Progress In Electromagnetics Research B, Vol. 40, 343–360, 2012

ENERGY DENSITY OF MACROSCOPIC ELECTRIC AND MAGNETIC FIELDS IN DISPERSIVE MEDIUM WITH LOSSES

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Abstract—Electric permittivity and magnetic permeability of linear passive dispersive medium were defined using the circuit equation of an electrically small antenna (scatterer) with resonant and antiresonant properties. It was shown that the average macroscopic energy stored by the scatterers is proportional to frequency derivative of the input admittance of corresponding antenna. It was found that the average macroscopic energy density of electric and magnetic fields in dispersive lossy medium is a function of frequency derivatives of its effective constitutive parameters in accordance with Poynting's theorem in dispersive lossy medium clarified for this case in the paper.

1. INTRODUCTION

Several methods were employed to define average density of the electromagnetic energy in dispersive lossy materials [1–6] which were classified [2, 3] as equivalent circuit (EC) and electrodynamic (ED) approaches. Both kinds of approaches gave the same specific dependence of the electric field energy density corresponding to the Lorentz model [1–5] while they led to different relations for the specific magnetic field energy density corresponding to the non-Lorentz [2] model of broken loop composites [1].

The energy densities of macroscopic electric and magnetic fields were defined by ED methods [2–4] together with power of losses on basis of assumptions about possible types of energy density functions, using equations of motion for polarization and magnetization [3] in Poynting's theorem. Employed ED approaches gave different relations for the magnetic field energy density in a lossy medium [2, 3] while, according to them, the magnetic field energy density in medium comprised of electrically small elements might be less than the energy

Received 27 March 2012, Accepted 24 April 2012, Scheduled 9 May 2012
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density in vacuum [1] without clear reasons. According with [2, 3], ED relations for the magnetic field energy density based on the non-Lorentz permeability are consistent with Brillouin's energy density in the limit of zero losses. However, the quality factor derived from relations obtained in [3] for the non-Lorentz permeability demonstrated a nonphysical behavior [1].

The input impedance of the solenoid defined through use of the non-Lorentz magnetic permeability of metamaterial was employed to find an equivalent circuit for quasi-static calculations of the stored energy in EC approach [1, 5]. Positivity of the energy stored by the solenoid loaded with metamaterial demonstrated positivity of corresponding magnetic field energy density. However, obtained relation for the magnetic field energy density is not consistent with Brillouin's energy density in the limit of zero losses according with [1].

Employed approaches for definition of the energy density were based on using of constitutive parameters for lossy medium, whereas, according with [1, 7], those macroscopic parameters is not sufficient, and knowledge of microscopic circuit structure is necessary to make sure that it does not include subcircuits with invisible energy on the level of the macroscopic description.

In order to determine specific issues limiting use of constitutive parameters for definition of the energy density, a generalized RLC circuit model of a moderately electrically small structural element was introduced in the paper in contrast with the Lorentz model. The generalized RLC circuit model takes into account performance properties of electrically small antennas and scatterers [8–11]. Obtained results show that the average macroscopic energy density of electric and magnetic fields in dispersive lossy medium defined through use of the energy stored by the ensemble of the independent electrically small antennas-scatterers with sufficiently high quality factors and not overlapping resonances is accurately approximated by functions of effective permittivity and permeability. The obtained average energy density of the macroscopic electric and magnetic fields in dispersive lossy medium corresponds to Poynting's theorem for linear dispersive lossy medium, which was clarified in the paper.

2. POYNTING'S THEOREM IN DISPERSIVE LOSSY MEDIUM

The energy density in dispersive lossy medium may be obtained using Poynting's theorem in the Abraham form, which implies that [12–14]

$$\operatorname{div}(\mathbf{E}\times\mathbf{H}) = -\mathbf{E}\frac{\partial\mathbf{D}}{\partial t} - \mathbf{H}\frac{\partial\mathbf{B}}{\partial t}. \quad (1)$$

In order to discuss dispersion, one has to use Fourier decomposition of the field vectors [12, 13]. In the case of the electric field, one can write

$$\mathbf{E}(t) = \int_{-\infty}^{+\infty} d\omega \mathbf{E}(\omega) e^{j\omega t}, \\ \mathbf{D}(t) = \int_{-\infty}^{+\infty} d\omega \mathbf{D}(\omega) e^{j\omega t}, \tag{2}$$

where $\mathbf{D}(\omega) = \varepsilon_0 \varepsilon(\omega) \mathbf{E}(\omega)$ for an isotropic linear medium.

Using Fourier components of the electric field, one can find

$$\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} = \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \mathbf{E}(\omega') \left[j\omega\varepsilon_0\varepsilon(\omega)\right] \mathbf{E}(\omega) e^{j(\omega+\omega')t}. \tag{3}$$

