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ON RELATIVISTIC POLARIZATION OF A ROTATING MAGNETIZED MEDIUM

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Abstract—We consider the relativistic polarization of a rotating magnetized medium in the framework of the approach suggested earlier [8], which is based on the charge conservation law and relativistic generalization of the first Kirchhoff law to a closed moving circuit carrying steady current. We show that the polarization of a magnet brought to a rotation differs, in general, from the relativistic polarization of a translationary moving magnet, and on this way we give one more explanation to the familiar Wilson & Wilson experiment, with the explicit demonstration of the implementation of the charge conservation law.

1. INTRODUCTION

In this paper we address to the classical electrodynamics of rotating media, in particular, with regards to the Wilson & Wilson (W&W) experiment carried out at the beginning of the past century [1]. As known, the idea of this experiment had been suggested by Einstein and Laub to verify the relativistic prediction on the development of an electric dipole moment by a translationally moving magnetic dipole. By technical reasons W&W substituted translational motion of a medium by its rotation, and the result of their experiment fully confirmed this relativistic prediction. During a long time this

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result was considered as a remarkable experimental verification of special relativity. However, at the modern time Pellegrini and Swift (P&S) questioned the correctness of relativistic analysis with respect to W&W experiment [2], stating that the expression for "relativistic polarization" of a moving magnetized medium (see, e.g., [3])

$$\mathbf{P}_{rel} = \frac{\mathbf{v} \times \mathbf{M}}{c} \tag{1}$$

(where $\mathbf{P}_{rel}$ is polarization, $\mathbf{M}$ is magnetization, and $\mathbf{v}$ is the translational velocity of a medium), seemingly tested in W&W experiment, comes into a contradiction with the charge conservation law, when Eq. (1) is extended to the rotation case. Based on this statement, P&S concluded that the theoretical analysis of W&W experiment must be substantially modified and that one needs to understand better the conditions, where the standard constitutive equations in a rotating medium are valid [2].

In the subsequent discussion (see, e.g., [4–6]) the conclusion of P&S was invalidated, and it was shown that the consistent formal analysis of W&W experiment in the framework of classical electrodynamics in rotating media does not reveal a discrepancy with the W&W result, which was additionally confirmed at a modern time [7]. Thus it was commonly adopted that the analysis of P&S [2] is not fully correct; however, to the moment nobody explicitly disproved the statement of P&S mentioned above that in a rotating medium Eq. (1) contradicts the charge conservation law.

The essence of this contradiction can be seen with the following simple example. Let us consider a homogeneously magnitized electrically neutral cylindrical magnet, where its magnetization can be presented by the effective current circulating over its rim. Then we assume that the magnet rotates around its axis of symmetry, and the tangential velocity $\mathbf{v}$ on its rim surface coincides with the direction of circulating current $\mathbf{I}$. We can attach the Lorentz frame K_0 to any point on cylinder's rim and carry out the transformation of charge density-current density four-vector (ρ, $\mathbf{j}$) from K_0 to the labframe K, i.e.,

$$\mathbf{j} = \gamma \left(\mathbf{j}_0 + \rho_0 \mathbf{v} \right), \tag{2}$$

$$\rho = \gamma \left(\rho_0 + \frac{\mathbf{j}_0 \cdot \mathbf{v}}{c^2} \right), \tag{3}$$

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ is the Lorentz factor. Since for the electrically neutral medium at rest $\rho_0 = 0$, we obtain the non-vanishing charge density

$$\rho = \gamma j v / c^2 \tag{4}$$

at the designated point on the surface of rotating cylinder, as viewed in the labframe. Further, with the same success we can attach a Lorentz frame to any other point on the surface of a rotating cylinder and obtain the same charge density (4) at the given location. Hence we conclude that *all* surface points of the rotating cylinder acquire the non-zero charge density (4), and thus the entire surface occurs homogeneously charged. One can show that this result fully agrees with Eq. (1); however it directly contradicts the charge conservation law, since we assumed that in the absence of rotation the cylinder is electrically neutral.

In order to resolve this paradox, P&S pointed out the difference between the cases of translational and rotational motions of a magnetic medium and suggested to introduce into consideration the rotating frame, whose origin of coordinates coincides with the origin of coordinates of a labframe and is located on the rotational axis of the medium. In this case the time coordinates are identical in both frames, i.e., $t' = t$, and hence $\rho' = \rho$ [2], so that no relativistic polarization of a surface of a rotating magnet emerges.

