m=-n}^n C_{nm} \left[ig_{n,TE}^m \mathbf{m}_{mn}^{r(1)}(kr, \theta, \phi) + g_{n,TM}^m \mathbf{n}_{mn}^{r(1)}(kr, \theta, \phi) \right] \quad (5)$$

$$\mathbf{H}^i = E_0 \frac{k}{\omega \mu} \sum_{n=1}^{\infty} \sum_{m=-n}^n C_{nm} \left[g_{n,TE}^m \mathbf{n}_{mn}^{r(1)}(kr, \theta, \phi) - ig_{n,TM}^m \mathbf{m}_{mn}^{r(1)}(kr, \theta, \phi) \right] \quad (6)$$

$$C_{nm} = \begin{cases} C_n & m \geq 0 \\ (-1)^{|m|} \frac{(n+|m|)!}{(n-|m|)!} C_n & m < 0 \end{cases} \quad (7)$$

$$C_n = i^{n-1} \frac{2n+1}{n(n+1)} \quad (8)$$

By equating the r components at the left and right sides of Eqs. (5) and (6) and using the orthogonality of the exponentials $e^{im\phi}$, associated Legendre functions $P_n^m(x)$ and spherical Bessel functions of the first kind $j_n(kr)$, the BSCs $g_{n,TE}^m$ and $g_{n,TM}^m$ can be calculated from the r components of the incident electromagnetic fields in the spherical coordinates (r, θ, ϕ) attached to $Oxyz$, by the following two triple integrals [1, 13]

$$g_{n,TE}^m = (-i)^{n-1} \frac{2n+1}{4\pi^2} \frac{(n-|m|)!}{(n+|m|)!} \int_0^\infty j_n(kr) k r d(kr) \int_0^\pi P_n^m(\cos \theta) \sin \theta d\theta \int_0^{2\pi} \frac{H_r^i}{H_0} e^{-im\phi} d\phi \quad (9)$$

$$g_{n,TM}^m = (-i)^{n-1} \frac{2n+1}{4\pi^2} \frac{(n-|m|)!}{(n+|m|)!} \int_0^\infty j_n(kr) k r d(kr) \int_0^\pi P_n^m(\cos \theta) \sin \theta d\theta \int_0^{2\pi} \frac{E_r^i}{E_0} e^{-im\phi} d\phi \quad (10)$$

Numerical evaluations of the BSCs by using Eqs. (9) and (10) is excessively time-consuming. When the particle coordinate system $Oxyz$ is parallel to the beam coordinate system $O'x'y'z'$ ($\beta = 0$), this difficulty can be overcome with the use of the localized approximation because of its high efficiency in computation. From a mathematical point of view, it takes the operations of $r \rightarrow \frac{1}{k}(n+1/2)$ and $\theta = \frac{\pi}{2}$ on the right sides of Eqs. (9) and (10) to simplify the computation of the integrals. Unfortunately, for the general case of oblique illumination ($\beta \neq 0$), such a localization principle fails to hold [13]. To overcome the difficulty, a method is provided by us to evaluate the BSCs for oblique illumination, which can be divided into two steps.

First, the incident Gaussian beam is expanded in terms of the spherical vector wave functions with respect to the system $Ox''y''z''$, and the corresponding BSCs can be efficiently evaluated by applying the localization principle.

Second, we use the addition theorem for the spherical vector wave functions under coordinate rotations to derive the expansion of Gaussian beam as in Eqs. (5) and (6), and, after some algebra, the

BSCs are explicitly expressed as follows [15, 16]:

$$C_{nm}g_{n,TE}^m = \sum_{s=-n}^n \rho(s, m, n) C_{ns} g_{n,TE}^{//m} \quad (11)$$

$$C_{nm}g_{n,TM}^m = \sum_{s=-n}^n \rho(s, m, n) C_{ns} g_{n,TM}^{//m} \quad (12)$$

where

$$\rho(s, m, n) = (-1)^{m+s} \left[\frac{(n+s)!(n-m)!}{(n-s)!(n+m)!} \right]^{\frac{1}{2}} u_{ms}(\beta) \quad (13)$$

$$u_{ms}(\beta) = \left[\frac{(n+m)!(n-m)!}{(n+s)!(n-s)!} \right]^{\frac{1}{2}} \sum_{\sigma} \binom{n+s}{n-m-\sigma} \binom{n-s}{\sigma} (-1)^{n-m-\sigma} \left(\cos \frac{\beta}{2} \right)^{2\sigma+m+s} \left(\sin \frac{\beta}{2} \right)^{2n-2\sigma-m-s} \quad (14)$$

and $g_{n,TE}^{//m}$, $g_{n,TM}^{//m}$ are the BSCs evaluated in the system $Ox''y''z''$, and, as pointed out above, can be efficiently calculated by using the localization approximation method.