In a general case, knowledge of $\varepsilon(\omega)$ in the frequency range up to the infinity is necessary in accordance with (2) and (3). Presence of dispersion corresponds to the complex and frequency-dependent electric susceptibility $\chi_e(\omega) = \varepsilon(\omega) - 1$, which in a narrow frequency range may be approximated by a linear dependence. In order to use (3) for the electric field with a narrow frequency range $\omega \approx \omega'$, one can write (3) in a symmetrical form

$$\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\int_{-\infty}^{+\infty} d\omega \int_{+\infty}^{-\infty} d\omega' \mathbf{E}^*\left(\omega'\right) \\ \times \left[j\omega\varepsilon_0\varepsilon(\omega) - j\omega'\varepsilon_0\varepsilon^*\left(\omega'\right)\right] \mathbf{E}(\omega) e^{j(\omega-\omega')t}, \tag{4}$$

where the Fourier decomposition of $\mathbf{E}(t)$ is presented by a function of $\mathbf{E}^*(\omega') = \mathbf{E}(\omega)$, $\omega' = -\omega$. In the case $\omega' \approx \omega$, one can find

$$\left[j\omega\varepsilon_0\varepsilon(\omega) - j\omega'\varepsilon_0\varepsilon^*\left(\omega'\right)\right] \\ = 2\omega\varepsilon_0\varepsilon''(\omega) + j\left(\omega - \omega'\right) \left.\frac{d\left(\omega'\varepsilon_0\varepsilon^*(\omega')\right)}{d\omega'}\right|_{\omega'=\omega} + o\left[\left(\omega' - \omega\right)\right]. \tag{5}$$

Using (5) in (4), one can find [13]

$$\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} = \int_{-\infty}^{+\infty} d\omega \int_{+\infty}^{-\infty} d\omega' \mathbf{E}^*\left(\omega'\right) \\ \times\omega\varepsilon_0\varepsilon''(\omega) \mathbf{E}(\omega) e^{j(\omega-\omega')t} + \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} d\omega \int_{+\infty}^{-\infty} d\omega' \mathbf{E}^*\left(\omega'\right) \\ \times\varepsilon_0 \mathbf{E}(\omega) e^{j(\omega-\omega')t} + \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} d\omega \int_{+\infty}^{-\infty} d\omega' \mathbf{E}^*\left(\omega'\right) \\ \times \left.\frac{d\left(\omega'\varepsilon_0\chi_e^*(\omega')\right)}{d\omega'}\right|_{\omega'=\omega} \times \mathbf{E}(\omega) e^{j(\omega-\omega')t}. \tag{6}$$

In order to illustrate obtained results, one can use the case of a monochromatic field $E(t) = E_0 \cos \omega t$. The first integrand (6) for this case gives the power of losses $\varepsilon_0 \omega \varepsilon'' E(t)^2$ in accordance with (3). The second integrand (6) gives the time derivative of the energy density of the electric field in free space $\frac{\varepsilon_0}{2} \frac{\partial}{\partial t} E(t)^2$. If permittivity is approximated by a constant and losses are small, then $\frac{d(\omega \varepsilon_0 \varepsilon')}{d\omega} = \varepsilon_0 \varepsilon'$ and the time derivative of the total electric field energy density is $\frac{\varepsilon_0 \varepsilon'}{2} \frac{\partial}{\partial t} E(t)^2 = -\frac{\varepsilon_0 \varepsilon' \omega}{2} E_0^2 \sin 2\omega t$ in accordance with (6). It is worth noticing that, in accordance with (3), $j \cos \omega t$ in (3) corresponds to $-\sin \omega t$ in (6).

All summands in (6) are real values for the quasi-monochromatic field except for the last integrand, which includes the complex value. The function $\omega \varepsilon(\omega)$ in the narrow frequency range is accurately approximated by the linear expansion (5), while we can clarify meaning of the imaginary component using the last integrand in (6) in the form similar to (3)

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \mathbf{E}(\omega') \times \left| \left| \frac{d(\omega' \varepsilon_0 \chi_e(\omega'))}{d\omega'} \right|_{\omega'=\omega} \right| \times \mathbf{E}(\omega) e^{j(\omega+\omega'-\alpha)t}, \tag{7}$$

where ω' was substituted for $-\omega'$, $\cos\alpha = \frac{\frac{d(\omega \varepsilon_0 (\varepsilon'(\omega)-1))}{d\omega}}{\left| \frac{d(\omega \varepsilon_0 \chi_e(\omega))}{d\omega} \right|}$, $\alpha > 0$. In the case of the monochromatic field, relation (7) corresponds to the time derivative of the energy density $\frac{1}{2} \left| \frac{\partial(\chi_e(\omega)\omega)}{\partial \omega} \right| \frac{\partial}{\partial t} (E_0 \cos(\omega t - \alpha/2))^2$ with the average value $\frac{\varepsilon_0}{4} \left| \frac{\partial(\chi_e(\omega)\omega)}{\partial \omega} \right| {E_0}^2$.

In the case of the monochromatic field, the real part of the last integrand (6) equal to $\frac{1}{2} \frac{d(\omega \varepsilon_0 (\varepsilon'(\omega)-1))}{d\omega} \frac{\partial}{\partial t} E(t)^2 \sim -\sin 2\omega t$ corresponds to the time derivative of the dispersive (Brillouin's) component of the electric field energy density stored by dispersive medium. The imaginary part of the integrand (6) $\frac{1}{4} \frac{d(\omega \varepsilon_0 \varepsilon''(\omega))}{d\omega} E_0^2 \frac{\partial}{\partial t} \sin 2\omega t \sim \cos 2\omega t$ corresponds to the time derivative of the dissipation component of the electric field energy density, which is necessary to introduce in accordance with (3) and (6). Nature of this component of the energy is discussed in the Section 4. Using a similar approach to $\mathbf{H} \frac{\partial \mathbf{B}}{\partial t}$, one can obtain the average energy density of the the quasi-monochromatic macroscopic magnetic field.