Here we emphasize that such argumentation would be correct only in the case, where the *measuring instrument* (a voltmeter in W&W experiment, which measures the potential difference between the external and internal surfaces of a rotating magnetic medium) *rigidly rotates* with the medium. However, in the conditions of W&W experiment, the voltmeter *rests* in a laboratory and is connected with the rotating surfaces of slab via the sliding contacts. If so, the electromagnetic fields, charges and currents should be directly determined for the laboratory frame, where the measuring instrument (voltmeter) is at rest. Therefore, the approach involving the analysis of classical electrodynamics in rotating systems can be considered, in the best case, as an auxiliary tool, since anyway the measured e.m.f. must be referred to a laboratory observer. It means that magnetization and polarization of a rotating slab *must* be determined in a laboratory frame via the introduction of a set of Lorentz frames co-moving with each rotating point, with further Lorentz transformation to a laboratory frame. In what follows, this approach is directly applied to the analysis of relativistic polarization of rotating magnetized media.

In the present contribution we intend to show that the resolution of the paradox in question (a seeming contradiction of Eq. (1) to the charge conservation law for the case of rotation) requires a closer insight to the origin and physical interpretation of the transformation rules (2), (3). In particular, as we mentioned earlier [8], the popular approach suggested by Feynman et al. [9] consisting in associating these transformations with the scale contraction effect in a moving straight wire is, in general, insufficient for a full understanding of the relativistic polarization of moving magnetic media. Moreover, when this approach is applied solely to closed circuits with steady currents [10], it leads to physically senseless results [8]. The majority of books on classical electrodynamics (see, e.g., [3, 11, 12]) directly bridge the transformations (2), (3) to the relativity of simultaneity of events in different inertial frames. This is a correct, but a too general statement, which cannot be directly used for the analysis of relativistic polarization of rotating magnetic media.

In our recent paper [8] we have shown that the transformations (2), (3) for a translational motion of a current carrying loop can be understood via the continuity equation for conduction electrons (named in Ref. [8] as the generalized first Kirchhoff law) combined with the charge conservation law. In Section 2 we apply this approach to a magnetic dipole brought to a rotational motion, using the known result regarding the fact that the entire dipole can be presented as a set of smaller (elementary) magnetic dipoles, filling its volume, where the boundary currents in adjacent elementary dipoles mutually cancel each other, giving rise to the resulting current, circulating over the rim of the dipole. Thus, when the magnetic medium is brought to a rotational motion at a constant angular frequency ω, each elementary magnetic dipole is characterized by its own tangential velocity $v(r) = \omega r$, where r is the radial coordinate. Next we apply the continuity equation to the carriers of current in each elementary dipole, demanding the implementation of the charge conservation law. On this way we determine the spatial distribution of charge density over the rotating magnetic medium and show that its surface actually acquires the charge density (4), as seen by a laboratory observer. At the same time, we also show the relativistic polarization of the bulk of rotating magnetic medium, so that the total bulk charge exactly counteracts the surface charge for a laboratory observer. Consequently, we obtain one more explanation to W&W experiment with the explicit demonstration of the implementation of the charge conservation law. Finally, in Section 3 we present our conclusions.

2. GENERALIZED KIRCHHOFF LAW AND POLARIZATION OF A ROTATING MAGNETIC MEDIUM

In this section we determine for the *laboratory observer* the relativistic polarization of a homogeneously magnetized slab (hollow cylindrical magnetic insulator), rotating at the constant angular frequency ω in the counter clockwise direction around its axis of symmetry (z-axis, see Fig. 1). We designate as R_{in} and R_{ex} the internal and external

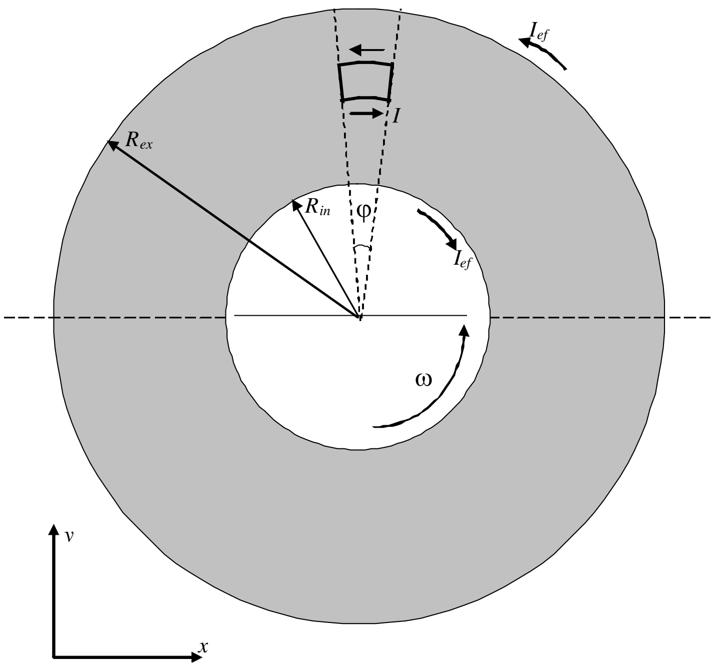

Figure 1. Cross-section in the xy-plane of the rotating magnetized slab. We mark inside the slab an elementary circuit (magnetic dipole), whose upper and lower parts lay in the azimuthal direction, and the side parts are directed along the radius.

radius of the slab, correspondingly, and suppose that its length along the axis z is equal to h, and that the magnetic moment of the slab is parallel to this axis.

