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FD^{2}TD ANALYSIS OF ELECTROMAGNETIC FIELD PROPAGATION IN MULTIPOLE DEBYE MEDIA WITH AND WITHOUT CONVOLUTION

M. Feliziani[1, *], S. Cruciani[1], V. De Santis[2], and F. Maradei[3]

[1]Department of Electrical Engineering, University of L'Aquila, L'Aquila, Italy

[2]IT'IS Foundation, Zurich, Switzerland

[3]Department of Electrical Engineering, Sapienza University of Rome, Rome, Italy

Abstract—This paper deals with the time-domain numerical calculation of electromagnetic (EM) fields in linearly dispersive media described by multipole Debye model. The frequency-dependent finite-difference time-domain (FD^{2}TD) method is applied to solve Debye equations using convolution integrals or by direct integration. Original formulations of FD^{2}TD methods are proposed using different approaches. In the first approach based on the solution of convolution equations, the exponential analytical behavior of the convolution integrand permits an efficient recursive FD^{2}TD solution. In the second approach, derived by circuit theory, the transient equations are directly solved in time domain by the FD^{2}TD method. A comparative analysis of several FD^{2}TD methods in terms of stability, dispersion, computational time and memory is carried out.

1. INTRODUCTION

The electromagnetic (EM) field propagation through dispersive, or frequency-dependent, materials has been subject of numerical investigations since many years, especially using techniques based on the finite-difference time-domain (FDTD) scheme [1]. These techniques, known as frequency-dependent finite-difference time-domain (FD^{2}TD) methods, differ for the way they incorporate the dispersive media behavior in the numerical solution scheme.

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* Corresponding author: Mauro Feliziani (mauro.feliziani@univaq.it).

In the present work, original FD^{2}TD methods are proposed to analyze linear frequency-dispersive media described by multi-pole Debye dispersions which are representative of many biological tissues and many materials used for dielectric substrates in electronic devices. These new methods are able to solve time-domain equations in presence and in absence of convolution integrals.

The FD^{2}TD techniques based on the solution of convolution equations are known as recursive convolution (RC) methods since the convolution integral between the frequency-dependent susceptibility and the electric field is efficiently implemented in a recursive way due to the exponential nature of the susceptibility function exhibited by many dispersive materials [2–4]. Among the recursive convolution approaches, the piecewise linear recursive convolution method proposed by Kelley-Luebbers (here named as KL-PLRC) is considered very accurate [4]. Recently, a new FD^{2}TD RC method has been proposed to analyze losses into dielectric substrates [5] or biological tissues [6]. In this approach, the transient polarization current is analytically derived by the inverse Laplace Transform (LT) and directly implemented into Maxwell's curl equation. The resulting convolution integral in time domain is then solved by an efficient recursive scheme based on piecewise constant (PC) approximation of the electric field. This method is named as LT-PCRC and it is here improved by substituting the numerical integration proposed in [6] with a closed analytical formula. In addition, the use of a piecewise linear (PL) approximation instead of the piecewise constant approximation is presented for the first time in the new LT-PLRC method.

In order to avoid the solution of the convolution equations, alternative FD^{2}TD methods have been also proposed in the past [7–21]. They mainly rely on auxiliary differential equation (ADE) methods [7–12], or Z-Transform (ZT) methods [13–17]. In this paper, a new and simple FD^{2}TD method without convolution equations is proposed. It is derived from the circuit theory, since the transient polarization current density in a Debye medium is obtained by the analogy with the transient current flowing into a resistive-capacitive circuit, whose time-domain solution is available. This original method is named as CIRC.

In the following sections, first the mathematical formulations of the new proposed FD^{2}TD methods are provided. Then, a comparative analysis with popular FD^{2}TD methods is presented in terms of numerical stability, accuracy, computational time, and memory storage. Finally, for the sake of simplicity, one-dimensional (1D) applications are examined to verify the performances of the several FD^{2}TD methods. It should be noted that the analysis of the 1D dimensional propagation is completely adequate since all the interest is in the time integration rather than in the spatial solution.

2. MATHEMATICAL FORMULATION

Electromagnetic fields in frequency domain are described by Maxwell's curl equations:

$$\nabla \times \mathbf{E}(\omega) = -j\omega\mu\mathbf{H}(\omega) \tag{1a}$$

$$\nabla \times \mathbf{H}(\omega) = j\omega\varepsilon_0\hat{\varepsilon}_r(\omega)\mathbf{E}(\omega) \tag{1b}$$

where ω is the angular frequency; $\mathbf{E}(\omega)$ and $\mathbf{H}(\omega)$ are the time-harmonic electric and magnetic fields, respectively; μ is the permeability; σ_0 is the free space permittivity; $\hat{\varepsilon}_r(\omega)$ is the frequency-dependent complex relative permittivity. In case of a N_p-pole Debye dispersive media, $\hat{\varepsilon}_r(\omega)$ is given by:

$$\hat{\varepsilon}_r(\omega) = \varepsilon_\infty - j\frac{\sigma_0}{\omega\varepsilon_0} + \sum_{m=1}^{N_p}\frac{\Delta\varepsilon_m}{1+j\omega\tau_m} \tag{2}$$

where ε_∞ is the infinite relative permittivity obtained as $\omega\to\infty$ and σ_0 the conductivity obtained as $\omega\to 0$. $\Delta\varepsilon_m$ and τ_m are the change in relative permittivity and relaxation time of the mth pole, respectively, being $m = 1, 2, \ldots, N_p$. Equation (1b) can be then rewritten as:

$$\nabla\times\mathbf{H}(\omega) = \mathbf{J}(\omega) \tag{3}$$

where the total current density $\mathbf{J}(\omega)$ is given in frequency domain by:

$$\mathbf{J}(\omega) = \mathbf{J}_\sigma(\omega) + \mathbf{J}_\varepsilon(\omega) + \sum_{m=1}^{N_p}\mathbf{J}_{p,m}(\omega) \tag{4}$$

with

$$\mathbf{J}_\sigma(\omega) = \sigma_0\mathbf{E}(\omega) \tag{5a}$$

$$\mathbf{J}_\varepsilon(\omega) = j\omega\varepsilon_0\varepsilon_\infty\mathbf{E}(\omega) \tag{5b}$$

$$\mathbf{J}_{p,m}(\omega) = j\omega\varepsilon_0\frac{\Delta\varepsilon_m}{1+j\omega\tau_m}\mathbf{E}(\omega) \tag{5c}$$

In (5), $\mathbf{J}_\sigma(\omega)$ is the conductive current density, $\mathbf{J}_\varepsilon(\omega)$ the displacement current density, and $\mathbf{J}_{p,m}(\omega)$ the polarization current density of the mth pole. Moving (1) into the time domain via (3)–(5) it yields:

$$\nabla \times \mathbf{E}(t) = -\mu\frac{\partial\mathbf{H}(t)}{\partial t} \tag{6a}$$

$$\nabla \times \mathbf{H}(t) = \sigma_0 \mathbf{E}(t) + \varepsilon_0 \varepsilon_\infty \frac{\partial \mathbf{E}(t)}{\partial t} + \sum_{m=1}^{N_p} \mathbf{J}_{p,m}(t) \tag{6b}$$

Equation system (6) can be numerically solved using finite differences in time and space. Without any loss of generality, let us consider a 1D problem with wave propagation along x-axis ($\mathbf{E} = E\hat{\mathbf{y}}, \mathbf{H} = H\hat{\mathbf{z}}, \mathbf{J} = J\hat{\mathbf{y}}$). Adopting the *leap-frog* scheme, (6a) is solved at the time instant $t = n\Delta t$ and at point $x = (i + 1/2)\Delta x$, while (6b) is solved at the time instant $t = (n + 1/2)\Delta t$ and at point $x = i\Delta x$, being Δt the constant time step, n the time iteration number, Δx the space interval time and i the spatial number. Then, using the Yee's notation [1], the discretized system has the following form:

$$\frac{E^n(i+1) - E^n(i)}{\Delta x} = -\mu \frac{H^{n+1/2}(i+1/2) - H^{n-1/2}(i+1/2)}{\Delta t} \tag{7a}$$

$$\frac{H^{n+1/2}(i+1/2) - H^{n+1/2}(i-1/2)}{\Delta x} = \sigma_0 \frac{E^{n+1}(i) + E^n(i)}{2} + \varepsilon_0 \varepsilon_\infty \frac{E^{n+1}(i) - E^n(i)}{\Delta t} + \sum_{m=1}^{N_p} J_{p,m}^{n+1/2}(i) \tag{7b}$$

The differences in the various $\mathrm{FD^2TD}$ methods are mainly related to the different way of dealing with the polarization current densities $\mathbf{J_{p,m}}$, as described in the following.