The definition of the energy density of the quasi-monochromatic field in dispersive medium is based on linear approximation of dependences $\omega\varepsilon(\omega)$, $\omega\mu(\omega)$ valid in a narrow frequency range. In the case of arbitrary changing fields discussed in [2–4], constitutive parameters have to be defined in frequency range up to infinity because they set constitutive relations necessary for a sufficient definition of Maxwell's equations [12]. Constitutive parameters of dispersive lossy medium defined in a limited frequency range are not fully represent properties of a medium because, Kramers-Kronig's formulae can be satisfied by different functions of constitutive parameters, which have common functional dependences in the mentioned frequency range. While the problem of definition of the energy density of an arbitrary changing field in arbitrary medium cannot be solved using models applicable in a limited frequency range, such models clarify physical mechanisms of interaction of the quasi-monochromatic field with dispersive lossy medium.

3. GENERALIZED CIRCUIT MODEL OF DISPERSIVE LOSSY MEDIUM

The Lorentz model is commonly used to approximate the electric permittivity of dispersive lossy medium. According to the model, motion of electrical charge in atoms [12–14] and structural elements of metamaterial in an applied field is described by similar differential equations [2, 4, 14] in accordance with mechanics equation of motion and resonant circuit model. That equation is transformed into a macroscopic equation for polarization [2–4, 12–14]. Using the equation for polarization, one can find Lorentz's relative permittivity, which represents properties of dispersive lossy medium on the macroscopic level

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + i\omega\Gamma}, \tag{8}$$

where $\varepsilon(\omega) = \varepsilon'(\omega) - i\varepsilon''(\omega)$ is the complex permittivity for an electric field with accepted in the paper time dependency $e^{i\omega t}$, ω_0 — the resonant angular frequency of atoms or structural elements of metamaterial, Γ — the damping coefficient, ω_p — the characteristic (plasma) frequency, which are constant.

It is known that the Lorentz model [12–14] does not take into account effects of radiative reaction [13], which are essential for structural elements of a non-infinitesimal electrical size. The line breadth of the radiating oscillator is not exactly predicted even by a modified Lorentz model with radiative damping [13] that demonstrates inherent limitations of the resonant model. Similar limitations arise in the case of applying of the resonant model to structural elements of metamaterial.

Despite noticed limitations, the Lorentz-type dependences of effective constitutive parameters [1–6, 14] were used in EC approach [1, 5, 7] to describe dispersive lossy metamaterial by macroscopic equivalent circuits. According with equivalent circuits, electrically small structural elements embedded in free space was presented by macroscopic resonant circuits. Insufficiency of the resonant model in the case of permeability was demonstrated for a lossy matrix material [1] that resulted in definition of a corrected Lorentz's dependence.

One can use a more general approach to definition of effective constitutive parameters, using a circuit model of an electrically small antenna (scatterer). Performance properties of those antennas are sufficiently known, while the antennas with resonant and antiresonant properties are accurately modeled by equivalent circuits [8–11, 15]. Using examples of equivalent circuits of an electrically small antenna with two resonances around antiresonance in Fig. 1 [11] and a scatterer with resonant and antiresonant frequencies in Fig. 2, one can write a circuit equation of an electrically small antenna in the form

$$L(\omega)\frac{d^2q}{dt^2} + R(\omega)\frac{dq}{dt} + \frac{q}{C(\omega)} = U(t), \tag{9}$$

where $q = q(t)$ — electric charge of the capacitance $C(\omega)$, $U(t) = U_0 e^{i\omega t}$ — voltage produced by an applied field [4]. The circuit Equation (9) is similar to the resonant circuit equation [4]. However, total inductance $L(\omega)$, capacitance $C(\omega)$, and resistance $R(\omega)$ are functions of frequency. As a result, the circuit Equation (9) corresponds to the generalised Lorentz effective permittivity. Similarity of the circuit Equation (9) and the resonant circuit equation [4] provides similarity of generalised Lorentz effective permittivity and Lorentz permittivity (2) while

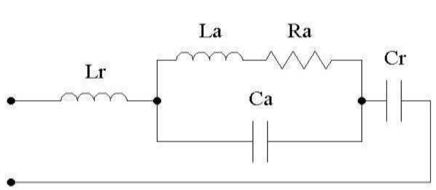

Figure 1. Equivalent circuit of an electrically small antenna with two resonances around antiresonance.

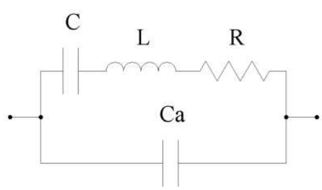

Figure 2. Equivalent circuit of an electrically small scatterer with resonant and antiresonant frequencies.

parameters of the generalized permittivity, such as the resonance angular frequency $\omega_0(\omega) = (L(\omega) C(\omega))^{-0.5}$, the damping coefficient $\Gamma(\omega) = R(\omega)/L(\omega)$, and plasma frequency are functions of frequency ω in accordance with (9). Frequency dependences of discussed parameters of the permittivity are especially significant when energy densities are expressed by functions of the parameters of equivalent circuits.

