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THE FIELD OF A MAGNETIC DIPOLE AND THE POLARIZABILITY OF A SUPERCONDUCTING OBJECT EMBEDDED IN THE INTERFACE BETWEEN MAGNETIC MATERIALS

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Abstract—In this paper, a careful study is made of the magnetostatic potential and field of a magnetic dipole embedded *in*, and with dipole moment parallel to, the interface between two magnetic regions. Unlike the case of a magnetic dipole perpendicular to the interface, the detailed position of the current of the dipole relative to the location of the interface has a profound effect on the value of the field produced away from the dipole. As a consequence, the question of defining and determining the magnetic polarizability of a superconducting object partially embedded in a magnetic interface is examined. The results of this paper are important for the proper modeling of arrays of scatterers embedded in an interface, such as frequency-selective surfaces (FSSs) and metafilms.

1. INTRODUCTION

In recent work [1, 2], Generalized Sheet Transition Conditions (GSTCs) for the electromagnetic fields at a metafilm (a surface array of small scatterers) have been obtained. Using these, plane wave reflection and transmission coefficients from the metafilm have been derived and their dependencies on scatterer geometry and incidence angle were investigated [3, 4]. The coefficients in the GSTCs depend on the density and polarizabilities of the scatterers that make up the metafilm. The analysis in [1, 3, 4] assumed that the metafilm is embedded in an infinite homogeneous medium, in which case the meaning of dipole moments

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and polarizabilities is well understood. When the metafilm is placed at an interface between two different media, the GSTCs must be modified to account for the influence of the interface. If the scatterers are infinitely thin, so that only tangential electric currents can be induced in them, the metafilm can only produce tangential electric dipoles and normal magnetic dipoles at the interface. In such a case, a simple modification of the GSTCs for a metafilm in a homogeneous medium will provide the correct result for the same metafilm at an interface. For the case when the scatterers are not thin, currents can be induced in the direction normal to the interface, and it is not clear what modifications must be made to the GSTCs.

In a previous paper [5], we have carried out a detailed study of the problem of a static electric dipole placed in the interface between dielectric media. In this study, it was found that either the excess or the net dipole moment may be the most important characteristic of an embedded dipole, depending on its orientation. The concepts learned from the study of the dipole moments were then used to provide a clear definition for the electric polarizability of a scatterer partially embedded in an interface. Our general result was illustrated for the case of a perfectly conducting spherical scatterer. Unlike the electric dipole case, however, little previous work has been done on the problem of a magnetic dipole in an interface between magnetic media [6–8], and a complete analysis is lacking. In this paper, we perform a detailed study of this problem. Our goals are to find a suitable unambiguous definition for the magnetic dipole moment and to determine correctly the resulting magnetic field. Based on these results, we will determine a proper specification of the polarizability of a superconducting object partially embedded in the interface between different magnetic media.

The paper is organized as follows. In Section 2, we obtain expressions for the magnetic vector potential and magnetic field for a certain model of a magnetic dipole located in a magnetic material interface. In Section 3, the distributions of free and magnetization currents resulting from this magnetic dipole are carefully considered and various possible definitions for the magnetic dipole moment are obtained. The jump condition for the fields at a surface distribution of magnetization in the interface is then studied in Section 4. In Section 5, the question of a proper definition for the magnetic polarizability is studied. For the special case of a superconducting object, symmetric with respect to a plane but otherwise arbitrary, whose magnetic field is known when placed in a uniform incident static magnetic field in free space, we find the field when the object is embedded symmetrically in the interface. From this, we are able to obtain the magnetic polarizability of the object in the interface in terms of its free space value. The case of a sphere is then presented as an illustrative example. We conclude with a discussion of the results and how they might apply to various electromagnetic modeling problems.