2.1. Convolution Based Models

The most popular recursive convolution method is based on the solution of (6b) via a direct time domain solution of (5c):

$$J_{p,m}(t) = \varepsilon_0 \frac{\partial}{\partial t} \left[\chi_m(t) * E(t) \right] \tag{8}$$

where $\chi_m(t) = L^{-1}[\Delta\varepsilon_m/(1+s\tau_m)]$ is the transient susceptibility of the mth pole, s the Laplace variable, L^{-1} the inverse Laplace transform, and $*$ denotes the convolution operator. Then, the convolution Equation (8) is efficiently solved by recursive calculations saving computational time and memory [2–4]. One of the most efficient versions of this method is the KL-PLRC approach described in [4], which is based on a piecewise linear approximation of the electric field.

A different but still efficient RC method has been recently proposed in [5, 6]. It is based on both recursive convolution and analytical Laplace Transform of the integrand. In this method the polarization current density (5c) is expressed in the Laplace domain as:

$$J_{p,m}(s) = y_m(s)E(s) \tag{9}$$

where $y_m(s) = s\varepsilon_0\Delta\varepsilon_m/(1 + s\tau_m)$ is the admittance of the mth pole. Equation (9) is then transformed in time domain as:

$$J_{p,m}(t) = y_m(t) * E(t) \tag{10}$$

The transient admittance $y_m(t)$ is analytically given by:

$$y_m\left(t\right) = L^{-1}\left[y_m\left(s\right)\right] = \frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\left[\delta\left(t\right) - \frac{1}{\tau_m}e^{-\frac{t}{\tau_m}}\right] \tag{11}$$

being δ the Dirac function. Then, the transient polarization current density $J_{p,m}(t)$ in (10) can be expressed as:

$$J_{p,m}(t) = \frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}E(t) - \frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m^2}\int_0^t e^{-(t-t')/\tau_m}E(t')dt' \tag{12}$$

or, in the 1D discretized form, as

$$J_{p,m}^{n+1/2}\left(i\right) = \frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\frac{E^{n+1}\left(i\right) + E^{n}\left(i\right)}{2} - \xi_m^{n+1/2}\left(i\right) \tag{13}$$

where $\xi_m^{n+1/2}(i)$ is the convolution integral term at the time instant $t = (n + 1/2)\Delta t$ given by:

$$\xi_m^{n+1/2}\left(i\right) = \frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m^2}\int_0^{(n+1/2)\Delta t} e^{-[(n+1/2)\Delta t-t']/\tau_m}E\left(t'\right)dt' \tag{14}$$

Due to the exponential form of the kernel, the convolution integral (14) is solved recursively as:

$$\begin{aligned}\xi_m^{n+1/2}(i) = {} & e^{-\frac{\Delta t}{\tau_m}}\xi_m^{n-1/2}(i) \\ & + \frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m^2}\int_{(n-1/2)\Delta t}^{(n+1/2)\Delta t} e^{-[(n+1/2)\Delta t-t']/\tau_m}E\left(t'\right)dt'\end{aligned} \tag{15}$$

Both piecewise constant and linear approximations can be adopted for solving (15), as done by the LT-PCRC and the LT-PLRC methods, respectively. In this paper, an improved formulation of the LT-PCRC method is provided, while the LT-PLRC method is proposed for the first time.

2.1.1. LT-PCRC FD²TD Method

Assuming the electric field to be constant during the time interval Δt, a piecewise constant recursive convolution integration scheme can be adopted, and (15) becomes:

$$\xi_m^{n+1/2}(i) = e^{-\frac{\Delta t}{\tau_m}} \xi_m^{n-1/2}(i) + \frac{\varepsilon_0 \Delta\varepsilon_m}{\tau_m}\left(1 - e^{-\frac{\Delta t}{\tau_m}}\right) E^n(i) \tag{16}$$

It should be noted that (16) differs from the formulas presented in [5, 6] for the analytical integration of the exponential term. Via (13) and (16), the equation system (7) can be updated by the following iterative equations:

$$H^{n+1/2}(i+1/2) = H^{n-1/2}(i+1/2) - \frac{\Delta t}{\mu}\frac{E^n(i+1) - E^n(i)}{\Delta x} \tag{17}$$

$$\begin{aligned} E^{n+1}(i) = &\left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{2\tau_m}\right)^{-1} \\ &\left[E^n(i)\left(-\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} - \sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{2\tau_m}\right)\right. \\ &\left.+\sum_{m=1}^{N_p}\xi_m^{n+1/2}(i) + \frac{H^{n+1/2}(i+1/2) - H^{n+1/2}(i-1/2)}{\Delta x}\right] \end{aligned} \tag{18}$$

Equations (16)–(18) represent a 1D FD²TD solution of the multipole Debye problem using the LT-PCRC approach. The main advantage of the proposed method is based on the definition of the specific admittance (11) that, in case of a Debye medium, is analytically known leading to an expression of the transient polarization current density (12) without any time derivative. Note that the main difference between the Kelley-Luebbers formulation [4] and the Laplace transform approach is that the former is based on (8) while the latter is based on (10). Therefore, the LT formulation avoids the numerical discretization of the time derivative in (8).

2.1.2. LT-PLRC FD²TD Method

Equation (15) can be solved also using a piecewise linear approximation for the electric field during the time interval Δt. The convolution term ξ_m in (13) must be calculated at time instant $(n+1/2)\Delta t$. Therefore, it can be approximated by the average value between two instants as:

$$\xi_m^{n+1/2}(i) = \frac{\xi_m^n(i) + \xi_m^{n+1}(i)}{2} \tag{19}$$

where [4]

$$\xi_m^n(i) = \sum_{q=0}^{n-1} \left[E^{n-q}(i)\vartheta_m^q + \left(E^{n-q-1}(i) - E^{n-q}(i)\right)\zeta_m^q\right] \tag{20}$$

being:

$$\vartheta_m^q = \frac{\varepsilon_0 \Delta\varepsilon_m}{\tau_m^2 \Delta t} \int_{q\Delta t}^{(q+1)\Delta t} e^{-t'/\tau_m} dt' = \frac{\varepsilon_0 \Delta\varepsilon_m}{\tau_m \Delta t} \left(1 - e^{-\Delta t/\tau_m}\right) e^{-q\Delta t/\tau_m} \tag{21}$$

$$\begin{aligned} \zeta_m^q &= \frac{\varepsilon_0 \Delta\varepsilon_m}{\tau_m^2} \frac{1}{\Delta t} \int_{q\Delta t}^{(q+1)\Delta t} \left(t' - q\Delta t\right) e^{-t'/\tau_m} dt' \\ &= \frac{\varepsilon_0 \Delta\varepsilon_m}{\tau_m \Delta t} \left[\left(1 - e^{-\Delta t/\tau_m}\right)\tau_m - e^{-\Delta t/\tau_m}\Delta t\right] e^{-q\Delta t/\tau_m} \end{aligned} \tag{22}$$