Such a magnetization of the slab can be described via the effective steady surface currents I_{ef}, flowing counter clock-wise on its external surface, and clock-wise on the internal surface (Fig. 1). We want to determine the charge distribution in the slab due to its relativistic polarization caused by the rotation.

As we have mentioned above, the total magnetic dipole moment of the slab can be described via a set of elementary magnetic dipoles, filling its entire volume. Using in a further analysis the cylindrical coordinates r, φ, z, we choose the elementary magnetic dipole in such a way that its upper and lower segments lie in the azimuthal direction φ, while the side segments lie in the radial direction r (see Fig. 1, where the direction of current in the circuit is also shown). The length of such a dipole along the axis z coincides with the length of the slab and is equal to h.

Further, using the approach suggested in [8], we want to write the continuity equation for carriers of current in the designated circuit. The physical meaning of such continuity equation for a steady current is straightforward [8]: For any cross-section area chosen at a given point of the circuit and *co-moving with* it, the number of charges passing across the mentioned area per unit period of time must be a fixed quantity for any point located on the perimeter of the circut. (We named this statement in Ref. [8] as the generalized first Kirchhoff law). Thus respectively designating ρ_0, u_0 and S_0 the proper charge density of carriers of current, their flow velocity and the area of cross-section, correspondingly, let us consider the cross-section in the lower right corner of the circuit†. Then the value of charge passing during a given unit period of time across the cross-sectional area in question, being delivered from the lower segment of the circuit, is equal to

$$\rho'_{low}\left(u'_{low}+v_{low}\right)S_0, \tag{5}$$

as seen by a laboratory observer. Here u'_{low} stands for the azimuthal component of velocity of carriers of current in the labframe, and we have taken into account that for the area S_0 *co-moving* with the circuit, the immovable charges of this circuit do not contribute to the net current. On the other hand, the value of charge coming to the right side segment of the circuit from this cross-sectional area during the given period of time is defined by the product

$$\rho'_s u'_s\left(S_0/\gamma_s\right), \tag{6}$$

where we have taken into account the relativistic contraction of the cross-sectional area of the circuit in its side segments. (Here u'_s is the radial component of velocity of carriers of current). Thus, due to the continuity requirement, the quantities (5) and (6) are equal to each other, i.e.,

$$\rho'_{low}\left(u'_{low}+v_{low}\right)S_0=\rho'_s u'_s\left(S_0/\gamma_s\right). \tag{7}$$

† Below we introduce further designations, referred to a laboratory observer:

ρ'_{low}, ρ'_{up}, ρ'_s are the charge densities of carriers of current in the lower, upper and side segments of the circuit, correspondingly;

ρ is the charge density of immovable charges of the circuit (e.g., positive ions for conducting circuits);

$\mathbf{u}'_{low}$, $\mathbf{u}'_{up}$, $\mathbf{u}'_s$ are the velocities of carriers of current in the lower, upper and side segments, correspondingly;

v_{low}, v_{up}, v_s are the tangential velocities along azimuthal direction φ in the lower, upper and side segments, correspondingly;

r_{low}, r_{up} are the radial coordinates respectively of the lower and uper segments of the circuit.

In addition, we designatel_{low}, l_{up}, l_s the proper lengths of the lower, upper and side segments of the circuit, correspondingly.

Using the Einstein law of velocity composition, we find

$$u'_{low}+v_{low}=\frac{u_0-v_{low}}{1-u_0 v_{low}/c^2}+v_{low}=\frac{u_0}{\gamma_{low}^2\left(1-u_0 v_{low}/c^2\right)},\quad u'_s=u_0/\gamma_{low}.$$