2. STATIC MAGNETIC DIPOLE AT THE INTERFACE BETWEEN TWO MEDIA

Let us first recall the static vector potential in an infinite homogeneous medium (whose permeability is μ) of a line element of current I flowing along a line segment described by the vector $\mathbf{d}$. As shown in many texts (see, for example, [9]):

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I \mathbf{d}}{4\pi d} \ln \left\{ \frac{\left|\mathbf{r} + \frac{\mathbf{d}}{2}\right| + \left|\mathbf{r} - \frac{\mathbf{d}}{2}\right| + d}{\left|\mathbf{r} + \frac{\mathbf{d}}{2}\right| + \left|\mathbf{r} - \frac{\mathbf{d}}{2}\right| - d} \right\} \tag{1}$$

where $\mathbf{r}$ is the position vector directed from the center of the line segment to the observation point. From the basic result (1) we can use superposition of several such current segments to obtain the vector potential of a closed current loop of area S. If we let $S \to 0$ and $I \to \infty$ in such as way that $m = IS$ remains constant, we obtain the vector potential of a magnetic dipole with magnetic dipole moment $\mathbf{m} = IS\mathbf{a}_n$ (where $\mathbf{a}_n$ is a unit vector perpendicular to the plane of the loop, in a direction given by the right-hand rule with respect to the current). This result will also be the far-field limit for the loop with nonzero area.

Omitting the details of the derivation (which can be found, e.g., in [10, p. 15, 11] or [12, p. 244]), we have in vector form

$$\mathbf{A}(\mathbf{r}) \simeq \frac{\mu \mathbf{m} \times \mathbf{r}}{4\pi r^3} \tag{2}$$

for $r = |\mathbf{r}| \gg h,\ d$, where $\mathbf{r}$ is the observation point measured from the center of the loop. The magnetic field can be calculated from the magnetic vector potential using

$$\mathbf{B} = \nabla \times \mathbf{A}\left(\mathbf{r}\right) \tag{3}$$

Since $\mathbf{m}$ is a constant, we may use a vector identity to evaluate

$$\nabla \times \left(\mathbf{m} \times \frac{\mathbf{r}}{r^3}\right) = \mathbf{m}\left(\nabla \cdot \frac{\mathbf{r}}{r^3}\right) - (\mathbf{m} \cdot \nabla)\frac{\mathbf{r}}{r^3} = -(\mathbf{m} \cdot \nabla)\frac{\mathbf{r}}{r^3} + 4\pi \mathbf{m}\delta(\mathbf{r}) \tag{4}$$

having used the fact that $\nabla \cdot \frac{\mathbf{r}}{r^3} = -\nabla^2(\frac{1}{r}) = 4\pi\delta(\mathbf{r})$, where $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional Dirac delta function. We thus have

$$\mathbf{B} = -\mu\left(\mathbf{m} \cdot \nabla\right) \frac{\mathbf{r}}{4\pi r^3} \qquad \text{for } r \neq 0 \tag{5}$$

In the remainder of this section, we examine how these results are affected by the presence of a plane interface between two magnetic materials.

2.1. Vector Potential of a Tangentially Oriented Magnetic Dipole

We will consider first a particular model for a magnetic dipole partially embedded in the interface between two media, with dipole moment parallel to the interface (a tangentially oriented dipole). The permeability is μ_1 in the upper medium ($z > 0$), and μ_2 in the lower medium ($z < 0$). We take the magnetic dipole to be a closed loop of six line current segments (labeled 1–6) carrying a current I in the directions shown in Fig. 1.

The current loop lies in the xz plane, and the width of the loop is d. The height of the portion of the loop in the upper medium is h_1, and h_2 in the lower medium, so that the partial loop areas are $S_1 = h_1 d$ and $S_2 = h_2 d$ in the upper and lower half-spaces respectively. We will find the static magnetic field as h_1, h_2 and $d \to 0$ and $I \to \infty$ such that the partial dipole moments $m_1 = S_1 I$ and $m_2 = S_2 I$ remain finite. To facilitate the calculation of the field, we will insert two additional current segments (labeled 7 and 8) just above and below the interface as shown, so that two loops (segments 1-2-3-7 and segments 4-5-6-8) are formed, each of which lies completely in one of the half-spaces. The contributions of these extra segments to the total field cancel each other out.