Due to their exponential form, (20)–(22) can be solved recursively as in [4]:

$$\xi_m^{n+1}(i) = e^{-\Delta t/\tau_m} \xi_m^n(i) + \left(\vartheta_m^0 - \zeta_m^0\right) E^{n+1}(i) + \zeta_m^0 E^n(i) \tag{23}$$

being $\vartheta_m^{q+1} = e^{-\Delta t/\tau_m}\vartheta_m^q$ and $\varsigma_m^{q+1} = e^{-\Delta t/\tau_m}\varsigma_m^q$. With some manipulations, (19) can be rearranged as:

$$\xi_m^{n+1/2}(i) = \left(\frac{1 + e^{-\Delta t/\tau_m}}{2}\right)\xi_m^n(i) + \left(\frac{\vartheta_m^0 - \zeta_m^0}{2}\right)E^{n+1}(i) + \left(\frac{\zeta_m^0}{2}\right)E^n(i) \tag{24}$$

and the updating equation for the electric field is finally given by:

$$\begin{aligned} E^{n+1}(i) = & \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{1}{2}\sum_{m=1}^{N_p} \frac{\varepsilon_0 \Delta\varepsilon_m}{\tau_m}\left(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}\right)\right)^{-1} \\ & \cdot \left[\sum_{m=1}^{N_p} \frac{1 + e^{-\Delta t/\tau_m}}{2}\xi_m^n(i) + \left(-\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t}\right.\right. \\ & \left. - \frac{1}{2}\sum_{m=1}^{N_p} \frac{\varepsilon_0 \Delta\varepsilon_m}{\tau_m}\left(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}\right)\right) E^n(i) \\ & \left. + \frac{H^{n+1/2}(i+1/2) - H^{n+1/2}(i-1/2)}{\Delta x}\right] \end{aligned} \tag{25}$$

2.2. FD^{2}TD Method without Using Convolution Equations

An alternative way to solve the field propagation in Debye media is based on the direct solution of time domain equations without

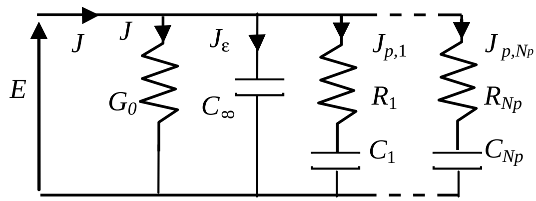


Figure 1. Circuit equivalent to a multipole Debye dispersion model.

solving convolution integrals. The original method here proposed is derived from the circuit theory and for this reason it is named in the following as CIRC method. According to the circuit theory the current density in (4)–(5) can be seen as the current flowing in the equivalent electrical circuit shown in Fig. 1 composed by linear time invariant (LTI) parameters.

The total current density $J(\omega)$ can be obtained through the analysis of the LTI electrical circuit, whose electric parameters are given by:

$$G_0 = \sigma_0, \quad C_\infty = \varepsilon_0 \varepsilon_\infty, \quad R_m = \tau_m / (\varepsilon_0 \Delta\varepsilon_m), \quad C_m = \varepsilon_0 \Delta\varepsilon_m \tag{26}$$

From a dimensional point of view, the electrical parameters of the equivalent circuit are per unit length (p.u.l), i.e., G_0 [S/m], C_∞ and C_m [F/m], R_m [Ω/m]. The total current density $J(t)$ in time domain is expressed as sum of various terms as:

$$J(t) = J_\sigma(t) + J_\varepsilon(t) + \sum_{m=1}^{N_p} J_{p,m}(t) \tag{27}$$

where

$$J_\sigma(t) = G_0 E(t) \tag{28a}$$

$$J_\varepsilon(t) = C_\infty \frac{\partial E(t)}{\partial t} \tag{28b}$$

The transient currents in G_0 and C_∞ are easily calculated with the FDTD method, while the transient polarization current density $J_{p,m}(t)$ can be obtained through the direct time-domain analysis of the LTI circuit.

From the circuit theory, it is well known that the constitutive equation between $E(t)$ and $J_{p,m}(t)$ of a simple resistor-capacitor circuit is given by:

$$E(t) = R_m J_{p,m}(t) + \frac{1}{C_m} \int_{-\infty}^{t} J_{p,m}(t) dt \tag{29}$$

Applying the time derivative to (29), it yields:

$$\frac{dE(t)}{dt} = R_m \frac{dJ_{p,m}(t)}{dt} + \frac{J_{p,m}(t)}{C_m} \tag{30}$$

Equation (30) can be solved numerically via finite differences at time $t = (n + 1/2)\Delta t$, and the following discretized form holds:

$$\frac{E^{n+1}(i) - E^{n}(i)}{\Delta t} = R_m \frac{J_{p,m}^{n+1/2}(i) - J_{p,m}^{n-1/2}(i)}{\Delta t} + \frac{J_{p,m}^{n+1/2}(i)}{C_m} \tag{31}$$

It should be noted that due to the different time instants at which the electric field and the current density are staggered, central and forward differences are respectively adopted to discretize the two time derivatives in (31). Rearranging these terms, the current density is given by:

$$J_{p,m}^{n+1/2}(i) = \frac{R_m C_m}{R_m C_m + \Delta t}\left[J_{p,m}^{n-1/2}(i) + \frac{E^{n+1}(i) - E^{n}(i)}{R_m}\right] \tag{32}$$

The updating equation for the electric field is then obtained in terms of circuit parameters as:

$$\begin{aligned} &E^{n+1}(i) \\ &= \left(\frac{G_0}{2} + \frac{C_\infty}{\Delta t} + \sum_{m=1}^{N_p} \frac{C_m}{R_m C_m + \Delta t}\right)^{-1} \left[\left(-\frac{G_0}{2} + \frac{C_\infty}{\Delta t} + \sum_{m=1}^{N_p} \frac{C_m}{R_m C_m + \Delta t}\right) E^{n}(i)\right. \\ &\quad \left. - \sum_{m=1}^{N_p} \frac{R_m C_m}{R_m C_m + \Delta t} J_{p,m}^{n-1/2}(i) + \frac{H^{n+1/2}(i+1/2) - H^{n+1/2}(i-1/2)}{\Delta x}\right] \end{aligned} \tag{33}$$

Equations (17), (32) and (33) represent the FD^{2}TD solution of the multipole Debye problem with the CIRC approach.

3. COMPARATIVE ANALYSIS

In this Section, a comparative analysis in terms of numerical stability, accuracy, and computational requirements is provided. Specifically, the LT-PCRC, LT-PLRC and CIRC methods are compared with other popular and similar methods (i.e., KL-PLRC [4], ADE-1 [11], and ADE-2 [10]). It should be noted that an analogous comparative analysis can be found in [24] but for the only FD^{2}TD methods available at that time. Moreover, the choice of the above selected methods was inevitably dictated by the resemblance of KL-PLRC with LT-PCRC and LT-PLRC, and of ADE-1 and ADE-2 with CIRC. The only ZT method has been excluded from this comparison due to its different mathematical formulation. Details on the ZT method can be found elsewhere [13–17].

3.1. Numerical Stability

The stability of a finite difference scheme can be derived in several ways [25–28]. In this paper, the same approach adopted in [27] is used and the von Neumann stability analysis based on the Fourier transform is carried out. Since we only deal with linear models, we can assume that the scheme handles a single (vector valued) variable $\mathbf{U}$ with spatial dependence $\mathbf{U}^n(i) = \mathbf{U}^n e^{jki\Delta x}$ where k is the wavenumber. The scheme is then described as $\mathbf{U}^{n+1} = \mathbf{G}\mathbf{U}^n$ where $\mathbf{G}$ is known as amplification matrix. A necessary stability criterion is related to the boundedness of this system, which yields to the following condition

$$\lambda_{\max} = \max\left(|\lambda_1(\mathbf{G})|, \ldots |\lambda_i(\mathbf{G})|, \ldots, |\lambda_M(\mathbf{G})|\right) \leq 1 \qquad (34)$$

being λ_i the ith eigenvalue of $\mathbf{G}$, and M the number of independent variables.