When the Davis-Barton model of the Gaussian beam is used [20, 22], the BSCs for a Gaussian beam with $\frac{1}{kw_0} > 0.1$ can be computed without any loss of accuracy by the localization approximation method [21].

For the representation of the incident Gaussian beam in spheroidal coordinates, the following equation is useful, which indicates the conversion relationship between the spherical and spheroidal vector wave functions [23]

$$\begin{bmatrix} \mathbf{m}_{mn}^{r(1)}(kR, \theta, \phi) \\ \mathbf{n}_{mn}^{r(1)}(kR, \theta, \phi) \end{bmatrix} = \sum_{l=m, m+1}^{\infty} \left[\frac{2(n+m)!}{(2n+1)(n-m)!} \cdot \frac{i^{l-n}}{N_{ml}} d_{n-m}^{ml}(c) \begin{bmatrix} \mathbf{M}_{ml}^{r(1)}(c, \zeta, \eta, \phi) \\ \mathbf{N}_{ml}^{r(1)}(c, \zeta, \eta, \phi) \end{bmatrix} \right] \quad (15)$$

where N_{mn} and $d_r^{mn}(c)$ are the normalization constants and expansion coefficients of the spheroidal angle functions $S_{mn}(\eta)$, respectively.

By using the relation $[\mathbf{M}_{ml} \ \mathbf{N}_{ml}] = [\mathbf{M}_{eml} \ \mathbf{N}_{eml}] + i[\mathbf{M}_{oml} \ \mathbf{N}_{oml}]$ when substituting Eq. (15) into Eqs. (5) and (6), after some manipulations the expansion of Gaussian beam in spheroidal

coordinates can be conveniently obtained as follows [16]:

$$\mathbf{E}^i = E_0 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \left[G_{n,TE}^m \mathbf{M}_{emn}^{r(1)}(c, \zeta, \eta, \phi) + i G_{n,TM}^m \mathbf{N}_{omn}^{r(1)}(c, \zeta, \eta, \phi) \right] \quad (16)$$

$$\mathbf{H}^i = E_0 \frac{k}{\omega \mu} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \left[G_{n,TM}^m \mathbf{M}_{omn}^{r(1)}(c, \zeta, \eta, \phi) - i G_{n,TE}^m \mathbf{N}_{emn}^{r(1)}(c, \zeta, \eta, \phi) \right] \quad (17)$$

where $c = kf$, and the BSCs $G_{n,TE}^m$ and $G_{n,TM}^m$, for the TE mode, are given by

$$\begin{bmatrix} G_{n,TE}^m \\ G_{n,TM}^m \end{bmatrix} = \sum_{r=0,1}^{\infty} \sum_{s=0}^{r+m} \frac{2(r+2m)!}{(2r+2m+1)r!} \frac{i^{-r-m}}{N_{mn}} d_r^{mn}(c) \begin{bmatrix} g_{r+m,TE}^{sm} \\ g_{r+m,TM}^{sm} \end{bmatrix} \quad (18)$$

$$\begin{aligned} \begin{bmatrix} g_{n,TE}^{sm} \\ g_{n,TM}^{sm} \end{bmatrix} &= \frac{2}{(1+\delta_{s0})(1+\delta_{m0})} i^n \frac{2n+1}{n(n+1)} (-1)^{s+m} \left[\frac{(n+s)!(n-m)!}{(n-s)!(n+m)!} \right]^{1/2} \\ &\quad \begin{bmatrix} g_{n,TE}^{//s} \\ i g_{n,TM}^{//s} \end{bmatrix} \left[(-1)^n u_{ms}^{(n)}(\pi + \beta) \pm u_{ms}^{(n)}(\beta) \right] \end{aligned} \quad (19)$$

The prime over the summation sign in Eq. (18) indicates that the summation is over even values of r when $n - m$ is even and over odd values of r when $n - m$ is odd. In Eq. (19), $\delta_{m0} = 0$ and $\delta_{s0} = 0$ when $m \neq 0$ and $s \neq 0$ respectively, and $\delta_{00} = 1$.

For the TM mode, the corresponding expansions of the Gaussian beam can be obtained by replacing $[\mathbf{M} \ \mathbf{N}]_{emn}$ in Eqs. (16) and (17) by $[\mathbf{M} \ \mathbf{N}]_{omn}$, $[\mathbf{M} \ \mathbf{N}]_{omn}$ by $[\mathbf{M} \ \mathbf{N}]_{emn}$, $G_{n,TE}^m$ by $G_{n,TM}^m$, and $G_{n,TM}^m$ by $-G_{n,TE}^m$.