Equation (2) and image theory can now be used to determine the vector potential of a current loop near a material interface. Consider first a current distribution $\mathbf{J}(x, y, z)$ in $z > 0$ corresponding to the loop 1-2-3-7. Its field at an observation point in the upper half-space can

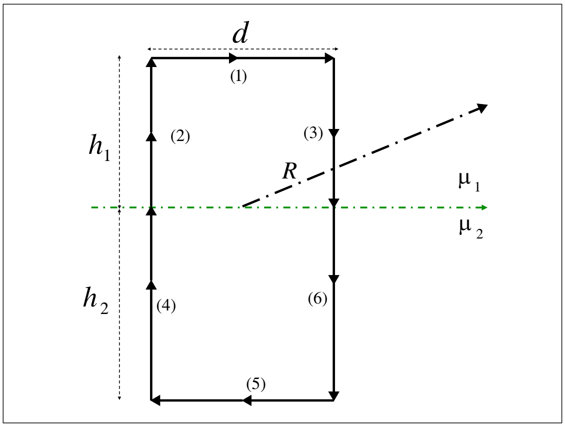

Figure 1. Current loop segments for a magnetic dipole in, and with dipole moment parallel to, a magnetic interface.

be found as the superposition of that due to the actual currents $\mathbf{J}$ and image currents $\mathbf{J}_{i1}$ given by [10, pp. 347–349, 13]:

$$\mathbf{J}_{i1}=\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\left[\mathbf{a}_x J_x\left(x,y,-z\right)+\mathbf{a}_y J_y\left(x,y,-z\right)-\mathbf{a}_z J_z\left(x,y,-z\right)\right] \quad (6)$$

acting in an infinite homogeneous space with permeability μ_1. The field observed in the lower half-space can be found using the image currents

$$\mathbf{J}_{i2}=\left(\frac{2\mu_1}{\mu_2+\mu_1}\right)\mathbf{J}\left(x,y,z\right) \quad (7)$$

acting in an infinite homogeneous space with permeability μ_2. A similar image representation holds for the field produced by currents located in the lower half-space (segments 4-5-6-8), and the total field is obtained by superposition.

We now calculate the vector potential $\mathbf{A}^{\parallel}$ of the partially embedded horizontal magnetic dipole of Fig. 1. In $z>0$, the potential comes from the upper loop (1-2-3-7) and its image acting in an infinite medium of permeability μ_1. The current in the horizontal segments of the image flows in the same direction as those of the original, while that in the vertical segments flows in the opposite direction, as shown in Fig. 2.

As a result, the direction of the magnetic dipole moment of the image loop is reversed from that of the original. The potential due to the lower loop (4-5-6-8) is found using (2) and (7), so that the total

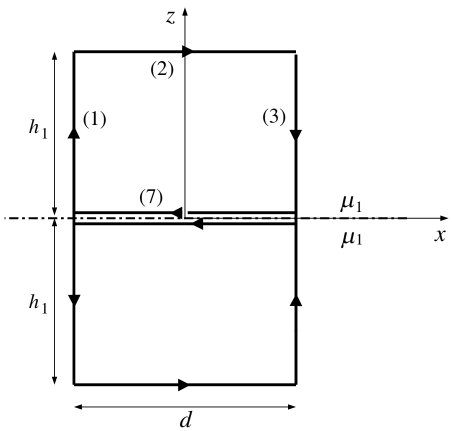

Figure 2. Current loop 1-2-3-7 and its image in a magnetic interface.

potential in $z > 0$ is found to be

$$\begin{aligned}\mathbf{A}^{\|} &\simeq \frac{\mu_1}{4\pi r^3}\left[1-\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\right]\mathbf{m}_1\times\mathbf{r}+\frac{\mu_2}{4\pi r^3}\left(\frac{2\mu_1}{\mu_2+\mu_1}\right)\mathbf{m}_2\times\mathbf{r}\\ &= \frac{\mu_1}{4\pi r^3}\left[\frac{2(\mu_1\mathbf{m}_1+\mu_2\mathbf{m}_2)}{\mu_2+\mu_1}\right]\times\mathbf{r}\end{aligned} \tag{8}$$

where $\mathbf{m}_1 = m_1\mathbf{a}_y$ and $\mathbf{m}_2 = m_2\mathbf{a}_y$. By a similar derivation, the potential in $z < 0$ is

$$\mathbf{A}^{\|} \simeq \frac{\mu_2}{4\pi r^3}\left[\frac{2(\mu_1\mathbf{m}_1+\mu_2\mathbf{m}_2)}{\mu_2+\mu_1}\right]\times\mathbf{r} \tag{9}$$