The goal of our investigation is to verify the stability performance of the methods here proposed compared to those already known in literature (i.e., KL-PLRC and ADE). To this aim, the amplification matrices and the eigenvalues must be evaluated. While the former have been analytically derived as reported in the Appendix A, the latter resulted prohibitive to be provided in an analytical closed form. Therefore, condition (34) was numerically verified when varying the media parameters and the temporal-spatial discretization within a range of significant values. The rationale behind the choice of these parameters was dictated by common practice and previous experience on the stability of FDTD schemes. Indeed, it is well-known that methods based on the leap-frog scheme must satisfy the Courant-Friedrichs-Lewy (CFL) stability condition given for a dispersive material by:

$$\kappa = c_\infty \Delta t / \Delta x \leq 1 \qquad (35)$$

where $c_\infty = 1/\sqrt{\mu\varepsilon_0\varepsilon_\infty}$. Also, the analysis conducted in [28] has shown that the stability of some ADE schemes is affected by the minimum relaxation time, i.e., $\tau_0 = \min(\tau_{1,}\tau_{2,\ldots,}\tau_{Np})$. To be consistent, the single-pole Debye parameters ε_∞, $\Delta\varepsilon$, σ_0, together with the parameters κ and $\tau_0/\Delta t$ have been selected for the sensitivity analysis and varied within the following ranges: $\varepsilon_\infty = 1$–1000, $\Delta\varepsilon = 1$–1000, $\sigma_0 = 0.001$–1, $\kappa = 0.9$–1.1, $\tau_0/\Delta t = 0.1$–1. Among these parameters, only $\tau_0/\Delta t$ and κ affect the stability of the several methods and the behavior of $\lambda_{\max}$ as function of these two parameters is shown in Fig. 2. As can be observed, all the considered FD^{2}TD methods are stable (i.e., $\lambda_{\max} \leq 1$) when $\kappa \leq 1$, with the exclusion of the ADE-2 method that must satisfy also the condition $\tau_0/\Delta t \geq 0.5$. It means that the CFL condition (i.e., $\kappa \leq 1$) does not guarantee the stability for the ADE-2 method.

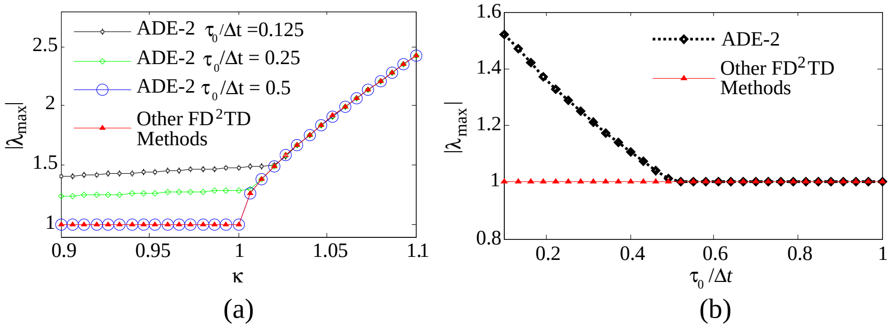


Figure 2. Influence of the Debye and discretization parameters on the numerical stability. (a) Effect of the κ factor on $\lambda_{\max}$. (b) Effect of $\tau_0/\Delta t$ on $\lambda_{\max}$.

This last method therefore can be assumed less robust than the other considered FD^{2}TD methods that are stable for $\kappa \leq 1$.

3.2. Numerical Dispersion

To quantify the accuracy of the aforementioned schemes, a rigorous analysis of the numerical dispersion errors should be considered. Let us consider a plane wave propagating in the x direction and in a homogeneous open domain, as described in [23, 24]. Then, the electric and magnetic fields are assumed to be given by:

$$E(x,t) = E(k,\omega)e^{j(\omega t - kx)} \tag{36a}$$

$$H(x,t) = H(k,\omega)e^{j(\omega t - kx)} \tag{36b}$$

In a Yee discretized 1D space, Maxwell's curl equations can be rewritten as [24]:

$$j\Omega\varepsilon_0\tilde{\varepsilon}_r E(k,\omega) = -j\tilde{K}H(k,\omega) \tag{37a}$$

$$j\Omega\mu H(k,\omega) = j\tilde{K}E(k,\omega) \tag{37b}$$

where $\tilde{\varepsilon}_r$ is the numeric relative complex permittivity and

$$\Omega = (2/\Delta t)\sin(\omega\Delta t/2) \tag{38a}$$

$$\tilde{K} = (2/\Delta x)\sin(k\Delta x/2) \tag{38b}$$

In the following, the numeric relative complex permittivity is initially provided only for new proposed formulations. Then, a comparison with the dispersion errors found in [23, 24] for the other FD^{2}TD methods is outlined. It should be noted that $\tilde{\varepsilon}_r$ of such methods was derived for a single pole Debye medium when neglecting the static conductivity σ_0 [23], and therefore the same assumptions are here adopted for a better comparison.

3.2.1. LT-PCRC Method

The equation for dispersion analysis of the LT-PCRC method is obtained by discretizing the convolution integral (14) without using the recursive algorithm, while adopting a piecewise constant approximation for the electric field. Therefore, via (6b) and (13)–(14) it yields:

$$\varepsilon_0\left[\varepsilon_\infty\frac{E^{n+1}-E^n}{\Delta t}+\frac{\Delta\varepsilon}{\tau_0}\frac{E^{n+1}+E^n}{2}\right.$$
$$\left.-\frac{\Delta\varepsilon}{\tau_0}\left(1-e^{-\Delta t/\tau_0}\right)\sum_{q=0}^{n}e^{-q\Delta t/\tau_0}E^{n-q}\right]=\nabla\times H^{n+1/2} \quad (39)$$

Using (38b), the curl of H at $x = i\Delta x$ and $t = (n + 1/2)\Delta t$ is given by:

$$\nabla\times H^{n+1/2}=-H(k,\omega)e^{j(\omega(n+1/2)\Delta t-ki\Delta x)}\left(\frac{e^{jk\Delta x/2}-e^{-jk\Delta x/2}}{\Delta x}\right) \quad (40)$$

Introducing (36) and (40) in (39), it yields:

$$\varepsilon_0E(k,\omega)e^{-jki\Delta x}\left[\varepsilon_\infty\frac{e^{j\omega(n+1)\Delta t}-e^{j\omega n\Delta t}}{\Delta t}+\frac{\Delta\varepsilon}{\tau_0}\frac{e^{j\omega(n+1)\Delta t}+e^{j\omega n\Delta t}}{2}\right.$$
$$\left.-\frac{\Delta\varepsilon}{\tau_0}\left(1-e^{-\Delta t/\tau_0}\right)\sum_{q=0}^{n}e^{-q\Delta t/\tau_0}e^{j\omega(n-q)\Delta t}\right]$$
$$=-j\tilde{K}H(k,\omega)e^{j\omega(n+1/2)\Delta t}e^{-jki\Delta x} \quad (41)$$

After some mathematic manipulations, the following equation is obtained:

$$\varepsilon_0E(k,\omega)\left[j\Omega\varepsilon_\infty+\frac{\Delta\varepsilon}{\tau_0}\Lambda-\frac{\Delta\varepsilon}{\tau_0}\left(1-e^{-\Delta t/\tau_0}\right)e^{-j\omega\Delta t/2}\sum_{q=0}^{n}e^{-q\Delta t(1+j\omega\tau_0)/\tau_0}\right]$$
$$=-j\tilde{K}H(k,\omega) \quad (42)$$

where $\Lambda = \cos(\omega\Delta t/2)$. By adopting the position $z = \exp\{-\Delta t(1 + j\omega\tau_0)/\tau_0\}$, and being $\lim\limits_{n\to\infty}\sum\limits_{q=0}^{n} z^q = (1-z)^{-1}(1-z^n) \approx (1-z)^{-1}$ then