Substituting these equalities into Eq. (7), we obtain:

$$\rho'_{low}=\rho'_s\left(1-u_0 v_{low}/c^2\right). \tag{8}$$

Applying a similar continuity equation to the right upper corner of the circuit, we derive:

$$\rho'_{up}=\rho'_s\left(1+u_0 v_{up}/c^2\right). \tag{9}$$

Here we remind that similar Eqs. (8) and (9) had been obtained in Ref. [8] for a translationary moving rectangular circuit (where by definition $v_{low}=v_{up}$), which, after the explicit determination of ρ'_s, ρ'_{low} and ρ'_{up} via the charge conservation law, yield Eq. (1). However, for the case of rotation we get $v_{low}\neq v_{up}$, and the analysis becomes more complicated. In addition, we point out that in the side segments, the Lorentz factor $\gamma_s=[1-v_s^2(r)/c^2]^{-1/2}$ becomes to be the function of r. However, for simplicity of further analysis we overlook the dependence of γ_s on r for the elementary circuit in question, which implies the accuracy of further calculations of the order $(v_s/c)^2$. Indeed, a polarization of a rotating magnetic media, which is the subject of our further calculations, cannot contain the terms of the order $(v_s/c)^2$, because it must vanish at $u_0=0$. Hence, the lowest order of the terms, containing the Lorentz factor, is $v_s^2 u_0/c^3$, which however lies beyond the adopted accuracy of calculations c^{-2}.

Thus, having obtained Eqs. (8) and (9), we now in the position to write the charge conservation law for the circuit in question, taking into account that for the immovable elementary circuit the charge of carriers of current is equal to $\rho_0 S_0 P_0$, where $P_0=l_{low}+l_{up}+2l_s$ is the perimeter of the circuit in its rest frame. This charge remains unchanged for the moving circuit, and hence we derive the equality:

$$\rho'_{low}S_0\left(l_{low}/\gamma\right)+2\rho'_s\left(S_0/\gamma\right)l_s+\rho'_{up}S_0\left(l_{up}/\gamma\right)=\rho_0 S_0 P_0. \tag{10}$$

Further substitution of Eqs. (8), (9) into Eq. (10) yields:

$$\rho'_s\left(1-u_0 v_{low}/c^2\right)l_{low}+2\rho'_s l_s+\rho'_s\left(1+u_0 v_{up}/c^2\right)l_{up}=\gamma\rho_0 P_0.$$

After some manipulations this equation can be presented in the form of

$$\rho'_s P_0+\rho'_s u_0/c^2\left(v_{up}l_{up}-v_{low}l_{low}\right)=\gamma\rho_0 P_0.$$

Involving the obvious equalities $l_{low}=\varphi r_{low}$, $l_{up}=\varphi r_{up}$, $v_{low}=\omega r_{low}$, $v_{up}=\omega r_{up}$ (where the angle φ is defined in Fig. 1), we obtain:

$$\rho'_s=\gamma\rho_0\left[1+\frac{2S_c}{P_0}\frac{u_0\omega}{c^2}\right]^{-1}, \tag{11}$$

where $S_c = \frac{1}{2}\varphi[r_{up}^2 - r_{low}^2]$ is the area of the circuit.

Substituting Eq. (11) into Eqs. (8) and (9), we determine the charge density of carriers of currents in the lower and upper segments, too:

$$\rho'_{low} = \gamma\rho_0 \left(1 - u_0 v_{low}/c^2\right) \left[1 + \frac{2S_c}{P_0}\frac{\omega u_0}{c^2}\right]^{-1}, \tag{12}$$

$$\rho'_{up} = \gamma\rho_0 \left(1 + u_0 v_{up}/c^2\right) \left[1 + \frac{2S_c}{P_0}\frac{\omega u_0}{c^2}\right]^{-1}. \tag{13}$$

A net charge density $\Delta\rho$ in each segment of the circuit is defined as the difference of charge density of carriers of current and charge density ρ of immovable charges of the circuit. Taking into account that $\rho = \gamma\rho_0$ in each segment, we derive with the adopted accuracy of calculations c^{-2}:

$$\Delta\rho_s = \rho'_s - \gamma\rho_0 \approx -\rho_0\frac{u_0\omega}{c^2}\frac{2S_c}{P_0}, \tag{14}$$

$$\Delta\rho_{low} = \rho'_{low} - \gamma\rho_0 \approx -\rho_0\frac{u_0\omega}{c^2}\left(r_{low} + \frac{2S_c}{P_0}\right), \tag{15}$$

$$\Delta\rho_{up} = \rho'_{up} - \gamma\rho_0 \approx \rho_0\frac{u_0\omega}{c^2}\left[r_{up} - \frac{2S_c}{P_0}\right]. \tag{16}$$