Denoting

$$\mathbf{m}_n = \frac{\mu_1\mathbf{m}_1+\mu_2\mathbf{m}_2}{\mu_0} \tag{10}$$

and

$$\mu_{\|} = \frac{\mu_1+\mu_2}{2} \tag{11}$$

we may write Equations (8)–(9) in the more compact form

$$\begin{aligned}\mathbf{A}^{\|}(\mathbf{r}) &\simeq \tfrac{\mu_1\mu_0}{\mu_{\|}}\tfrac{\mathbf{m}_n\times\mathbf{r}}{4\pi r^3} \quad \text{for} \quad z>0\\ &\simeq \tfrac{\mu_2\mu_0}{\mu_{\|}}\tfrac{\mathbf{m}_n\times\mathbf{r}}{4\pi r^3} \quad \text{for} \quad z<0\end{aligned} \tag{12}$$

2.2. Vector Potential of a Normally Oriented Magnetic Dipole

A normally oriented magnetic dipole can be modeled by two identically shaped current loops parallel to the interface (the xy plane). The first has a dipole moment $\mathbf{m}_1 = \mathbf{a}_z m_1$ and lies just above the interface, while the second has dipole moment $\mathbf{m}_2 = \mathbf{a}_z m_2$ and lies just under the interface. We can use image theory and superposition as in the case of the tangential dipole to compute the magnetic potential of the resulting composite loop in $z > 0$ or $z < 0$. For $|\mathbf{r}| = r$ large compared to the dimensions of the loop, the vector potential of an embedded magnetic dipole perpendicular to the interface can written as:

$$\mathbf{A}^{\perp}(\mathbf{r}) \simeq \frac{\mu_\perp}{4\pi r^3}(\mathbf{m}_e\times\mathbf{r}) \tag{13}$$

where

$$\mathbf{m}_e = \mathbf{m}_1+\mathbf{m}_2 \tag{14}$$

and

$$\mu_\perp = \frac{2\mu_1\mu_2}{\mu_1+\mu_2} \tag{15}$$

2.3. Magnetic Field of Tangentially and Normally Oriented Magnetic Dipoles in the Interface

Using the same procedure by which we obtained the magnetic field (5) of a dipole in an infinite homogeneous medium, we may calculate the field of a dipole partially embedded in an interface. The field of a tangentially oriented dipole can be written as:

$$\begin{aligned} \mathbf{B}^{\|}(r) &\simeq -\tfrac{\mu_1\mu_0}{4\pi\mu_{\|}}(\mathbf{m}_n\cdot\nabla)\tfrac{\mathbf{r}}{r^3} \quad \text{for} \quad z>0 \\ &\simeq -\tfrac{\mu_2\mu_0}{4\pi\mu_{\|}}(\mathbf{m}_n\cdot\nabla)\tfrac{\mathbf{r}}{r^3} \quad \text{for} \quad z<0 \end{aligned} \tag{16}$$

while

$$\mathbf{H}^{\|}(\mathbf{r}) \simeq -\frac{\mu_0}{4\pi\mu_{\|}}(\mathbf{m}_n\cdot\nabla)\frac{\mathbf{r}}{r^3} \tag{17}$$

In a similar way, the magnetic fields for the normally oriented dipole can be be obtained by substituting from (13) into (3), resulting in

$$\mathbf{B}^{\perp}(\mathbf{r}) \simeq -\frac{\mu_{\perp}}{4\pi}(\mathbf{m}_e\cdot\nabla)\frac{\mathbf{r}}{r^3} \tag{18}$$

and

$$\begin{aligned} \mathbf{H}^{\perp}(\mathbf{r}) &\simeq -\tfrac{\mu_{\perp}}{4\pi\mu_1}(\mathbf{m}_e\cdot\nabla)\tfrac{\mathbf{r}}{r^3} \quad \text{for} \quad z>0 \\ &\simeq -\tfrac{\mu_{\perp}}{4\pi\mu_2}(\mathbf{m}_e\cdot\nabla)\tfrac{\mathbf{r}}{r^3} \quad \text{for} \quad z<0 \end{aligned} \tag{19}$$