Equation (42) can be rewritten as:

$$\varepsilon_0 E(k,\omega)\left[j\Omega\varepsilon_\infty+\frac{\Delta\varepsilon}{\tau_0}\Lambda-\frac{\Delta\varepsilon}{\tau_0}\frac{\left(1-e^{-\Delta t/\tau_0}\right)e^{-j\omega\Delta t/2}}{1-e^{-\Delta t(1+j\omega\tau_0)/\tau_0}}\right]=-j\tilde{K}H(k,\omega) \quad (43)$$

The numeric relative complex permittivity $\tilde{\varepsilon}_r$ is finally obtained by comparing (37a) and (43) as:

$$\tilde{\varepsilon}_r=\varepsilon_\infty+\frac{\Delta\varepsilon}{j\Omega\tau_0}\left[\Lambda-\frac{\sinh\left(\Delta t/(2\tau_0)\right)}{\sinh\left(\Delta t(1+j\omega\tau_0)/(2\tau_0)\right)}\right] \quad (44)$$

3.2.2. LT-PLRC Method

Following the same procedure as described above, the numeric relative complex permittivity for the LT-PLRC method can be written as

$$\tilde{\varepsilon}_r=\varepsilon_\infty+\frac{\Delta\varepsilon}{j\Omega\tau_0^2}e^{-j\omega\frac{\Delta t}{2}}\left(\frac{\alpha\beta+\gamma}{2}\right)\left[1+\frac{\cosh(\Delta t/(2\tau_0))\,e^{-j\omega\frac{\Delta t}{2}}}{\sinh(\Delta t(1+j\omega\tau_0)/(2\tau_0))}\right]+\frac{\Delta\varepsilon}{j\Omega\tau_0}-\frac{\Delta\varepsilon}{j\Omega\tau_0^2}\beta\Lambda \quad (45)$$

where $\alpha = \exp\{-\Delta t/\tau_0\}$, $\beta = [\Delta t - \tau_0(1-\alpha)]\tau_0/\Delta t$ and $\gamma = [\tau_0(1-\alpha)-\alpha\Delta t]\tau_0/\Delta t$.

3.2.3. CIRC Method

After some mathematical manipulations and following the same procedure described above, the numerical relative permittivity for the CIRC method is obtained via (26) as:

$$\tilde{\varepsilon}_r=\varepsilon_\infty+\Delta\varepsilon e^{j\omega\frac{\Delta t}{2}}\Big/\left(j\Omega\tau_0+e^{j\omega\frac{\Delta t}{2}}\right) \quad (46)$$

3.2.4. Comparison

To compare the dispersion errors of the several methods, it is convenient to define the relative error functions as in [24]:

$$err^{real} = |\mathrm{Re}\{k-k^{num}\}/\mathrm{Re}\{k\}| \quad (47a)$$

$$err^{imag} = |\mathrm{Im}\{k-k^{num}\}/\mathrm{Im}\{k\}| \quad (47b)$$

where $k = \omega\sqrt{\mu\varepsilon_0\varepsilon_r}$ and k^{num} is the numerical solution of the dispersion equation given for a 1D propagation by [23, 24]:

$$k^{num}=\frac{2}{\Delta x}\sin^{-1}\left[\frac{\Delta x\sqrt{\tilde{\varepsilon}_r}}{c\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right)\right] \quad (48)$$

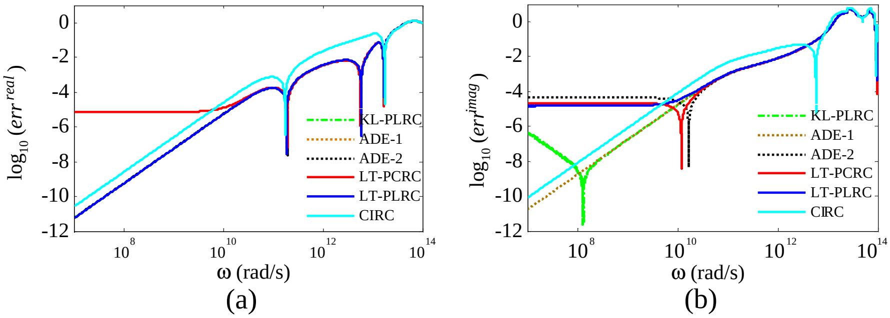


Figure 3. The logarithm of the (a) phase error err^{real} and (b) attenuation error err^{imag} as a function of angular frequency.

Under the assumption that the time-harmonic waves are of the form $\exp\{j(\omega t - kx)\}$, it is evident that err^{real} is a measure of the numerical phase error and err^{imag} is a measure of the numerical attenuation error.

To have an idea of the discrepancies among the different FD^{2}TD methods considered here, the same 1D test case adopted in [24] is examined. The numerical phase and attenuation errors are reported in Figs. 3(a) and (b) for a material characterized by a single-pole Debye dispersion relation with $\varepsilon_\infty = 1.8$, $\Delta\varepsilon = 79.2$, $\tau_0 = 9.4\,\text{ps}$ and assuming the discretization parameters equal to $\Delta x = 37.5\,\mu\text{m}$ and $\Delta t = 2.5\,\text{fs}$.

From these plots it is noted that the numerical phase error of the LT-PCRC method is higher than that of the other FD^{2}TD methods at lower frequencies, even if the absolute error is in any case very low ($\sim 10^{-5}$), at least for the considered material. As regards the attenuation error, ADE-1 and CIRC methods are very efficient in the whole frequency spectrum, while the convolution-based methods (KL-PLRC, LT-PCRC and LT-PLRC) and the ADE-2 method present higher error at lower frequencies, although the absolute value is once again in the 10^{-4}–10^{-5} order of magnitude.

3.3. Computational Requirement

The computational requirements of a numerical procedure depend essentially on the number of algebraical operations and on the memory requirements. For the sake of clarity, let us examine the LT-PCRC equations (see Appendix B) to evaluate the electric field at each point $x = i\Delta x$ and at any time instant $n\Delta t$. Equation (B4a) must be computed by a number of multiplications (M) equal to $M = 2$ and additions (A) equal to $A = 1$ for each Debye pole, while solution of

(B4b) requires $M = 2$ and $A = 2$ plus $A = N_p$ operations to calculate the sum of the convolution term, i.e., $\Sigma\xi_m$. Assuming N_x as the total number of the spatial points in the x-direction and N_t the total number of the time steps, the number of algebraical operations of the LT-PCRC method is equal to $[(2A+2M)\ N_p+2A+2M]\ N_tN_x$. Following the same procedure described above, it is possible to calculate the number of algebraical operations for the different FD^{2}TD methods under investigation. The result is summarized in Table 1, where the different methods are listed in order of increasing cost. As can observed, the difference in the computational cost among the several methods increases as the number of poles N_p. The magnetic field calculation by (17) is equal for all the considered FD^{2}TD methods and therefore it is omitted in this method comparison.

In Table 1 it is also reported the memory requirement for each time iteration, which is almost the same for all considered FD^{2}TD methods. The numerical cost for three-dimensional (3D) configurations (not reported here for the sake of brevity) is analogous to that of the 1D case.