Equations (14)–(16) describe the charge distribution over the perimeter of the designated small magnetic dipole depicted in Fig. 1. We point out that in contrast to the case of translational motion of closed rectangular circuit (see, e.g., [8]), in the side segments, where the direction of current is orthogonal to tangentional (translational) velocity, the charge density of relativistic polarization (14) is not equal to zero and proportional to the ratio $2S_c/P_0$. Correspondingly, in the low and upper segments of the circuit, where the direction of current is collinear to the tangentional (translational) velocity, the charge density of relativistic polarization also contains the terms proportional to $2S_c/P_0$ (Eqs. (15), (16)). Further on, it is important to notice that in the calculation of relativistic polarization $\mathbf{P}_{rel}$ in a *given spatial point* $\mathbf{r}$ of a rotating magnetic medium, we have to consider a point-like magnetic dipole located in this point, and for such infinitely small dipole the ratio $2S_c/P_0$ tends to zero. In addition, we have $\omega r_{low} \approx \omega r_{up} = \omega v(\mathbf{r})$, so that

$$\Delta\rho_s \to 0, \quad \Delta\rho_{low} \to -\rho_0\frac{u_0 v(\mathbf{r})}{c^2} = -\frac{jv(\mathbf{r})}{c^2}, \quad \Delta\rho_{up} \to \rho_0\frac{u_0 v(\mathbf{r})}{c^2} = \frac{jv(\mathbf{r})}{c^2}, \tag{17}$$

and in this limit further straightforward calculations indicate a perfect fulfilment of Eq. (1). Hence for the case of W&W experiment, the radial electric polarization of an insolating magnetic medium with the electric permittivity ε in the considered point $\mathbf{r}$ is equal to

$$P_r = \frac{(\varepsilon - 1)}{4\pi}\left(E_r + \frac{\omega r B_z}{c}\right) + (P_r)_{rel}\,. \tag{18}$$

Here E_r is the radial electric field, B_z is the magnetic field caused by the rotation of charged cylinder enclosing the magnetic medium, and we have taken into account that $v(\mathbf{r}) = \omega r$. Then the radial displacement takes the form

$$D_r = E_r + 4\pi P_r = \varepsilon E_r + (\varepsilon - 1)\frac{\omega r B_z}{c} + \left(1 - \frac{1}{\mu}\right)\frac{\omega r B_z}{c}, \tag{19}$$

where μ is the relative magnetic permeability, and we have used Eq. (1) in the form

$$(P_r)_{rel} = \frac{\omega r M_z}{c}. \tag{20}$$

Further taking into account that for a long magnetic insulator cylinder the displacement is equal to zero, we derive from Eq. (19)

$$E_r = -\left(1 - \frac{1}{\varepsilon\mu}\right)\frac{\omega r B_z}{c}. \tag{21}$$

This equation indicates that the potential difference measured in W&W experiment between the internal and external surfaces of a rotating magnetic slab should be proportional to the factor $(1 - \frac{1}{\varepsilon\mu})$, which is actually the case.

Equations (18)–(21) had been proposed by Pellegrini [13] with the purpose to demonstrate that the suggested interpretation of W&W experiment, which involves the radial relativistic polarization (20), issuing from Eq. (1), does contradict the charge conservation law and thus cannot be recognized satisfactory.

However, based on Eq. (17) we already have shown that Eq. (1) is well fulfilled for each point of a magnetic medium brought to a rotation. Thus, the next principal problem is to demonstrate that this equation could be well compatible with the charge conservation law, when we derive with Eqs. (14)–(16) the macroscopic charge distribution inside a rotating magnetic medium.

In order to determine this charge distribution, we have to sum up the charge densities, defined by Eqs. (14)–(16), over a full set of elementary magnetic dipoles, which finally give rise to the resultant effective surface current I_{ef}. And only after the implementation of this operation we will obtain the macroscopic charge distribution in the rotating magnetic medium shown in Fig. 1.

First we sum up the charge densities in the adjacent elementary magnetic dipoles along the radial coordinate (i.e., at fixed φ) and get the subsequent sums of Eqs. (15) and (16), where in Eq. (15) the radial coordinate r_{low} is every time replaced by r_{up}. Then one can see that any such summation yields the value $2\Delta\rho_s$, i.e., it reproduces the radial distribution of charge density, derived from Eq. (14). The fully non-compensating contribution to the charge density (16) emerges only on the external and internal surfaces of the sector of rotating magnetic cylinder, defined by the angle φ, whose area is equal to

$$S_c = \varphi\left(R_{ex}^2 - R_{in}^2\right)/2,$$

and perimeter

$$P_0 = 2\left(R_{ex} - R_{in}\right) + \varphi\left(R_{ex} + R_{in}\right).$$

Substituting these equalities into Eqs. (14)–(16), we obtain for this sector:

$$\begin{aligned} \Delta\rho_s &= -\rho_0 \frac{u_0\omega}{c^2} \frac{\varphi\left(R_{ex}^2 - R_{in}^2\right)}{2\left(R_{ex} - R_{in}\right) + \varphi\left(R_{ex} + R_{in}\right)} \\ &= -\rho_0 \frac{u_0\omega}{c^2} \frac{\varphi\Delta R}{2\Delta R/(R_{ex} + R_{in}) + \varphi}, \end{aligned} \tag{22}$$

$$\Delta\rho_{low} = -\rho_0 \frac{u_0\omega}{c^2} \left(R_{in} + \frac{\varphi\Delta R}{2\Delta R/(R_{ex} + R_{in}) + \varphi}\right), \tag{23}$$

$$\Delta\rho_{up} = \rho_0 \frac{u_0\omega}{c^2} \left[R_{ex} - \frac{\varphi\Delta R}{2\Delta R/(R_{ex} + R_{in}) + \varphi}\right], \tag{24}$$

where $\Delta R = R_{ex} - R_{in}$.