3. FREE AND BOUND CURRENT AND MAGNETIC DIPOLE MOMENTS

The magnetic dipole moment $\mathbf{m}_e$ appearing in the previous section is an excess dipole moment, computed from the conduction current (call it $\mathbf{J}_e$ here) only and not dependent on the permeabilities of the half spaces. However, in addition to the conduction currents there will also be a bound (magnetization) current distribution at the interface $z = 0$ between the two media as well as at the locations of $\mathbf{J}_e$. If these bound currents due to magnetization of the half-spaces are taken into account, we will obtain somewhat different total magnetic dipole moments, which we will see for some purposes more naturally describe the potentials and fields. We will find that $\mathbf{m}_n$, a weighted combination of excess dipole moments that does depend on the properties of the two media, is in fact a total dipole moment.

Let us first obtain a complete expression for the bound current density. The induced magnetization density $\mathbf{M}$ can be written as

$$\mathbf{M} = \left(\frac{\mu}{\mu_0} - 1\right)\mathbf{H} = \left\{\frac{\mu_1}{\mu_0}\vartheta(z) + \frac{\mu_2}{\mu_0}\left[1-\vartheta(z)\right] - 1\right\}\mathbf{H} \tag{20}$$

where ϑ is the unit step function

$$\begin{aligned} \vartheta(z) &= 1 \qquad z > 0 \\ &= 0 \qquad z < 0 \end{aligned} \tag{21}$$

The induced magnetization current density $\mathbf{J}_m$ is

$$\mathbf{J}_m = \nabla \times \mathbf{M} \tag{22}$$

Substituting (20) into (22) and using the vector identity for evaluating $\nabla \times (f\mathbf{A})$ for a scalar function f and a vector function $\mathbf{A}$, we obtain

$$\mathbf{J}_m = \left\{\frac{\mu_1}{\mu_0}\vartheta(z) + \frac{\mu_2}{\mu_0}[1 - \vartheta(z)] - 1\right\}\nabla \times \mathbf{H} + \left(\frac{\mu_1}{\mu_0} - \frac{\mu_2}{\mu_0}\right)\delta(z)\mathbf{a}_z \times \mathbf{H} \tag{23}$$

No delta function contribution arises from the discontinuity in H_z because only tangential derivatives of this field component are taken. The first term can be expressed in terms of the excess current density $\nabla \times \mathbf{H} = \mathbf{J}_e$. Rearranging the terms of (23) a little, we have

$$\mathbf{J}_m = -\mathbf{J}_e + \left\{\frac{\mu_1}{\mu_0}\vartheta(z) + \frac{\mu_2}{\mu_0}[1 - \vartheta(z)]\right\}\mathbf{J}_e + \left(\frac{\mu_1}{\mu_0} - \frac{\mu_2}{\mu_0}\right)\delta(z)\mathbf{a}_z \times \mathbf{H} \tag{24}$$

We now calculate the portion $\mathbf{m}_b$ of the magnetic dipole moment due only to the bound current density using (24):

$$\mathbf{m}_b = \frac{1}{2}\int \mathbf{r} \times \mathbf{J}_m dV = -\mathbf{m}_e + \mathbf{m}_n + \frac{1}{2}\left(\frac{\mu_1}{\mu_0} - \frac{\mu_2}{\mu_0}\right)\int \mathbf{r} \times (\mathbf{a}_z \times \mathbf{H}) dS \tag{25}$$

where $\mathbf{r}$ is the position vector, dS is a surface element in the xy-plane,

$$\mathbf{m}_e = \frac{1}{2}\int \mathbf{r} \times \mathbf{J}_e \, dV \tag{26}$$

is the excess magnetic dipole moment and

$$\mathbf{m}_n = \frac{1}{2}\frac{\mu_1}{\mu_0}\int_{z>0} \mathbf{r} \times \mathbf{J}_e \, dV + \frac{1}{2}\frac{\mu_2}{\mu_0}\int_{z<0} \mathbf{r} \times \mathbf{J}_e dV = \frac{\mu_1 \mathbf{m}_1 + \mu_2 \mathbf{m}_2}{\mu_0} \tag{27}$$

is the weighted dipole moment defined in (10). The last term of (25) is equal to zero. To show this for a dipole tangential to the interface, we examine (17), which tells us that the integrand is an odd function of one of the variables x and y and an even function of the other, causing the integral over the xy-plane to be zero. For a dipole normal to the interface, we find from (19) that $\mathbf{a}_z \times \mathbf{H}(z = 0)$ is zero, whence the integral is zero as well. Equation (25) thus reduces to

$$\mathbf{m}_b = -\mathbf{m}_e + \mathbf{m}_n \tag{28}$$

In other words, the weighted dipole moment $\mathbf{m}_n$ defined above is in fact the *net* (excess plus bound) dipole moment:

$$\mathbf{m}_n = \mathbf{m}_b + \mathbf{m}_e \tag{29}$$

and is the same as the weighted dipole moment used in the previous section.