3. SCATTERING OF GAUSSIAN BEAM BY A SPHEROIDAL PARTICLE

In the framework of the GLMT for a spheroid, once the Gaussian beam expansion is obtained, the scattered fields as well as the fields within the spheroidal particle can be expanded in terms of appropriate spheroidal vector wave functions as follows [17]:

$$\mathbf{E}^s = E_0 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \left[\beta_{mn} \mathbf{M}_{emn}^{r(3)}(c, \zeta, \eta, \phi) + i \alpha_{mn} \mathbf{N}_{omn}^{r(3)}(c, \zeta, \eta, \phi) \right] \quad (20)$$

$$\mathbf{E}^w = E_0 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \left[\delta_{mn} \mathbf{M}_{emn}^{r(1)}(c_1, \zeta, \eta, \phi) + i \gamma_{mn} \mathbf{N}_{omn}^{r(1)}(c_1, \zeta, \eta, \phi) \right] \quad (21)$$

where α_{mn} , β_{mn} , δ_{mn} and γ_{mn} are unknown expansion coefficients to be determined by using the boundary conditions, and $c_1 = f k_1$, $k_1 = k \tilde{n}$

and \tilde{n} is the refractive index of the material of the spheroidal particle relative to that of free space.

The corresponding expansions of the magnetic fields in spheroidal coordinates can be obtained with the following relations

$$\mathbf{H} = \frac{1}{i\omega\mu} \nabla \times \mathbf{E} \quad (22a)$$

$$\begin{bmatrix} \mathbf{M}_e^{r(j)} & \mathbf{N}_e^{r(j)} \\ o^{mn} & o^{mn} \end{bmatrix} = \frac{1}{k} \nabla \times \begin{bmatrix} \mathbf{N}_e^{r(j)} & \mathbf{M}_e^{r(j)} \\ o^{mn} & o^{mn} \end{bmatrix} \quad (22b)$$

In all these equations, the superscript j takes the value of 1 or 3, depending on the usage of the radial function $R_{mn}^{(j)}(c, \zeta)$ of the first or third kind in the spheroidal vector wave functions.

The boundary conditions for a spheroidal particle are described by

$$\left. \begin{aligned} E_\eta^i + E_\eta^s &= E_\eta^w, & E_\phi^i + E_\phi^s &= E_\phi^w \\ H_\eta^i + H_\eta^s &= H_\eta^w, & H_\phi^i + H_\phi^s &= H_\phi^w \end{aligned} \right\} \quad \text{at } \zeta = \zeta_0 \quad (23)$$

where ζ_0 is the radial coordinate of the spheroidal particle surface.

By virtue of the field expansions in Eqs. (16), (17) and in Eqs. (20), (21), the above boundary conditions in Eq. (23) can be written as

$$\sum_{n=m}^{\infty} i^n [\Gamma] \begin{bmatrix} \alpha_{mn} \\ \beta_{mn} \\ \delta_{mn} \\ \gamma_{mn} \end{bmatrix} = \sum_{n=m}^{\infty} i^n \begin{bmatrix} -G_{n,TE}^m U_{mn}^{(1),t}(c) - G_{n,TM}^m V_{mn}^{(1),t}(c) \\ -G_{n,TM}^m U_{mn}^{(1),t}(c) - G_{n,TE}^m V_{mn}^{(1),t}(c) \\ -G_{n,TE}^m X_{mn}^{(1),t}(c) - G_{n,TM}^m Y_{mn}^{(1),t}(c) \\ -G_{n,TM}^m X_{mn}^{(1),t}(c) - G_{n,TE}^m Y_{mn}^{(1),t}(c) \end{bmatrix} \quad (24)$$

where the matrix $[\Gamma]$ is given by

$$[\Gamma] = \begin{bmatrix} V_{mn}^{(3),t}(c) & U_{mn}^{(3),t}(c) & -U_{mn}^{(1),t}(c_1) & -V_{mn}^{(1),t}(c_1) \\ U_{mn}^{(3),t}(c) & V_{mn}^{(3),t}(c) & -\tilde{n}V_{mn}^{(1),t}(c_1) & -\tilde{n}U_{mn}^{(1),t}(c_1) \\ Y_{mn}^{(3),t}(c) & X_{mn}^{(3),t}(c) & -X_{mn}^{(1),t}(c_1) & -Y_{mn}^{(1),t}(c_1) \\ X_{mn}^{(3),t}(c) & Y_{mn}^{(3),t}(c) & -\tilde{n}Y_{mn}^{(1),t}(c_1) & -\tilde{n}X_{mn}^{(1),t}(c_1) \end{bmatrix} \quad (25)$$

The parameters $U_{mn}^{(j),t}$, $V_{mn}^{(j),t}$, $X_{mn}^{(j),t}$ and $Y_{mn}^{(j),t}$ ($j = 1$ or 3 according to the radial function $R_{mn}^{(j)}(c, \zeta)$ in them of the first or third kind) are given by Asano and Yamamoto in [24], and Eq. (24), given the value of $m \geq 0$, is valid for each of $t \geq 0$. By taking t to be sufficiently large, an adequate number of relations between the unknown expansion coefficients α_{mn} , β_{mn} , δ_{mn} and γ_{mn} can be generated. With the use of the standard numerical techniques, the expansion coefficients, and then from Eqs. (20) and (21) the scattered and internal fields can be determined [17, 24].