4. ELECTRIC FIELD ENERGY DENSITY IN THE MODELED MEDIUM

The problem of definition of the electric field energy density can be solved through use of the quality factor of electrically small antenna, which is defined by the ratio of the energy stored by an antenna to power of losses and radiation. The quality factor of an electrically small antenna is proportional to the inverse conductance bandwidth [9] while the bandwidth is defined by conductance of the antenna [8]. Using the conductance of an antenna, one can obtain the quality factor as a function of the input impedance of the antenna [9]. However, it was noticed in [8–10, 16] that those relations had low accuracy in antiresonant ranges of some antennas. Fortunately, reasons of that have been recently clarified, using the case of electrically small antennas with double resonances [11, 15].

It was shown that the time-averaged energy of non-propagating electromagnetic field of an antenna with two coupled resonances [15] presented by the equivalent circuit in Fig. 1 is accurately approximated by the sum of the energies of the resonant and antiresonant subcircuits [11]

$$\overline{W}(\omega) = \frac{1}{4}\left(\frac{\partial X_r}{\partial\omega} + \sqrt{\left[\frac{\partial R}{\partial\omega}\right]^2 + \left[\frac{\partial X_a}{\partial\omega}\right]^2 + \Delta^2}\right) I_0^2, \qquad (10)$$

where I_0 is the amplitude of the input current of the antenna, $\frac{\partial X_r}{\partial\omega} = \frac{|X_{rC}|+X_{rL}}{\omega}$ — frequency derivative of reactance of the resonant subcircuit comprised of C_r and L_r with reactance X_{rC} and X_{rL}; $\frac{\partial X_a}{\partial\omega}$ and $\frac{\partial R}{\partial\omega}$ is frequency derivatives of reactance and resistance of the antiresonant subcircuit comprised of R_a, L_a and C_a. The sum of negative X_c and positive X_l components of reactance produced by the antiresonant subcircuit is [11]

$$|X_c| + X_l = \omega\sqrt{\left[\frac{\partial R}{\partial\omega}\right]^2 + \left[\frac{\partial X_a}{\partial\omega}\right]^2 + \Delta^2}, \qquad (11)$$

where $\Delta = \frac{2R}{\omega_0}$, $\Delta^2 \ll \left[\frac{\partial R}{\partial \omega}\right]^2 + \left[\frac{\partial X_a}{\partial \omega}\right]^2$ for the antenna with the electrical size $ka \cong 0.5$ [11], k — the wave number, a — the minimal radius of the sphere circumscribing the antenna. According with (11), frequency derivatives of reactance and resistance define the effective reactance of the antiresonant subcircuit, while the energy is stored by reactive elements.

In order to evaluate relative importance of addends in (11), one can use the so called approximate quality factor (or the impedance quality factor) of an electrically small antenna tuned to antiresonance [8–11]

$$Q = \left(\left[\frac{\partial R}{\partial \omega}\right]^2 + \left[\frac{\partial X_a}{\partial \omega}\right]^2 \right)^{0.5} / \Delta. \tag{12}$$

As a result, $\sqrt{\left[\frac{\partial R}{\partial \omega}\right]^2 + \left[\frac{\partial X_a}{\partial \omega}\right]^2 + \Delta^2} \cong \sqrt{\left[\frac{\partial R}{\partial \omega}\right]^2 + \left[\frac{\partial X_a}{\partial \omega}\right]^2}$ with relative accuracy $0.5Q^{-2}$, while $Q \leq (ka)^{-1} + (ka)^{-3}$ [17].

The energy stored by the antenna (10) is calculated through use of reactance of resonant and antiresonant circuits because the frequency derivative of reactance of an antiresonant subcircuit of an antenna becomes negative in the vicinity of the antiresonant frequency [11, 15]. This aspect is essential for antennas with closely spaced resonances because of comparable magnitudes of frequency derivatives of reactance of resonant and antiresonant subcircuits. Decrease of the electrical size of the antenna corresponds to increase in the quality factor and frequency ranges between resonances and antiresonance of the antenna [11, 15]. Increase in the quality factor of the antiresonant subcircuit of the antenna of a comparatively small size $ka < 0.1$ as compared with $ka \cong 0.5$ corresponds to a more than 102 times increase in the frequency derivative of reactance of the antiresonant circuit [18]. Therefore, $\frac{\partial X_a}{\partial \omega} \cong \frac{\partial X}{\partial \omega}$, $\left|\frac{\partial X_a}{\partial \omega} / \frac{\partial R}{\partial \omega}\right| \cong 1$ and the energy stored by electrically small antennas $ka \leq 0.1$ in antiresonant ranges is defined by $\left|\frac{\partial Z}{\partial \omega}\right|$ with accuracy better than Q^{-2}.

In resonant ranges, equivalent circuits of antennas with $ka < 0.1$ boil down to a resonant circuit that provides $\frac{\partial R}{\partial \omega} \ll \left|\frac{\partial X}{\partial \omega}\right|$. An additional resistance connected in parallel with capacitance of the resonant circuit [1] provide $R(\omega)$ of the equivalent series circuit with comparatively small frequency derivative $\frac{\partial R}{\partial \omega} \cong \left|\frac{\partial X}{\partial \omega}\right| / Q$ that corresponds accuracy better than Q^{-2}.

Overall, in accordance with calculations of the quality factors of antennas with not closely spaced resonances [8–11, 15, 16], the average energy stored by an antenna is

$$\overline{W} = \frac{1}{4}\sqrt{\left[\frac{\partial R}{\partial \omega}\right]^2 + \left[\frac{\partial X}{\partial \omega}\right]^2} {I_0}^2 = \frac{1}{4}\left|\frac{\partial Z}{\partial \omega}\right| {I_0}^2. \tag{13}$$

Comparison of the impedance and exact quality factors with inverse antenna bandwidths shows that the relation (13) connected with the impedance quality factor is more reliable for antennas in dispersive lossy medium [8].