We can summarize the results of this section and the previous one as follows:

1. The magnetic potential and magnetic field produced by a dipole parallel to the interface is proportional not to the excess magnetic dipole moment but to the net magnetic dipole moment.
2. The magnetic potential and magnetic field produced by a dipole normal to the interface is proportional to the excess magnetic dipole moment, regardless of how those dipole moments are split microscopically on either side of the interface.
3. If $\mu_1 > \mu_0$ and $\mu_2 > \mu_0$, then $|\mathbf{m}_n| > |\mathbf{m}_e|$ (a magnification effect occurs).

4. JUMP CONDITION AT SURFACE MAGNETIC DIPOLE LAYERS (MAGNETIZATION SHEETS)

We next consider a surface distribution of magnetic dipoles at a magnetic material interface. We follow the method of [14], but we will carefully distinguish between excess and bound surface magnetization density. Consider first an excess surface magnetization density $\mathbf{M}_{Se}$ (density of magnetic dipole moment per unit area) is located in a homogeneous medium of permeability μ at the plane $z = 0$. The excess magnetization density is then

$$\mathbf{M}_e = \mathbf{M}_{Se}\delta(z) \tag{30}$$

As in [14], we postulate representations for $\mathbf{H}$ as the sum of a term with a delta functions, one with a step function and the remainder continuous in z:

$$\mathbf{H} = \mathbf{H}_\delta\delta(z) + \mathbf{H}_\vartheta\vartheta(z) + \mathbf{H}_c \tag{31}$$

A similar form is postulated for $\mathbf{B} = \mu\mathbf{H}$. Note that the coefficients $\mathbf{M}_{Se}$, $\mathbf{H}_\delta$ and $\mathbf{B}_\delta$ of the delta functions are taken to be independent of z without loss of generality.

We substitute (30) and (31) into Maxwell's equations for the static magnetic field, and equate terms with the same order of singularity at $z = 0$ to each other. From $\nabla\cdot\mathbf{B} = 0$ we obtain

$$\mathbf{a}_z\cdot\mathbf{B}_\delta = 0 \tag{32}$$

$$\mathbf{a}_z\cdot\mathbf{B}_\vartheta + \nabla\cdot\mathbf{B}_\delta = 0 \tag{33}$$

from the terms in $\delta'(z)$ and $\delta(z)$ respectively. By substituting into $\nabla\times\mathbf{H} = \mathbf{J}_e = \nabla\times\mathbf{M}_e$, we find that

$$\mathbf{a}_z\times\mathbf{H}_\delta = \mathbf{a}_z\times\mathbf{M}_{Se} \tag{34}$$

$$\mathbf{a}_z\times\mathbf{H}_\vartheta + \nabla\times\mathbf{H}_\delta = \nabla\times\mathbf{M}_{Se} \tag{35}$$

From (32) we have

$$B_{\delta z} = 0 \tag{36}$$

and from (34) we get

$$\mathbf{H}_{\delta t} = \mathbf{M}_{Set} \tag{37}$$

where the subscript t denotes the tangential (xy) components of a vector. Equations (36) and (37) can now be substituted into (33) and (35) to get the step function parts of tangential **H** and normal **B** across the magnetization sheet:

$$\begin{aligned} \mathbf{a}_z \times \mathbf{H}_{\vartheta t} &= -\mathbf{a}_z \times \nabla_t M_{Sez} && (38) \\ B_{\vartheta z} &= -\mu \nabla_t \cdot \mathbf{M}_{Set} && (39) \end{aligned}$$