4. SCATTERING OF GAUSSIAN BEAM BY A SPHEROIDAL PARTICLE WITH CONCENTRIC NON-CONFOCAL DIELECTRIC COATING

For this scattering problem, the scattering geometry can be obtained by replacing the spheroidal particle in Fig. 1 with a coated one. The spheroidal particle and dielectric coating are concentric but not necessarily confocal, so that the particle and coating surfaces can be in different spheroidal coordinate systems. We denote the semifocal distance, semimajor and semiminor axes by f_1 , a_1 and b_1 for the spheroidal particle surface, and by f_2 , a_2 and b_2 for the outer surface of the dielectric coating.

As in Eqs. (16), (17) and (20), the electromagnetic fields of the incident Gaussian beam as well as the scattered fields can be represented by infinite series with the spheroidal vector wave functions attached to the dielectric coating, i.e., the value of the parameter c taken to be kf_2 in the spheroidal vector wave functions.

The electromagnetic fields within the spheroidal particle can be represented by

$$\mathbf{E}^{w(1)} = E_0 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n \left[\delta_{mn}^{(1)} \mathbf{M}_{emn}^{r(1)}(c_1, \zeta, \eta, \phi) + i\gamma_{mn}^{r(1)} \mathbf{N}_{omn}^{r(1)}(c_1, \zeta, \eta, \phi) \right] \quad (26)$$

where $c_1 = f_1 k_1$, $k_1 = k\tilde{n}_1$ and \tilde{n}_1 is the refractive index of the material of the spheroidal particle relative to that of free space.

To overcome the difficulty of applying the boundary conditions on the particle and coating surfaces which are concentric non-confocal, the electromagnetic fields within the dielectric coating are expanded in terms of the spheroidal vector wave functions attached to the spheroid surface and coating surface, respectively, as follows:

$$\begin{aligned} \mathbf{E}^w = E_0 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n & \left[\delta_{mn} \mathbf{M}_{emn}^{r(1)}(c_2, \zeta, \eta, \phi) + \chi_{mn} \mathbf{M}_{emn}^{r(3)}(c_2, \zeta, \eta, \phi) \right. \\ & \left. + i\gamma_{mn} \mathbf{N}_{omn}^{r(1)}(c_2, \zeta, \eta, \phi) + i\tau_{mn} \mathbf{N}_{omn}^{r(3)}(c_2, \zeta, \eta, \phi) \right] \end{aligned} \quad (27)$$

$$\begin{aligned} \mathbf{E}^w = E_0 \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} i^n & \left[\delta'_{mn} \mathbf{M}_{emn}^{r(1)}(c'_2, \zeta, \eta, \phi) + \chi'_{mn} \mathbf{M}_{emn}^{r(3)}(c'_2, \zeta, \eta, \phi) \right. \\ & \left. + i\gamma'_{mn} \mathbf{N}_{omn}^{r(1)}(c'_2, \zeta, \eta, \phi) + i\tau'_{mn} \mathbf{N}_{omn}^{r(3)}(c'_2, \zeta, \eta, \phi) \right] \end{aligned} \quad (28)$$

where $c_2 = k_2 f_1$, $c'_2 = k_2 f_2$, $k_2 = k\tilde{n}_2$, and \tilde{n}_2 is the refractive index of the material of the dielectric coating relative to that of free space.

With ζ_1 and ζ_2 as the radial coordinates of the boundary surfaces of the spheroid and coating respectively, the boundary conditions on

the surface $\zeta = \zeta_2$ are described by.

$$\left. \begin{aligned} E_\eta^i + E_\eta^s &= E_\eta^w, & E_\phi^i + E_\phi^s &= E_\phi^w \\ H_\eta^i + H_\eta^s &= H_\eta^w, & H_\phi^i + H_\phi^s &= H_\phi^w \end{aligned} \right\} \quad \text{at } \zeta = \zeta_2 \quad (29)$$

and on the surface $\zeta = \zeta_1$ by

$$\left. \begin{aligned} E_\eta^{w(1)} &= E_\eta^w, & E_\phi^{w(1)} &= E_\phi^w \\ H_\eta^{w(1)} &= H_\eta^w, & H_\phi^{w(1)} &= H_\phi^w \end{aligned} \right\} \quad \text{at } \zeta = \zeta_1 \quad (30)$$