Here we emphasize that Eqs. (23), (24) are relevant only for a thin surface layer δ, where the effective surface current I_{ef} flows, and for insulator magnetic materials δ can be a value of an atomic scale)[‡]. At $R_{ex} + \delta < r < R_{ex} - \delta$ (i.e., in the bulk of magnetic medium) the charge density is determined by Eq. (22).

Further we notice that for any magnetic material there is always the smallest value of $\varphi = \varphi_0$ defined by its magnetic structure (e.g., for a ferromagnetic material the limited value of φ_0 depends on the ratio of typical size of magnetic domain to typical size of a magnet). Thus, the angle φ_0 determines the minimal size of "elementary sector" for a given composition of magnetic medium. For real magnetic materials the value of φ_0 is usually very small, so that the radial (bulk) charge

[‡] Due to a purely relativistic origin of Eqs. (22)–(24), one can ask the question: Is it legitimate to involve the material properties (e.g., the thickness of the surface layer δ, where the effective current flows) to the analysis of these equations, given that, for example, in the frame co-rotating to the medium, the charge densities (22)–(24) are all vanishing? We give a positive answer to this question, following the assertion of Einstein: All relativistic effects are real in the same extent, like any other physical phenomena, as soon as they become a subject of measurement for a given inertial observer.

density (22) is very small, too. At the same time, as we will see below, its non-vanishing value is important for the implementation of the charge conservation law in a rotating magnetic medium.

Next we determine the total charge on the surfaces of the elementary segment:

$$Q_{low}=\Delta\rho_{low}\varphi_0 R_{in} S_0=-\varphi_0\frac{I_{ef}v_{in}}{c^2}\left(R_{in}+\frac{\varphi_0\Delta R}{2\Delta R/(R_{ex}+R_{in})+\varphi_0}\right), \quad (25)$$

$$Q_{up}=\Delta\rho_{up}\varphi_0 R_{ex} S_0=\varphi_0\frac{I_{ef}v_{ex}}{c^2}\left[R_{ex}-\frac{\varphi_0\Delta R}{2\Delta R/(R_{ex}+R_{in})+\varphi_0}\right], \quad (26)$$

as well as in the radial segments:

$$Q_s = 2\Delta\rho_s \Delta R S_0 = -\frac{2I_{ef}\left(v_{ex}-v_{in}\right)}{c^2}\frac{\varphi_0\Delta R}{2\Delta R/(R_{ex}+R_{in})+\varphi_0}, \quad (27)$$

where $S_0 = \delta h$, $I_{ef} = \rho_0 u_0 S_0$, and $v_{ex} = \omega R_{ex}$, $v_{in} = \omega R_{in}$ are the tangential velocities on the external and internal surfaces of the rotating sector, correspondingly.

Further on we notice that the magnet contains $(2\pi/\varphi_0)$ elementary segments. Hence the multiplication of Eqs. (25), (26) and (27) by this factor gives the total charge located on the internal surface, external surface and in the bulk of rotating magnet, correspondingly. Taking also into account that the angle φ_0 is very small, and using the limit $\varphi_0 \to 0$, we derive the following:

$$\begin{aligned} Q_{\text{internal}} &= \frac{2\pi}{\varphi_0}Q_{low} = -\frac{2\pi I_{ef}v_{in}}{c^2}\left(R_{in}+\frac{\varphi_0\Delta R}{2\Delta R/(R_{ex}+R_{in})+\varphi_0}\right) \\ &\to -\frac{I_{ef}v_{in}}{c^2}2\pi R_{in}, \end{aligned} \quad (28)$$

$$\begin{aligned} Q_{\text{internal}} &= \frac{2\pi}{\varphi_0}Q_{up} = \frac{2\pi I_{ef}v_{ex}}{c^2}\left[R_{ex}-\frac{\varphi_0\Delta R}{2\Delta R/(R_{ex}+R_{in})+\varphi_0}\right] \\ &\to \frac{I_{ef}v_{ex}}{c^2}2\pi R_{ex}, \end{aligned} \quad (29)$$

$$\begin{aligned} Q_{\text{bulk}} &= \frac{2\pi}{\varphi_0}Q_s = -\frac{2I_{ef}\left(v_{ex}-v_{in}\right)}{c^2}\frac{2\pi\Delta R}{2\Delta R/(R_{ex}+R_{in})+\varphi_0} \\ &\to -\frac{I_{ef}\left(v_{ex}-v_{in}\right)}{c^2}2\pi\left(R_{ex}+R_{in}\right). \end{aligned} \quad (30)$$