Using the admittance $Y = Z^{-1}$ in (6), one can find

$$\overline{W} = \frac{1}{4}\left|\frac{\partial Y}{\partial \omega}\right| {U_0}^2, \tag{14}$$

where $I_0 = U_0/|Z|$, $\left|\frac{\partial Z}{\partial \omega}\right| = \left|\frac{\partial Y}{\partial \omega}\right|/|Z|^2$, $Y = G + iB$, $G = \frac{R}{R^2+X^2}$ is the conductance, and $B = -\frac{X}{R^2+X^2}$ — the susceptance of an electrically small antenna [19]. Making use of (14), one can find the time-averaged energy stored by a circuit comprised of N subcircuits connected in parallel

$$\overline{W} = \frac{1}{4}\sum\nolimits_{i=1}^{N}\left|\frac{\partial Y_i}{\partial \omega}\right| {U_0}^2. \tag{15}$$

It should be noticed that dependence of the stored energy from frequency derivative of resistance (13) demonstrates importance of antiresonant properties along with resonant ones in a general case in accordance with the generalized Lorentz model, whereas it becomes vague for the formula (15) and other below-derived relations.

RLC circuit in Fig. 2 represents an equivalent circuit of a scatterer with resonant and antiresonant properties, e.g., an electrically small pair of needles directed along the electric field [1, 14], which antiresonant properties is provided by capacitance of free space [10, 18] C_a around needles. Localization of resonant admittance of the circuit in Fig. 2 in a frequency range $\Delta\omega \propto 1/Q$ [8, 9] allows neglecting by interference of resonant and antiresonant properties in the vicinity of a resonant frequency if $Q \gg 1$, as a result, $\sum_{i=1}^{N}\left|\frac{\partial Y_i}{\partial \omega}\right| \cong \left|\frac{\partial Y}{\partial \omega}\right|$. That is also applied to the case of many resonant frequencies of different scatterers (inclusions in [1]) since corresponding series circuits do not significantly interact if resonances are not considerably overlapped.

Reactance of an electrically small antenna in accordance with the circuit Equation (9) is

$$X(\omega) = \omega L(\omega) - \frac{1}{\omega C(\omega)} = \omega L\left(1 - \frac{\omega_0^2}{\omega^2}\right), \tag{16}$$

where for brevity $L = L(\omega)$, $\omega_0 = \omega_0(\omega)$.

Using (16), one can find conductance and susceptance corresponding to a scatterer.

$$\begin{aligned} G(\omega) &= \left(\frac{\omega}{L}\right) \frac{\Gamma\omega}{\Gamma^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}, \\ B(\omega) &= -\left(\frac{\omega}{L}\right) \frac{\left(\omega^2 - \omega_0^2\right)}{\Gamma^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}. \end{aligned} \tag{17}$$

Comparing (8) and (17), one can connect the admittance and the effective permittivity of a medium comprised of an ensemble of independent electrically small scatterers

$$\begin{aligned} B(\omega) &= \varepsilon_0 \left(\varepsilon' - 1\right) \omega S/d, \\ G(\omega) &= \varepsilon_0 \varepsilon'' \omega S/d, \end{aligned} \tag{18}$$

where $S\left(m^2\right)$ and $d\,(m)$ is dimensional constants, $\omega_p^2 = d/\left(\varepsilon_0 LS\right)$ in accordance with (8) and (17). It is worth noticing that the average energy stored by LC circuit is proportional to $\left|\frac{\partial B}{\partial \omega}\right|$ (14) while the energy density of the electric field in dispersive lossless medium is proportional to $\frac{\partial(\varepsilon'\omega)}{\partial\omega}$ in accordance with Brillouin's formula [12–14]. The time-averaged power accepted by an RLC circuit or an antenna [8, 9] is proportional to G while the power of losses of the energy flux of an applied field in dispersive lossy medium depends on $\varepsilon''\omega$ in accordance with (1). Even though the electrical size of structural elements is small, ε'' depends on radiation resistance proportional to $(ka)^2$ [21].

Spacial averaging of the microscopic electromagnetic field results in nontrivial attribution of the electromagnetic energy to macroscopic electric and magnetic fields. If pair parts of microscopic inductances of structural elements of a medium create magnetic fields in opposite directions, then macroscopic magnetization of the medium is zero. That would result in absence of the macroscopic magnetic field energy stored by the medium with constitutive parameters $\mu = 1$, $\varepsilon \neq 1$ while the energy of the microscopic magnetic field is taken into account as positive energy of the macroscopic electric field.