Similar to the case of a spheroidal particle described in Section 3, by considering the use of Eqs. (27) and (28) when writing Eqs. (29) and (30) respectively, we can express the above boundary conditions in Eq. (29) explicitly as

$$\sum_{n=m}^{\infty} i^n [\Gamma] \begin{bmatrix} \alpha_{mn} \\ \beta_{mn} \\ \delta'_{mn} \\ \chi'_{mn} \\ \gamma'_{mn} \\ \tau'_{mn} \end{bmatrix} = \sum_{n=m}^{\infty} i^n \begin{bmatrix} -G_{n,TE}^m U_{mn}^{(1),t}(c) - G_{n,TM}^m V_{mn}^{(1),t}(c) \\ -G_{n,TM}^m U_{mn}^{(1),t}(c) - G_{n,TE}^m V_{mn}^{(1),t}(c) \\ -G_{n,TE}^m X_{mn}^{(1),t}(c) - G_{n,TM}^m Y_{mn}^{(1),t}(c) \\ -G_{n,TM}^m X_{mn}^{(1),t}(c) - G_{n,TE}^m Y_{mn}^{(1),t}(c) \end{bmatrix} \quad (31)$$

and the boundary conditions in Eq. (30) as

$$\begin{aligned} & \sum_{n=m}^{\infty} i^n \begin{bmatrix} U_{mn}^{(1),t}(c_2) & U_{mn}^{(3),t}(c_2) & V_{mn}^{(1),t}(c_2) & V_{mn}^{(3),t}(c_2) \\ \tilde{n}_2 V_{mn}^{(1),t}(c_2) & \tilde{n}_2 V_{mn}^{(3),t}(c_2) & \tilde{n}_2 U_{mn}^{(1),t}(c_2) & \tilde{n}_2 U_{mn}^{(3),t}(c_2) \\ X_{mn}^{(1),t}(c_2) & X_{mn}^{(3),t}(c_2) & Y_{mn}^{(1),t}(c_2) & Y_{mn}^{(3),t}(c_2) \\ \tilde{n}_2 Y_{mn}^{(1),t}(c_2) & \tilde{n}_2 Y_{mn}^{(3),t}(c_2) & \tilde{n}_2 X_{mn}^{(1),t}(c_2) & \tilde{n}_2 X_{mn}^{(3),t}(c_2) \end{bmatrix} \begin{bmatrix} \delta_{mn} \\ \chi_{mn} \\ \gamma_{mn} \\ \tau_{mn} \end{bmatrix} \\ &= \sum_{n=m}^{\infty} i^n \begin{bmatrix} U_{mn}^{(1),t}(c_1) & V_{mn}^{(1),t}(c_1) \\ \tilde{n}_1 V_{mn}^{(1),t}(c_1) & \tilde{n}_1 U_{mn}^{(1),t}(c_1) \\ X_{mn}^{(1),t}(c_1) & Y_{mn}^{(1),t}(c_1) \\ \tilde{n}_1 Y_{mn}^{(1),t}(c_1) & \tilde{n}_1 X_{mn}^{(1),t}(c_1) \end{bmatrix} \begin{bmatrix} \delta_{mn}^{(1)} \\ \gamma_{mn}^{(1)} \end{bmatrix} \end{aligned} \quad (32)$$

The matrix $[\Gamma]$ in Eq. (31) is given by

$$[\Gamma] = \begin{bmatrix} V_{mn}^{(3),t}(c) & U_{mn}^{(3),t}(c) & -U_{mn}^{(1),t}(c'_2) & -U_{mn}^{(3),t}(c'_2) \\ U_{mn}^{(3),t}(c) & V_{mn}^{(3),t}(c) & -\tilde{n}_2 V_{mn}^{(1),t}(c'_2) & -\tilde{n}_2 V_{mn}^{(3),t}(c'_2) \\ Y_{mn}^{(3),t}(c) & X_{mn}^{(3),t}(c) & -X_{mn}^{(1),t}(c'_2) & -X_{mn}^{(3),t}(c'_2) \\ X_{mn}^{(3),t}(c) & Y_{mn}^{(3),t}(c) & -\tilde{n}_2 Y_{mn}^{(1),t}(c'_2) & -\tilde{n}_2 Y_{mn}^{(3),t}(c'_2) \\ -V_{mn}^{(1),t}(c'_2) & -V_{mn}^{(3),t}(c'_2) & & \\ -\tilde{n}_2 U_{mn}^{(1),t}(c'_2) & -\tilde{n}_2 U_{mn}^{(3),t}(c'_2) & & \\ -Y_{mn}^{(1),t}(c'_2) & -Y_{mn}^{(3),t}(c'_2) & & \\ -\tilde{n}_2 X_{mn}^{(1),t}(c'_2) & -\tilde{n}_2 X_{mn}^{(3),t}(c'_2) & & \end{bmatrix} \quad (33)$$