The obtained Eqs. (28)–(30) give the total charge of the rotating magnetic medium due to relativistic polarization on its both surfaces and in the bulk. First of all, one can see that the sum of the charges (28)–(30) is equal to zero, i.e.,

$$Q_{\text{internal}} + Q_{\text{internal}} + Q_{\text{bulk}} = 0,$$

as it should be due to the charge conservation law. We can add that in the particular case of a rotating cylinder ($R_{in} = 0$, $R_{ex} = R$), $Q_{in} = 0$, while

$$Q_{\text{internal}} = -Q_{\text{bulk}} = \frac{I_{ef}v}{c^2} 2\pi R.$$

Further, let us present Eqs. (28)–(30) in the equivalent forms, which establish the relationship between charge density and effective current density. Taking first Eq. (28), we get

$$Q_{\text{internal}} = -\frac{j_{ef}v_{in}}{c^2} 2\pi R_{in} S_0 = -\frac{j_{ef}v_{in}}{c^2} V_{in},$$

or

$$\rho_{\text{internal}} = \frac{Q_{\text{internal}}}{V_{in}} = -\frac{j_{ef}v_{in}}{c^2}, \tag{31}$$

where $j_{ef} = I_{ef}/S_0$ is the effective current density, $V_{in} = 2\pi R_{in} S_0$ the volume of the internal surface layer, where the effective current flows, and ρ_{internal} is the charge density of polarization charges in the internal surface layer.

In a similar way we derive for the external surface layer:

$$\rho_{\text{internal}} = \frac{j_{ef}v_{ex}}{c^2}, \tag{32}$$

and we see that in the adopted accuracy of calculations c^{-2}, both Eqs. (31), (32) are in a full agreement with the relativistic expression (4), resulting from the Lorentz transformations (2), (3).

Analogously, for the charge density in the bulk of rotating medium we derive:

$$\rho_{\text{bulk}} = \frac{Q_{\text{bulk}}}{\pi \left(R_{ex}^2 - R_{in}^2\right) h} = -\frac{2I_{ef}\omega}{c^2 h} = -\frac{2j_{ef}\omega\delta}{c^2}. \tag{33}$$

We see that the bulk charge density (33) is much less than the surface charge densities (31), (32), when $\delta \ll R_{ex}$, R_{in}. Even so, the presence of the non-vanishing charge density inside a rotating magnetic medium has the principal importance for the implementation of the charge conservation law, since the surface charges of relativistic polarization of a rotating magnetic slab on its internal and external surfaces, in general, differ from each other (compare Eqs. (28) and (29)).

Next we emphasize that the mentioned above compliance between Eqs. (31), (32) and Eq. (4) represents, in general, a non-trivial fact, because the Lorentz transformations (2), (3) are directly applicable only to the case, where *all points* of a circuit with a steady current move at *the same translational* velocity. When it is not the case (e.g., for a rotating magnetic medium), we can apply transformations (2), (3) only to elementary circuits (where the approximation of constant translational velocity for all its points is fulfilled), and then to use the continuity equation and the charge conservation law, in order to derive a macroscopic polarization of such a medium. In this case the obtained results should lose a direct relationship with the Lorentz transformations (2), (3). In these conditions the compatibility of Eqs. (4), (31) and (32) is actually amazing. At the same time, we stress that in no way the bulk charge density (33) can be directly derived from transformations (2) and (3); rather it follows from the application of continuity equation to the current in each elementary magnetic cirtuit with further summation of currents/charge densities over a set of elementary magnetic dipoles, constituting a macroscopic magnetic medium.

Finally, let us explicitly present the charge density of relativistic polarization ρ_{rel} of a rotating magnetic slab as the function of radial coordinate r, taking into account that in the macroscopic scale the value of δ can be usually considered infinitely small, and the surface charge density can be expressed via the δ-function. Using Eqs. (28), (29) and (33), we obtain for the given value of effective surface current:

$$\rho_{rel}\left(r\right)=\frac{I_{ef}v_{ex}}{c^{2}h}\frac{R_{ex}\delta\left(r-R_{ex}\right)}{r}-\frac{I_{ef}v_{in}}{c^{2}h}\frac{R_{in}\delta\left(r-R_{in}\right)}{r}-\frac{2I_{ef}\omega}{c^{2}h}, \quad (34)$$

which is defined at $R_{in} \leq r \leq R_{ex}$, and where we have taken into account that the volume element is $dV = 2\pi hrdr$. We point out that in the adopted accuracy of calculations c^{-2}, the polarization charge density in the bulk of rotating magnetic medium does not depend on r. As example, in Fig. 2 we present a conditional graphic representation of the function (34) for the magnetic slab depicted in Fig. 1.