Using (14) and (18), one can find the energy of the macroscopic electric field stored by electrically small scatterers arranged on a lattice as a function of the effective susceptibility

$$\overline{W_e} = \frac{\varepsilon_0}{4} \left|\frac{\partial\left(\chi_e\omega\right)}{\partial\omega}\right| \frac{S}{d} (\,E_0 d)^2 = \frac{\varepsilon_0}{4} \left|\frac{\partial\left(\chi_e\omega\right)}{\partial\omega}\right| V {E_0}^2, \tag{19}$$

where $U_0 = E_0 d$, E_0 is the amplitude of an applied electric field, d — effective length of a unit cell of the lattice in the direction of the electric field, and $V = Sd$ — volume of the unit cell. Using the amplitude of a slowly varying envelope $E_0(t)$ [12–14] instead of E_0, one can find the total average energy density of the quasi-monochromatic electric field in dispersive lossy medium, including the energy density in free space

$$\overline{U_{et}(t)} = \frac{\overline{W_e}}{V} + \frac{1}{4}\varepsilon_0 |E_0(t)|^2 = \frac{\varepsilon_0}{4}\left(1 + \left|\frac{\partial\left((\varepsilon - 1)\,\omega\right)}{\partial\omega}\right|\right)|E_0(t)|^2, \quad (20)$$

where $\frac{\varepsilon_0}{4}|E_0(t)|^2$ is the field energy density in free space taken into account in the macrostructural circuit model [5] through use of capacitance C_0. In accordance with (20), the energy density does not depend on the conductance of a matrix material $\sigma = const$ because corresponding $\Delta\varepsilon'' \sim \sigma/\omega$ [12].

According with Kramers-Kronig's formulae based on the causality principle, real and imaginary parts of constitutive parameters are Hilbert transforms of each other [14]. This implies in accordance with (18) interconnection of conductance and susceptance (or resistance and reactance) of scatterers with $Q \gg 1$. That clarifies dependence of the stored energy (20) from the factor $\left|\frac{\partial((\varepsilon-1)\omega)}{\partial\omega}\right|$.

In the limit of zero losses $\overline{U_{et}(t)} = \frac{\varepsilon_0}{4}\frac{\partial(\varepsilon'\omega)}{\partial\omega}|E_0(t)|^2$ in accordance with (20) and Brillouin's formula because $\frac{\partial(\varepsilon'\omega)}{\partial\omega} > 1$ [5]. Using the Lorentz permittivity (8) in (20), one can find

$$\overline{U_{es}} = \frac{\varepsilon_0}{4}\left(1 + \frac{\omega_p^2\left(\omega^2 + \omega_0^2\right)}{\Gamma^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}\right){E_0}^2. \quad (21)$$

The specific energy density of the electric field (21) obtained through use of the generalized Brillouin's formula (20) coincides with previously obtained results for the specific energy density of the electric field [1–6] in dispersive lossy medium comprised of resonant non-radiative structural elements.

5. MAGNETIC FIELD ENERGY DENSITY

The equivalent circuit of the electrically small antenna in Fig. 1 with resonant and antiresonant properties generalizes serial RLC circuits used to model structural elements of metamaterials, such as broken loops [1, 5, 14]. Emf in the loop of the shorted antenna is $U(t) = -\frac{d\Phi}{dt}$ [13]; Φ is the magnetic flux created through the loop by an applied magnetic field. Under condition of uniformity of the magnetic field within an electrically small unit cell, one can find emf amplitude

$$U_0 = \mu_0\omega S_\perp H_0, \quad (22)$$

where H_0 is the magnetic field amplitude, $S_\perp$ — projection area of the of the loop on a plane orthogonal to the lines of the magnetic field.

Using relations (14) and (22), one can find the average electromagnetic energy stored by the loop

$$\overline{W_m} = \frac{1}{4}\left|\frac{\partial Y}{\partial \omega}\right| \mu_0^2 \omega^2 S_\perp^2 {H_0}^2. \tag{23}$$

In order to use (23) on the macroscopic level, it is necessary to connect the admittance of the loop with macroscopic parameters. In accordance with [1–5], one can introduce the generalised relative non-Lorentz magnetic permeability as effective medium parameter of broken loops arrays

$$\mu(\omega) = 1 + \frac{A\omega^2}{\omega_0^2 - \omega^2 + i\omega\Gamma}, \tag{24}$$

where $A(\omega) = \mu_0 S_\perp^2 / (L(\omega) V)$, $\mu(\omega) = \mu'(\omega) - i\mu''(\omega)$. Using (17) and (24), one can obtain

$$\begin{aligned} G(\omega) &= V\mu'' / \left(\omega\mu_0 S_\perp^2\right), \\ B(\omega) &= V\left(\mu' - 1\right) / \left(\omega\mu_0 S_\perp^2\right), \end{aligned} \tag{25}$$

where V is the volume of a unit cell of broken loops arrays. Using (24) and (25) in (23), one can find the average magnetic energy density

$$\overline{W_m} = \frac{\mu_0}{4}\left|\frac{\partial(\chi_m/\omega)}{\partial\omega}\right| \omega^2 V {H_0}^2, \tag{26}$$

where $\chi_m(\omega) = \mu(\omega) - 1$ [12]. Using the amplitude of a slowly varying envelope $|H(t)|$ [12–14] instead of H_0, one can find the total energy density of the magnetic field including the energy density in free space

$$\overline{U_{mt}(t)} = \frac{\langle W_m \rangle}{V} + \frac{\mu_0 {H_0}^2}{4} = \frac{\mu_0}{4}\left(1 + \left|\frac{\partial((\mu-1)/\omega)}{\partial\omega}\right| \omega^2\right)|H(t)|^2. \tag{27}$$