Obviously, it is not sufficient to solve Eqs. (31) and (32) for the unknown coefficients α_{mn} , β_{mn} , δ_{mn} , χ_{mn} , γ_{mn} , τ_{mn} , δ'_{mn} , χ'_{mn} , γ'_{mn} , τ'_{mn} , $\delta_{mn}^{(1)}$ and $\gamma_{mn}^{(1)}$, and the other relations between them can be generated by using a transformation from spheroidal vector wave functions to spherical ones, which is of the form [26]

$$\begin{bmatrix} \mathbf{M}_{e_{mn}}^{r(j)}(c^{(h)}, \zeta, \eta, \phi) \\ \mathbf{N}_{e_{mn}}^{r(j)}(c^{(h)}, \zeta, \eta, \phi) \end{bmatrix} = \sum_{q=0,1}^{\infty} i^{m+q-n} d_q^{mn}(c^{(h)}) \begin{bmatrix} \mathbf{m}_{e_{m+m+q}}^{r(j)}(k_2r, \theta, \phi) \\ \mathbf{n}_{e_{m+m+q}}^{r(j)}(k_2r, \theta, \phi) \end{bmatrix} \quad (34)$$

where the superscript j takes, as already mentioned, the value of 1 or 3 according to the radial functions $R_{mn}^{(j)}(c^{(h)}, \zeta)$ and $R_{mn}^{(j)}(k_2r)$ of the first or third kind, and $c^{(h)}$ is c_2 or c'_2 .

By substituting Eq. (34) into Eqs. (27) and (28), which express the electric fields within the dielectric coating in the same spherical coordinate system, we can find that, for every q , the following formulae hold because of the orthogonality and linear independence of the spherical vector wave functions $\mathbf{m}_{e_{m+m+q}}^{r(j)}(k_2R, \theta, \phi)$, $\mathbf{n}_{e_{m+m+q}}^{r(j)}(k_2R, \theta, \phi)$

($j = 1, 3, q = 0, 1, 2 \dots \infty$)

$$\sum_{n=m,m+1}^{\infty} \begin{bmatrix} d_q^{mn}(c_2)\delta_{mn} - d_q^{mn}(c'_2)\delta'_{mn} \\ d_q^{mn}(c_2)\chi_{mn} - d_q^{mn}(c'_2)\chi'_{mn} \\ d_q^{mn}(c_2)\gamma_{mn} - d_q^{mn}(c'_2)\gamma'_{mn} \\ d_q^{mn}(c_2)\tau_{mn} - d_q^{mn}(c'_2)\tau'_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (35)$$

The prime over the summation sign indicates that q takes even values when $n - m$ is even and odd values when $n - m$ is odd [16].

For every value of m , we can truncate the infinite system of equations consisting of Eqs. (31), (32) and (35) by setting $n = m, m + 1, \dots, m + N$, $t = 0, 1, \dots, N$ and $q = 0, 1, \dots, N$, N being a suitable large number for a convergent solution, so that the total number of unknown coefficients is $12 \times (N + 1)$. From the above truncated system, an adequate number of relations between the unknown coefficients can be generated, and the standard numerical techniques may be employed to solve for them.

5. NUMERICAL RESULTS

Usually, one is more interested in the behavior of the scattered wave at relatively large distances from the scatterer (far field), which can

be deduced by taking the asymptotic form of \mathbf{E}^s , as $c\zeta \rightarrow \infty$. In $\mathbf{M}_{emn}^{r(3)}(c, \zeta, \eta, \phi)$ and $\mathbf{N}_{omn}^{r(3)}(c, \zeta, \eta, \phi)$, as $c\zeta \rightarrow \infty$, there can be neglected terms of order higher than $1/r$, then, from Eq. (20) the asymptotic forms of the scattered electric field \mathbf{E}^s can be obtained as [24]

$$-E_{\eta}^s = E_0 \frac{i\lambda}{2\pi r} \exp\left(i\frac{2\pi r}{\lambda}\right) \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left[\alpha_{mn} \frac{dS_{mn}(c, \cos \theta)}{d\theta} + m\beta_{mn} \frac{S_{mn}(c, \cos \theta)}{\sin \theta} \right] \sin m\phi \quad (36)$$

$$E_{\phi}^s = E_0 \frac{i\lambda}{2\pi r} \exp\left(i\frac{2\pi r}{\lambda}\right) \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left[m\alpha_{mn} \frac{S_{mn}(c, \cos \theta)}{\sin \theta} + \beta_{mn} \frac{dS_{mn}(c, \cos \theta)}{d\theta} \right] \cos m\phi \quad (37)$$