4. RESULTS AND DISCUSSION

Simple numerical test-cases are analyzed to verify the performances of the different FD^{2}TD methods. For all the considered applications, the canonical 1D configuration (plane wave propagation along x-axis, $\mathbf{E} = E_y\hat{\mathbf{y}}$, and $\mathbf{H} = H_z\hat{\mathbf{z}}$) is intentionally chosen for the availability of the analytical solution in frequency domain. In the first application a homogeneous single-pole Debye media is considered. The waveform of the incident electric field $E^i(t)$ source, located in $x = 0$, is described by a monopole Gaussian pulse $E^i(t) = \exp\{-(t/t_0 - 4)^2\}$ where $t_0 =$ 62.5 ps. The calculation parameters used for the FD^{2}TD analysis are: $\Delta x = \min(\lambda'/10,\ \delta/8)$, $\Delta t = \Delta x/(2c_\infty)$, being δ the penetration

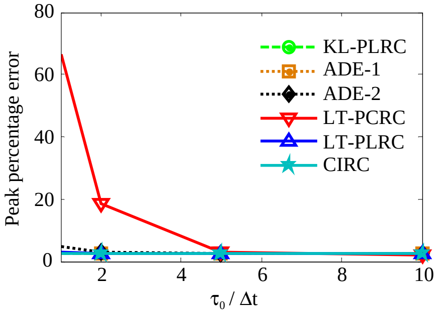
Table 1. Computational Requirements of different 1D FD^{2}TD methods.

Method	*Number of algebraical operations*	*Memory storage*
LT-PCRC	$[(2A+2M)\ N_p+\ 2A+2M]\ N_tN_x$	$(2+N_p\ )\ N_x$
ADE-2	$[(2A+2M)\ N_p+\ 2A+2M]\ N_tN_x$	$(2+N_p\ )\ N_x$
ADE-1	$[(2A+2M)\ N_p+$ 3A $+2M]\ N_tN_x$	$(2+N_p\ )\ N_x+1$
CIRC	$[(2A+2M)\ N_p+$ 3A $+2M]\ N_tN_x$	$(2+N_p)N_x+1$
LT-PLRC	[(3A + 3M) $N_p+\ 2A+2M]\ N_tN_x$	$(2+N_p\ )\ N_x+1$
KL-PLRC	[(3A + 3M) $N_p+\ 2A+2M]\ N_tN_x$	$(2+N_p\ )\ N_x+1$

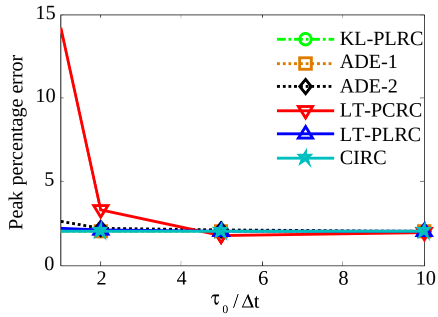
depth, $A' = 0.001$ the required attenuation of the excitation pulse, $\lambda = c'/f_{\max}$, $c' = 1/\sqrt{\mu_0\varepsilon_0\varepsilon'_r}$, $f_{\max} = \sqrt{-\log(A')}/(\pi t_0)$, $\varepsilon'_r = \varepsilon_\infty + \Sigma_m \Delta\varepsilon_m/(1 + (2\pi f_{\max}\tau_m)^2)$. The numerical transient computation is terminated before than the EM pulse reaches the boundary extremities of the considered domain in order to avoid any reflection which could affect the benchmark validity among the considered $\mathrm{FD^2TD}$ methods.

The influence of the ratio $\tau_0/\Delta t$ has been evaluated for two Debye media named as material #1 (single-pole with $\varepsilon_\infty = 4$, $\Delta\varepsilon = 28$, $\sigma_0 = 0.01\,\mathrm{S/m}$ and $\tau_0 = 0.7\,\mathrm{ps}$) and material # 2 (single-pole with $\varepsilon_\infty = 4$, $\Delta\varepsilon = 2$, $\sigma_0 = 0.01\,\mathrm{S/m}$ and $\tau_0 = 0.7\,\mathrm{ps}$). These two ideal materials have been chosen since it is possible to obtain values of Δt comparable with τ_0 while for typical Debye media used in practical applications the time step Δt imposed by the *CFL* condition is usually much lower than the minimum relaxation time τ_0. The error calculated at $x = 0.1\,\mathrm{m}$ with different $\mathrm{FD^2TD}$ methods is plotted as function of the ratio $\tau_0/\Delta t$ in Fig. 4 for both materials. Specifically, the error is referred to the peak value of the transient "exact" solution $E(x,t)$ obtained by inverse transform of the analytical solution $E(x, \omega)$ in the frequency domain, as described in [23]. From these figures it is noted that the LT-PCRC methods is strongly sensitive to the ratio $\tau_0/\Delta t$. However, good accuracy is achieved for all the considered $\mathrm{FD^2TD}$ methods when the ratio $\tau_0/\Delta t > 5$.

Finally, the accuracy and the computational efficiency of the proposed $\mathrm{FD^2TD}$ methods is also analyzed for a multipole Debye medium (material #3: $\varepsilon_\infty = 3.92$, $\Delta\varepsilon_1 = 83.65$, $\Delta\varepsilon_2 = 2.77$, $\sigma_0 = 0\,\mathrm{S/m}$, $\tau_1 = 17.67\,\mathrm{ps}$ and $\tau_2 = 0.9\,\mathrm{ps}$) using the slab configuration of Fig. 5. The cell size is $\Delta x = 50\,\mu\mathrm{m}$ while the time step is $\Delta t = 0.166\,\mathrm{ps}$. The incident electric field is modelled by a soft source located in free space in position $x = 0$ and given by $E^i(t) = A\exp\{-(t/t_0 - 4)^2\}$



(a)



(b)

Figure 4. Percentage error in the peak value for: (a) material #1, (b) material #2.

with $t_0 = 6.024\,\text{ps}$ and $A = 2$. The Debye slab of material #3 is placed in cells 301–1200, while free space occupies cells numbers 1–300 and 1201–1500. Mur's absorbing conditions [29] are used at both the boundaries of the domain.

The value of the electric fied inside the slab for the proposed methods is compared with the analytical solution as shown in Fig. 6(a) at the time $t = 500\Delta t$ and in Fig. 6(b) at $t = 1500\Delta t$.

The frequency domain analytical solution of the wavenumber $k(\omega) = \omega\sqrt{\mu\varepsilon_0\varepsilon_r(\omega)}$ is compared with that calculated numerically [16] by:

$$k_{FDTD} = -\frac{1}{j(x_2 - x_1)} ln\left(\frac{E_{FFT}(\omega, x_2)}{E_{FFT}(\omega, x_1)}\right) \tag{49}$$

where x_1 and x_2 are two points located inside the Debye slab, E_{FFT} $(\omega,\, x_i)$ is the FFT transform of the electric field $E_z(t)$ at position x_i for $i = 1, 2$. As example, assuming $x_1 = 310\Delta x$ and $x_2 = 315\Delta x$, the trends of k_{FDTD} is shown in Fig. 7 for the proposed FD^{2}TD methods.

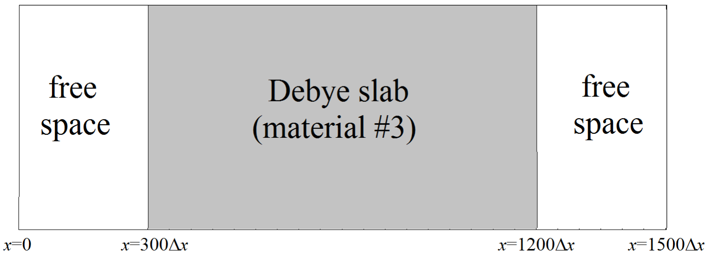


Figure 5. Test configuration for the material #3.