Another point is the physical interpretation of a non-vanishing charge density in the bulk of rotating magnetic medium. In this connection one should notice that, in general, the straightforward physical interpretation of trasformations (2), (3) can be given in the cases, where we deal with a real transport of charges (conduction current). However, it is not the case for non-conducting magnetic materials, where, classically speaking, the effective current is composed either by "spinning" or "orbiting" electrons in each elementary magnetic dipole, and the details of such magnetism can be rigorously described only at the quantum level. Nevertheless, the fact of the appearance of non-vanishing charge density in the bulk of a dielectric rotating magnet can be qualitatively understood in the semi-classical approach, if we take into account that for two neighbour elementary magnetic cells (e.g., for atomic dipoles), located respectively in the points r and $r + dr$, there appears a difference in the time dilation effect due to their different tangential velocities ωr and $\omega\left(r + dr\right)$.

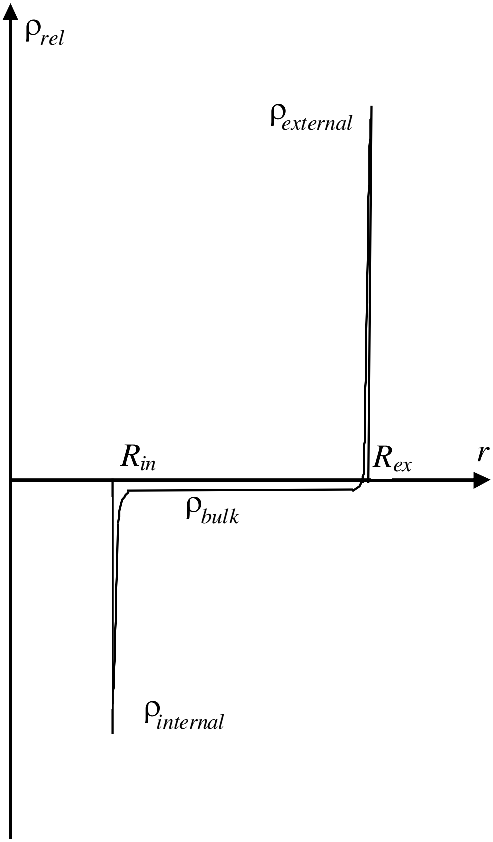

Figure 2. Charge density of relativistic polarization ρ_{rel} as the function of radial coordinate r for the rotating magnetic slab shown in Fig. 1. For the convenience of presentation, we show the value of ρ_{bulk} much larger than its actual magnitude. The integration over the area of $\rho(r)$ distribution gives zero due to the charge conservation law.

Hence the "orbital frequencies" of electrons around a nucleus in atomic dipoles located in the points r and $r + dr$, are not exactly equal to each other for a laboratory observer. Therefore, the "currents" generated by these electrons do not fully compensate each other in the adjacent elementary dipoles. Hence the non-vanishing charge density emerges on the boundary between such dipoles, leading finally to the non-vanishing charge density in the bulk of rotating magnetic medium.

We can add that in the case, where the magnetism is created by the aligned electron's spins, the non-vanishing charge density in the bulk of rotating magnet can be explained by different Thomas rotation for spin of electrons, having different radial coordinates. However, further analysis of this problem requires more information on the origin of magnetism in magnetic materials and falls outside the scope of the present paper.

3. CONCLUSION

Thus, using the physical interpretation of the Lorentz transformations for charge and current densities (2), (3), which we suggested earlier, and which is based on the continuity equation for the carriers of current in a closed moving circuit, we reanalysed the effect of relativistic polarization of rotating magnetic media. The consistency of such analysis is provided by the evaluation of the electromagnetic fields, charges and currents in the laboratory frame, where the measuring instrument (voltmeter) is at rest. We demonstrated the validity of application of Eq. (1) to any point of a rotating medium, which one more confirms the correctness of the result of W&W experiment. In addition, we have found that the relativistic polarization of the internal and external surfaces of a rotating magnetic slab (the first and second terms in the rhs of Eq. (34) is accompanied by the appearance of non-vanishing charge density in the bulk of the slab (the third term in rhs of Eq. (34), so that the net charge of the slab remains to be equal to zero. This result finally proves that the effect of relativistic polarization of moving magnetic dipoles (1) is fully compatible with the charge conservation law, both in the cases of translational and rotational motion of the dipoles.

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