By virtue of Eqs. (36) and (37), we can have the differential scattering cross section which is defined by

$$\sigma(\theta, \phi) = 4\pi r^2 \left| \frac{\mathbf{E}^s}{E_0} \right|^2 = \frac{\lambda^2}{\pi} \left(|T_1(\theta, \phi)|^2 + |T_2(\theta, \phi)|^2 \right) \quad (38)$$

where

$$T_1(\theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left[m\beta_{mn} \frac{S_{mn}(c, \cos \theta)}{\sin \theta} + \alpha_{mn} \frac{dS_{mn}(c, \cos \theta)}{d\theta} \right] \sin m\phi \quad (39)$$

$$T_2(\theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left[m\alpha_{mn} \frac{S_{mn}(c, \cos \theta)}{\sin \theta} + \beta_{mn} \frac{dS_{mn}(c, \cos \theta)}{d\theta} \right] \cos m\phi \quad (40)$$

The normalized differential scattering cross section $\pi\sigma(\theta, \phi)/\lambda^2$ is thereafter evaluated in the coordinate system attached to the spheroidal particle. In the following calculations, the incident Gaussian beam is TE polarized and $x_0 = x_0 = z_0 = 0$. Since the representation of the radial functions of the third kind $R_{mn}^{(3)}(c, \zeta)$ in terms of the spherical Hankel functions of the first kind converges slowly [Flammer, (4.1.16)], difficulties arise in the calculations for larger or smaller ka and larger a/b . By adopting an algorithm using the recursive matrix equation for the computation of the expansion coefficients $d_q^{mn}(c)$ [27], numerical values presented until now have been for spheroids with $1 < ka < 20$ and $a/b < 3$.

Figure 2 shows the normalized differential scattering cross sections $\pi\sigma(\theta, \phi)/\lambda^2$ in three scattering planes ($\phi = 0, \frac{\pi}{4}, \frac{\pi}{2}$), for incidence of a Gaussian beam with $w_0 = 2\lambda$ (wavelength $\lambda = 0.6328 \mu\text{m}$) on a

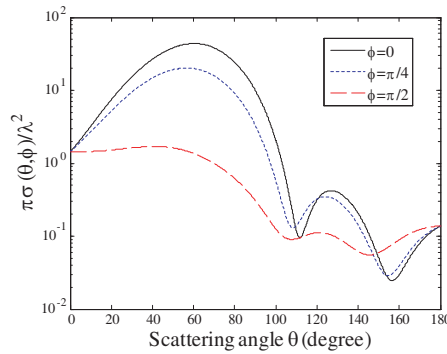


Figure 2. Normalized differential scattering cross sections $\pi\sigma(\theta, 0)/\lambda^2$, $\pi\sigma(\theta, \frac{\pi}{4})/\lambda^2$ and $\pi\sigma(\theta, \frac{\pi}{2})/\lambda^2$ for a dielectric spheroid ($ka = 5$, $\tilde{n} = 1.33$, $a/b = 2$, $\beta = -\frac{\pi}{3}$) illuminated by a Gaussian beam of $w_0 = 2\lambda$.

dielectric spheroidal particle. In the $\phi = 0$ plane, parallel to the incident plane, the spheroid has the maximum forwardscattering, i.e., in the angle regions around $\theta = \frac{\pi}{3}$ due to the transmitted light by the particle which dominates the scattering profile, and oscillations due to interference of light diffracted with light transmitted by the particle. With the increase of the angle by which the scattering plane incline from the incident plane, for example from $\phi = \frac{\pi}{4}$ to $\phi = \frac{\pi}{2}$, the curve oscillates with a smaller amplitude.

The radiation force and torque generated by a Gaussian beam on a particle can be used for optical trapping or manipulation, in which one fundamental problem concerns the description of Gaussian beam scattering by the particle, especially in an analytical form. In theoretical predictions of light scattering, it is probably appropriate to model a biological cell in water solution as a spheroidal particle having a spherical inclusion (a_1/b_1 taken to be 1.0001 in computation for the scattering geometry in Section 4), illuminated by one Ar laser beam. The incident Ar laser beam has a wavelength of $\lambda = 0.3868 \mu\text{m}$ in water and is approximated by a Gaussian beam with $w_0 = 2\lambda$. Fig. 3 shows the normalized differential scattering cross sections $\pi\sigma(\theta, 0)/\lambda^2$ for such a theoretical model. In Fig. 3, $\tilde{n}_1 = 1.109$ and $\tilde{n}_2 = 1.045$ are the refractive indices of the material of the inner sphere and outer spheroid, but relative to that of water, and the wavelength λ of the incident Gaussian beam is $0.3868 \mu\text{m}$.