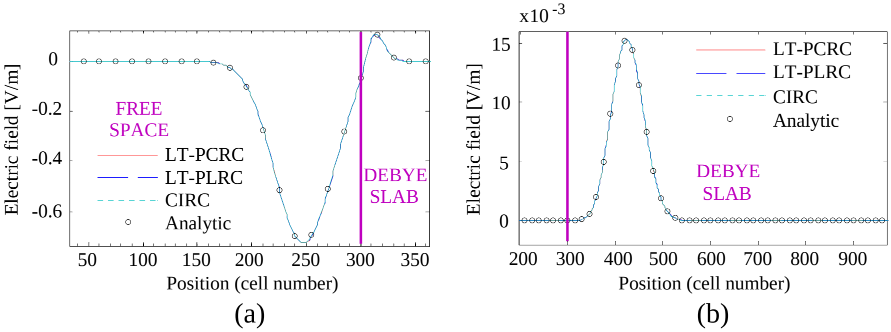


Figure 6. FDTD vs. analytic solution at the time (a) $t = 500\Delta t$ and (b) $t = 1500\Delta t$ for the material #3.

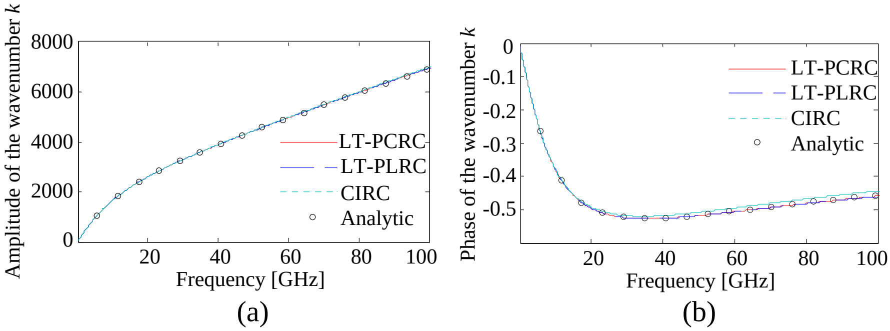


Figure 7. Wavenumber k: (a) amplitude and (b) phase for the material #3.

As can be observed from both Figs. 6 and 7, a good agreement between numerical and analytical solutions is achieved confirming the accuracy of the proposed techniques.

5. CONCLUSION

Three FD^2TD formulations have been proposed for the analysis of electromagnetic field propagation in multipole Debye media with (LT-PCRC, LT-PLRC) and without (CIRC) convolution equations. Two of the three FD^2TD methods hereby proposed are completely original (i.e., LT-PLRC and CIRC) while the third method (i.e., LT-PCRC), already presented in [5, 6], has been here improved.

The performances of LT-PCRC, LT-PLRC, and CIRC methods have been compared with other similar FD^2TD methods (KL-PLRC [4], ADE-1 [11], and ADE-2 [10]). The result of the comparative analysis has shown that all methods are very similar in terms of stability, accuracy and memory storage, while the number of algebraic operations is slightly different. Specifically, the LT-PCRC is the most efficient method in terms of calculation time together with the ADE-2, but this last must satisfy a much more restrictive stability condition. On the other hand, the accuracy of both methods is dependent on the ratio between the minimum relaxation time and the selected time step $\tau_0/\Delta t$. For values of $\tau_0/\Delta t > 5$ their precision in the calculations is comparable with that of the other FD^2TD methods, at least in the considered test cases. This condition is not a big limitation, since it is easily satisfied by many materials as biological tissues or dielectrics used in electronic components. However, for more generic analyses, the other methods are recommended, with particular focus on the CIRC and ADE-1 due to their slightly better computational efficiency. It should be also noted that this is the first time that the influence of the ratio $\tau_0/\Delta t$ has been deeply investigated in the field analysis of the $\mathrm{FD^2TD}$ methods.

APPENDIX A.

In this Appendix the amplification matrix of the proposed methods is given considering a single pole Debye medium for the sake of brevity, but the expressions for multipole Debye media can be easily obtained.

The amplification matrix for the LT-PCRC is:

$$\mathbf{G}=\begin{bmatrix} a_\xi & 0 & b_\xi \\ 0 & 1 & b_H\left(e^{jk\Delta x}-1\right) \\ b_E a_\xi & c_E\left(1-e^{-jk\Delta x}\right) & a_E+b_E b_\xi+c_E b_H\left(e^{jk\Delta x}+e^{-jk\Delta x}-2\right) \end{bmatrix} \quad \text{(A1)}$$

where:

$$b_H = -\Delta t/(\mu\Delta x) \quad \text{(A2a)}$$

$$a_E = \left(\frac{\varepsilon_0\varepsilon_\infty}{\Delta t}-\frac{\sigma_0}{2}-\frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\right)\left(\frac{\varepsilon_0\varepsilon_\infty}{\Delta t}+\frac{\sigma_0}{2}+\frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\right)^{-1} \quad \text{(A2b)}$$

$$b_E = \left(\frac{\sigma_0}{2}+\frac{\varepsilon_0\varepsilon_\infty}{\Delta t}+\frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\right)^{-1} \quad \text{(A2c)}$$

$$c_E = \frac{1}{\Delta x}\left(\frac{\sigma_0}{2}+\frac{\varepsilon_0\varepsilon_\infty}{\Delta t}+\frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\right)^{-1} \quad \text{(A2d)}$$

$$a_\xi = e^{-\Delta t/\tau_0} \quad \text{(A2e)}$$

$$b_\xi = \varepsilon_0\Delta\varepsilon\left(1-e^{-\Delta t/\tau_0}\right)\Big/\tau_0 \quad \text{(A2f)}$$

The amplification matrix of the LT-PLRC method is:

$$\mathbf{G}=\begin{bmatrix} a_\xi+\gamma_2 b_{E1} & \gamma_2 c_{E1}\left(1-e^{-jk\Delta x}\right) & \gamma_3 \\ 0 & 1 & b_H\left(e^{jk\Delta x}-1\right) \\ b_{E1} & c_{E1}\left(1-e^{-jk\Delta x}\right) & \gamma_4 \end{bmatrix} \quad \text{(A3)}$$

where:

$$\gamma_2=\vartheta^0-\zeta^0 \quad \text{(A4a)}$$

$$\gamma_3=\left(\vartheta^0-\zeta^0\right)\left(a_{E1}-c_{E1}b_H\left(e^{jk\Delta x}+e^{-jk\Delta x}-2\right)\right)+\zeta^0 \quad \text{(A4b)}$$

$$\gamma_4=a_{E1}-c_{E1}b_H\left(e^{jk\Delta x}+e^{-jk\Delta x}-2\right) \quad \text{(A4c)}$$

$$\vartheta^0 = (1 - a_\xi)\,\varepsilon_0 \Delta\varepsilon/\tau_0 \tag{A4d}$$

$$\zeta^0 = \frac{\tau_0}{\Delta t}\left(\vartheta^0 - \Delta t a_\xi \varepsilon_0 \Delta\varepsilon/\tau_0^2\right) \tag{A4e}$$

$$a_{E1} = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\left(1 - \frac{\zeta^0}{\tau_0}\right)\right)^{-1}\left(\frac{\varepsilon_0\varepsilon_\infty}{\Delta t} - \frac{\sigma_0}{2} - \frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\left(1 - \frac{\zeta^0}{\tau_0}\right)\right) \tag{A4f}$$

$$b_{E1} = \frac{1}{2}\left(1 + a_\xi\right)\left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\left(1 - \frac{\zeta^0}{\tau_0}\right)\right)^{-1} \tag{A4g}$$

$$c_{E1} = \frac{1}{\Delta x}\left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\varepsilon_0\Delta\varepsilon}{2\tau_0}\left(1 - \frac{\zeta^0}{\tau_0}\right)\right)^{-1} \tag{A4h}$$

The amplification matrix of the CIRC method is:

$$\mathbf{G} = \begin{bmatrix} a_J + b_J b_{E2} & b_J c_{E2}\left(1 - e^{-jk\Delta x}\right) & \begin{matrix} b_J[a_{E2} + c_{E2} b_H (e^{jk\Delta x} \\ +e^{-jk\Delta x} - 2) - 1] \end{matrix} \\ 0 & 1 & b_H\left(e^{jk\Delta x} - 1\right) \\ b_{E2} & c_{E2}\left(1 - e^{-jk\Delta x}\right) & \begin{matrix} a_{E2} + c_{E2} b_H \\ \left(e^{jk\Delta x} + e^{-jk\Delta x} - 2\right) \end{matrix} \end{bmatrix} \tag{A5}$$

where:

$$a_{E2} = \left(\frac{\varepsilon_0\varepsilon_\infty}{\Delta t} - \frac{\sigma_0}{2} + \frac{\varepsilon_0\Delta\varepsilon}{\tau_0 + \Delta t}\right)\left(\frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\sigma_0}{2} + \frac{\varepsilon_0\Delta\varepsilon}{\tau_0 + \Delta t}\right)^{-1} \tag{A6a}$$

$$b_{E2} = \frac{\tau_0}{\tau_0 + \Delta t}\left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\varepsilon_0\Delta\varepsilon}{\tau_0 + \Delta t}\right)^{-1} \tag{A6b}$$

$$c_{E2} = \frac{1}{\Delta x}\left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{\varepsilon_0\Delta\varepsilon}{\tau_0 + \Delta t}\right)^{-1} \tag{A6c}$$

APPENDIX B.