The non-Lorentz permeability is a physically sound parameter in the quasi-static limit ($\lim_{\omega\to 0}\mu(\omega) = 1$), whereas the Lorentz permeability is adequate in the high frequency limit [1, 3, 5]. Combining properties of Lorentz and non-Lorentz permeability, one can introduce the quasi-Lorentz magnetic permeability

$$\tilde{\mu}(\omega) = 1 + \frac{A\omega_m^2}{\omega_0^2 - \omega^2 + i\omega\Gamma}, \tag{28}$$

where in the high frequency limit, in which the condition of electrically small size of structural elements is satisfied [14], $\omega_m^2(\omega) = \omega_{0m}^2$, ω_{0m}^2 is a constant parameter of the Lorentz permeability so that

$\lim_{\omega\to\infty} \tilde{\mu}(\omega) = 1$. In accordance with properties of the the non-Lorentz permeability (24), $\omega_m^2(\omega) \approx \omega^2$ is an approximating step-function defined in low frequency range $\left(\frac{\partial\omega_m^2}{\partial\omega} = 0\right)$. Using (24) and (28) in (27), one can find

$$\overline{\mathcal{U}_{mt}(t)} = \frac{\mu_0}{4}\left(1 + \left|\frac{\partial\left((\tilde{\mu}-1)\,\omega\right)}{\partial\omega}\right| \frac{\omega^2}{\omega_m^2}\right)|H_0(t)|^2. \tag{29}$$

In accordance with (27) and (29), the macroscopic magnetic field energy density can be expressed through use of the Lorentz permeability (the causal model [1], $\omega_m^2 = \omega_{0m}^2$) (29) and non-Lorentz permeability (27).

Using the condition of quasi-monochromaticity $\omega_m \cong \omega$ for a slowly varying envelope $H(t)$ in the denominator of (29), one can find in accordance with definition of the quasi-Lorentz permeability (28)

$$\overline{\mathcal{U}_{mt}(t)} = \frac{\mu_0}{4}\left(1 + \left|\frac{\partial\left((\tilde{\mu}-1)\,\omega\right)}{\partial\omega}\right|\right)|H_0(t)|^2. \tag{30}$$

Using (30) in the limit of zero losses, one can obtain Brillouin's formula for the magnetic energy density in the form, which is valid in low and higher frequency ranges

$$\overline{U_{mt}(t)} = \frac{\mu_0}{4}\frac{\partial\left(\tilde{\mu}'\omega\right)}{\partial\omega}|H_0(t)|^2. \tag{31}$$

In the limit of zero losses, relation (30) and (31) are equivalent because $\frac{\partial(\tilde{\mu}'\omega)}{\partial\omega} > 1$ [5]. Therefore, relations (27), (30), and (31) are equivalent under stated conditions.

Inserting non-Lorentz permeability ($A = const$, $\omega_0 = const$, $\Gamma = const$) (24) in (27) or using analogy between (20) and (30), one finds the specific magnetic field energy density [1, 5]

$$\overline{U_{ms}} = \frac{\mu_0}{4}\left(1 + \frac{A\omega^2\left(\omega^2+\omega_0^2\right)}{\Gamma^2\omega^2 + \left(\omega^2-\omega_0^2\right)^2}\right){H_0}^2. \tag{32}$$

In fact, there is natural similarity of (21) and (32). In both cases we have the same energy of electromagnetic oscillations; while the factor ω^2 in (32) is explained by proportionality $U_0 \sim \omega$ (22). Relations (21) and (32) are valid in the case of constant parameters of the Equation (9) when analogy with mechanical vibrations [22] is lawful.

Macroscopic equations of motion for magnetization used in ED approaches [2–4] are based on (9), whereas the obtained specific energy density functions inconsistent with (32). Conformity of the obtained energy density in the limit of zero losses with the formula $\overline{U_{mt}} = \frac{\mu_0}{4}\frac{\partial(\mu'\omega)}{\partial\omega}\overline{|H(t)|}^2$ was assumed as a criterium of internal consistency of the ED approaches [2–4], whereas the latter relation and (31) do not coincide. Using the non-Lorentz permeability (24) in (30) by analogy with ED approach, one can find

$$\overline{\mathcal{U}_{ms}} = \frac{\mu_0}{4}\left(1 + \frac{A\omega^2\left(4\Gamma^2\omega^2 + \left(\omega^2 - 3\omega_0^2\right)^2\right)^{0.5}}{\Gamma^2\omega^2 + \left(\omega^2 - \omega_0^2\right)^2}\right) H_0{}^2. \qquad (33)$$

Relation (33) closely approximates the energy density of the magnetic field (26) [3] in a lower frequency range $\omega < \sqrt{3\omega_0^2 + \Gamma^2}$. The magnetic field energy density (26) [3] and (28) [2] obtained with ED approaches is smaller than the energy density in free space in a higher frequency range [1], whereas (33) is free from this disadvantage. In the case of zero losses, relations (33), (19) [3], and (21) [2] are identical. However, all those relations, including (33), inconsistent with (9).

Using the model of the quasi-Lorentz magnetic permeability (28), one can transform the specific magnetic field energy density obtained through use of ED approach with the Lorentz magnetic permeability [6, 14] into relations corresponding to the non-Lorentz magnetic permeability. The above mentioned energy relation, similarly as the relation for the electric field (21), includes a constant factor in the numerator [3, 6, 14] in the form of the squared frequency ω_{0m}^2. If ω_{0m}^2 (28) is presented by a step-function approximating ω^2, then that specific magnetic field energy density is transformed in (32). Choosing frequency ranges of steps of the step-function wider than the frequency range of the quasi-monochromatic field, one boils down ED approach in the magnetic field case to ED approach successfully employed for the electric field.