Figure 3 shows that, with the increase of the angle between the propagation direction of the incident Gaussian beam and the major axis of the spheroid, i.e., the increase of the absolute value of β in

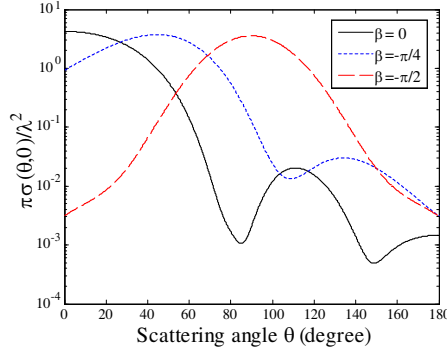


Figure 3. Normalized differential scattering cross sections $\pi\sigma(\theta, 0)/\lambda^2$ for a spheroidal particle with a spherical inclusion ($ka_1 = 2$, $ka_2 = 6$, $a_1/b_1 = 1.0001$, $a_2/b_2 = 2$, $\tilde{n}_1 = 1.109$, $\tilde{n}_2 = 1.045$), for incidence of a Gaussian beam with $w_0 = 2\lambda$ ($\beta = 0$, $\beta = -\frac{\pi}{4}$, $\beta = -\frac{\pi}{2}$).

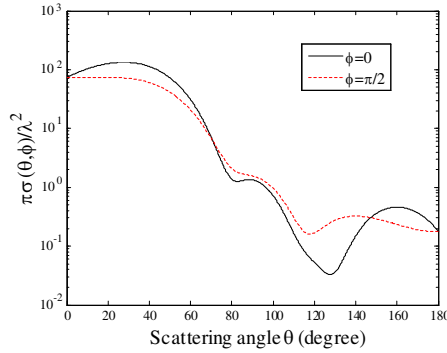


Figure 4. Normalized differential scattering cross sections $\pi\sigma(\theta, 0)/\lambda^2$ and $\pi\sigma(\theta, \frac{\pi}{2})/\lambda^2$ for a coated spheroid ($ka_1 = 4$, $ka_2 = 6$, $a_1/b_1 = a_2/b_2 = 2$, $\tilde{n}_1 = 1.5$, $\tilde{n}_2 = 1.33$, $\beta = -\frac{\pi}{6}$) for incidence of a Gaussian beam with $w_0 = 2\lambda$.

Fig. 1, the curve shows fewer oscillations. According to the concept of geometrical shadow adopted by Asano and Yamamoto [24, 25], the geometrical shadow of the spheroidal particle is $\pi b_2(a_2^2 \sin^2 \beta + b_2^2 \cos^2 \beta)^{1/2}$ and that of the spherical inclusion is πa_1^2 . It is obvious that, with the increase of the absolute value of β , the ratio of the geometrical shadow of the spherical inclusion to that of the spheroidal particle becomes smaller, and then the interference effects decrease of light diffracted and transmitted by the spherical inclusion with light by the spheroidal particle, thus leading to a smoother curve with fewer

oscillations.

Figure 4 shows the normalized differential scattering cross sections $\pi\sigma(\theta, \phi)/\lambda^2$ in two scattering planes $\phi = 0$ (plane 1) and $\phi = \frac{\pi}{2}$ (plane 2), parallel to and normal to the incident plane respectively, for a spheroid with a concentric non-confocal dielectric coating illuminated by a Gaussian beam of $w_0 = 2\lambda$. Compared to plane 1, plane 2 shows smoother oscillations.

6. CONCLUSION

As an extension of the GLMT for spheres, the theory of Gaussian beam scattering from a spheroidal particle is given, including the expansion of Gaussian beam in terms of spheroidal vector wave functions, scattering of Gaussian beam by an arbitrarily oriented spheroidal particle, and by a spheroidal particle with non-confocal dielectric coating. Numerical results for the normalized differential scattering cross section are presented. The plotted curve has the maximum forwardscattering, and oscillates with a smaller amplitude as the angle between the scattering plane and incident plane increases. For a coated spheroid, with the increase of the absolute value of β , the curve becomes smoother with fewer oscillations. Due to the difficulties in computation of the radial functions of the third kind $R_{mn}^{(3)}(c, \zeta)$, applications of the method to larger or smaller ka and larger a/b is at present limited. As a result, this study provides an important analytical model of the scatterer, and is suggestive and useful for interpretation of shaped beam scattering phenomena for homogenous and inhomogeneous spheroidal particles.

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