The methods described in the paper can be rewritten in order to reduce the number of operation needed. Consider:

$$y_m^{n+1} = a_m y_m^n + b_m u^n \tag{B1a}$$

$$x^{n+1} = \alpha x^{n+1} + \sum_{m=1}^{M} c_m y_m^{n+1} = \alpha x^{n+1} + \sum_{m=1}^{M} c_m (a_m y_m^n + b_m u^n)$$

$$= \alpha x^{n+1} + \sum_{m=1}^{M} c_m a_m y_m^n + \sum_{m=1}^{M} c_m b_m u^n \tag{B1b}$$

Defining $w_m^{n+1} = c_m y_m^{n+1}$ (B1b) become:

$$x^{n+1} = \alpha x^{n+1} + \sum_{m=1}^{M} a_m w_m^n + \sum_{m=1}^{M} c_m b_m u^n \tag{B2}$$

Consequently, (B1a)–(B1b) can be rewritten as:

$$w_m^{n+1} = a_m w_m^n + d_m u^n \tag{B3a}$$

$$x^{n+1} = \alpha x^{n+1} + \sum_{m=1}^{M} w_m^{n+1} \tag{B3b}$$

where $d_m = c_m b_m$. These passages can be applied to the proposed methods. For the LTPCRC method (16) and (18) defining $w_m = (\sigma_0/2 + \varepsilon_0\varepsilon_\infty/\Delta t + \sum_{m=1}^{N_p}(\varepsilon_0\Delta\varepsilon_m/(2\tau_m)))^{-1}\xi_m$, become:

$$w_m^{n+1/2}(i) = a_{w,m}(i) w_m^{n-1/2}(i) + b_{w,m}(i) E^n(i) \tag{B4a}$$

$$E^{n+1}(i) = a_E(i)E^n(i) + b_E(i)\Big(H^{n+1/2}(i+1/2) - H^{n+1/2}(i-1/2)\Big) + \sum_{m=1}^{N_p} w_m^{n+1/2}(i) \tag{B4b}$$

being

$$a_{w,m} = \exp\{-\Delta t/\tau_m\} \tag{B5a}$$

$$b_{w,m} = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{2\tau_m}\right)^{-1} \frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\left(1 - e^{-\frac{\Delta t}{\tau_m}}\right) \tag{B5b}$$

$$a_E = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{2\tau_m}\right)^{-1}\left(-\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} - \sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{2\tau_m}\right) \tag{B5c}$$

$$b_E = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{2\tau_m}\right)^{-1} \frac{1}{\Delta x} \tag{B5d}$$

For the LTPLRC method the Equations (24) and (25), defining $w_m = (\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{1}{2}\sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}))^{-1}\frac{1+e^{-\Delta t/\tau_m}}{2}\xi_m$, become:

$$w_m^{n+1}(i) = a_{w,m}(i) w_m^n(i) + b_{w,m}(i) E^{n+1}(i) + c_{w,m}(i) E^n(i) \tag{B6a}$$

$$E^{n+1}(i) = a_E(i)E^n(i) + b_E(i)\Big(H^{n+1/2}(i+1/2) - H^{n+1/2}(i-1/2)\Big) + \sum_{m=1}^{N_p} w_m^n(i) \tag{B6b}$$

being

$$a_{w,m} = \exp\{-\Delta t/\tau_m\} \tag{B7a}$$

$$b_{w,m} = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{1}{2}\sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\left(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}\right)\right)^{-1}$$

$$\frac{1+e^{-\Delta t/\tau_m}}{2}\left(\vartheta_m^0 - \zeta_m^0\right) \tag{B7b}$$

$$c_{w,m} = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{1}{2}\sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\left(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}\right)\right)^{-1}$$

$$\frac{1+e^{-\Delta t/\tau_m}}{2}\zeta_m^0 \tag{B7c}$$

$$a_E = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{1}{2}\sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\left(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}\right)\right)^{-1}$$

$$\left(\frac{\varepsilon_0\varepsilon_\infty}{\Delta t} - \frac{\sigma_0}{2} - \frac{1}{2}\sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\left(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}\right)\right) \tag{B7d}$$

$$b_E = \left(\frac{\sigma_0}{2} + \frac{\varepsilon_0\varepsilon_\infty}{\Delta t} + \frac{1}{2}\sum_{m=1}^{N_p}\frac{\varepsilon_0\Delta\varepsilon_m}{\tau_m}\left(1 - \frac{\vartheta_m^0 - \zeta_m^0}{\tau_m}\right)\right)^{-1}\frac{1}{\Delta x} \tag{B7e}$$

For the CIRC method (32) and (33) defining $w_m = (\frac{G_0}{2} + \frac{C_\infty}{\Delta t} + \sum_{m=1}^{N_p}\frac{C_m}{R_mC_m+\Delta t})^{-1}(\frac{R_mC_m}{R_mC_m+\Delta t})J_{p,m}$ we have:

$$w_m^{n+1/2}(i) = a_{w,m}(i)w_m^{n-1/2}(i) + b_{w,m}(i)\left(E^{n+1}(i) - E^n(i)\right) \tag{B8a}$$

$$E^{n+1}(i) = a_E(i)E^n(i) + b_E(i)\left(H^{n+1/2}(i+1/2) - H^{n+1/2}(i-1/2)\right)$$

$$-\sum_{m=1}^{N_p} w_m^{n-1/2}(i) \tag{B8b}$$

being

$$a_{w.m} = R_mC_m/(R_mC_m + \Delta t) \tag{B9a}$$

$$b_{w,m} = R_m\left(\frac{C_m}{R_mC_m+\Delta t}\right)^2\Bigg/\left(\frac{G_0}{2} + \frac{C_\infty}{\Delta t} + \left(\sum_{m=1}^{N_p}\frac{C_m}{R_mC_m+\Delta t}\right)\right) \tag{B9b}$$

$$a_E = \left(\frac{G_0}{2}+\frac{C_\infty}{\Delta t}+\sum_{m=1}^{N_p}\frac{C_m}{R_m C_m+\Delta t}\right)^{-1}\left(\frac{C_\infty}{\Delta t}-\frac{G_0}{2}+\sum_{m=1}^{N_p}\frac{C_m}{R_m C_m+\Delta t}\right) \quad \text{(B9c)}$$

$$b_E = \left(\frac{G_0}{2}+\frac{C_\infty}{\Delta t}+\sum_{m=1}^{N_p}\frac{C_m}{R_m C_m+\Delta t}\right)^{-1}\frac{1}{\Delta x} \quad \text{(B9d)}$$

Similar passages can be applied also to the methods LK-PLRC, ADE-1 and ADE-2 which have similar structures to, respectively, LTPLRC, CIRC and LTPCRC leading to Table 1.

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