6. DISCUSSION

Macroscopic electric and magnetic properties of lossless medium comprised of electrically small elements are fully described by constitutive parameters. However, constitutive parameters might be insufficient for definition of the average field energy density in some lossy mediums in accordance with [1, 5, 7].

In fact, using pieces of winded cable, one can design an artificial medium with structural elements with electrical sizes larger than the electrical size defined by their external geometry. Obviously, such medium is described by constitutive parameters in a low frequency range where the true electrical size of structural elements is electrically small. Even if one does not know topology of the structural elements *a priori*, the true electrical size of structural elements is revealed through interaction of such medium with quasi-monochromatic radiation because of reflections in the cable. The fact of absence of losses in dispersive medium does not provide validity of constitutive parameters or the quasi-static EC approach in a high frequency range.

If reflection in the cable loaded by corresponding resistor is absent, then the stored energy in the cable is invisible on the macroscopic level and may be counted as a kind of the pro-thermal energy. That part of the microscopic field energy is inevitably dissipated while macroscopic electromagnetic properties of the medium do not depend on mechanisms of dissipation. As a result, the electrical size of the structural elements does not depend on the cable length.

In a number of cases, the energy stored by RLC circuits can be found through presenting the impedance of the circuit by a sum of the impedance of simple subcircuits analogously the methods of synthesis of lossless circuits. That was used in the Section 4 for the antennas with closely spaced resonances around antiresonance of the antenna. Using that example, it was shown that the energy of RLC circuits is defined by frequency derivative of admittance of the whole circuit with accuracy dependent on the quality factor of the circuit.

There is a particular example of a two-pole RLC circuit, which was used in [1, 7] to demonstrate impossibility of finding of the electromagnetic energy stored by the circuit through use of the input impedance. The circuit differs from the antiresonant subcircuit in Fig. 1 by an additional series resistance R_a in the branch with C_a so that $\sqrt{L_a/C_a} = R_a$. That implements critical damping in the circuit and provides constant value of the input impedance with zero reactance at least in some frequency range [20]. The quality factor of the circuit is small ($Q = 0.5$) as well as the ratio of the stored energy to resistance. If resistor in a resonant circuit with the quality factor $Q \gg 1$ had properties of such circuit the quality factor of the resonant circuit would change insignificantly $\Delta Q \approx 1$. Moreover, as in the case of resistor connected through use of the cable, the reactive energy of the critically damped RLC circuit should be classified as the pro-thermal energy. It means that macroscopic electromagnetic field does not depend on the energy of the microscopic electromagnetic field in reactive elements of the resistor subcircuit. Therefore, invisibility of the small part of the microscopic energy on the level of the macroscopic description of the field means absence of contribution of the invisible energy in the macroscopic field and macroscopic electromagnetic properties.

As was shown in Section 2, we can use Poynting's theorem not only in the case of dispersive medium without losses but also with losses, taking into account the dissipative part of the stored energy (7). One can measure the total monochromatic flux through the surface of sample volume and find the power of losses as a function of the flux. Using a quasi-monochromatic field, we can find variation of the stored energy provided by slow variation of the amplitude of the quasi-monochromatic field, taking into account the power of losses. In order to detect dispersive and dissipative parts of the stored energy separately, one can use their dependence from frequency.

7. CONCLUSION

In accordance with the causality principle [12], there is one-to-one association between microscopic structure of a medium and its (effective) constitutive parameters defined in all frequency range up to infinity. Such definition cannot be consistently deduced from Maxwell's equations because it has to reflect macroscopic properties of a particular medium. However, in order to define the energy density of the quasi-monochromatic macroscopic electric and magnetic fields in any dispersive lossy medium, a linear approximation of dependences $\omega\varepsilon(\omega)$, $\omega\mu(\omega)$ is sufficient. That allows general definition of the average macroscopic field energy density comprised of the dispersive and dissipative parts in accordance with Poynting's theorem in dispersive lossy medium discussed in Section 2.

Using the circuit model of a structural element of dispersive lossy medium corresponding to the circuit model of an electrically small antenna, effective constitutive parameters were described by generalized Lorentz-type dependences with frequency dependent parameters. The resonant model corresponding to the Lorentz-type dependences represents a particular case of generalized Lorentz-type dependences in the case of neglecting by antiresonant and radiative properties of structural elements.

It was shown that concept of electrically small structural element in the definition of constitutive parameters [12–14] may be refined. Accuracy of the definition of the average macroscopic field energy depends on the the quality factor of corresponding structural elements. According to the model, representation of resonant and antiresonant properties is necessary for an accurate description of properties of structural elements in accordance with the generalized Lorentz model.

Modelling of dispersive lossy medium by an ensemble of electrically small antennas in Sections 4 and 5 showed that constitutive parameters of the medium are defined by the admittance of electrically small antennas. The energy density of the electric (20) and magnetic (27), (30) fields in dispersive lossy medium comprised of electrically small structural elements with non-overlapping resonances $Q \gg 1$ was defined by functions of frequency derivatives of constitutive parameters in accordance with Poynting's theorem in dispersive lossy medium. The obtained energy densities of macroscopic electric and magnetic fields in dispersive lossy medium were described by similar types of relations, which, consistent with (9), generalise Brillouin's formula for the case of lossy dispersive medium.

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