Now suppose an excess surface magnetization density $\mathbf{M}_{Se1}$ is located in medium 1 at the plane $z = z_1 > 0$, and $\mathbf{M}_{Se2}$ in medium 2 at the plane $z = z_2 < 0$. Jumps in tangential **H** and normal **B** occur across each sheet, given by the appropriate modifications of (38) and (39) respectively. Letting z_1 and z_2 go to zero, and since tangential **H** and normal **B** are continuous at the material interface, we add these jumps to get the total discontinuities across a composite magnetization sheet partially embedded in a material interface:

$$\begin{aligned} \mathbf{a}_z \times \mathbf{H}_t\big|_{z=0^-}^{0^+} &= -\mathbf{a}_z \times \nabla \left(M_{Se1z} + M_{Se2z}\right) && (40) \\ B_z\big|_{z=0^-}^{0^+} &= -\nabla_t \cdot \left(\mu_1 \mathbf{M}_{Se1t} + \mu_2 \mathbf{M}_{Se2t}\right) && (41) \end{aligned}$$

Note that the combination of magnetization appearing in (40) is analogous to (14) that naturally describes an embedded normal magnetic dipole, while that appearing in (41) is analogous to (10) for the tangential dipole.

5. MAGNETIC POLARIZABILITY OF A SYMMETRIC SCATTERER IN AN INTERFACE

We finally turn our attention to the question of how to define the magnetic polarizability dyadic for an object partially embedded in a magnetic interface, making clear all assumptions and definitions that we make.

5.1. Magnetic Polarizability of a Scatterer in an Infinite Homogeneous Medium

For simplicity, we limit our consideration to a superconducting scatterer (on which the normal component of **B** vanishes). First, let it be located in free space, subjected to a constant incident magnetic field given by $\mathbf{H}^i = \mathbf{B}^i/\mu_0$. The incident field induces free surface currents on the scatterer. Considering these conduction currents as excess currents, the excess magnetic dipole moment induced on the scatterer is

$$\mathbf{m}_e = \frac{1}{2}\int \mathbf{r} \times \mathbf{J}_{Se}\, dS = \frac{1}{2}\int \mathbf{r} \times (\mathbf{a}_n \times \mathbf{H})\; dS \tag{42}$$

where $\mathbf{H}$ and $\mathbf{B}$ are the *total* (incident plus induced) magnetic fields at the scatterer. The dyadic magnetic polarizability of the scatterer $\overleftrightarrow{\alpha}_{M0}$ in free space is then conventionally defined by the relation

$$\mathbf{m}_e = \frac{1}{\mu_0}\overleftrightarrow{\alpha}_{M0} \cdot \mathbf{B}^i \tag{43}$$

In free space, the excess magnetic dipole moment $\mathbf{m}_e$ and the net magnetic dipole moment $\mathbf{m}_n$ are the same.

If the same scatterer were placed in an infinite homogeneous magnetic material of permeability μ and subjected again to the same incident field $\mathbf{H}^i$ (but a generally different $\mathbf{B}^i$), the same induced free surface current density on the scatterer and the same excess magnetic dipole moment $\mathbf{m}_e$ will result, while the net magnetic dipole moment is different:

$$\mathbf{m}_n = \frac{\mu}{\mu_0}\mathbf{m}_e \tag{44}$$

In terms of the excess magnetic dipole moment, we have

$$\mathbf{m}_e = \overleftrightarrow{\alpha}_{M0} \cdot \mathbf{H}^i \tag{45}$$

while for the net dipole moment

$$\mathbf{m}_n = \frac{1}{\mu_0}\overleftrightarrow{\alpha}_{M0} \cdot \mathbf{B}^i \tag{46}$$

It would seem that either (45) or (46) could serve as the natural generalization of magnetic dipole moment to this case, and would have the advantage that the magnetic polarizability would be the same as in free space. But we could also write

$$\mathbf{m}_e = \frac{1}{\mu}\overleftrightarrow{\alpha}_{M0} \cdot \mathbf{B}^i \tag{47}$$

or

$$\mathbf{m}_n = \frac{\mu}{\mu_0}\overleftrightarrow{\alpha}_{M0} \cdot \mathbf{H}^i \tag{48}$$

As we will see below, there are advantages to these other definitions, especially when a magnetic interface must be taken into account.

5.2. Polarizability of a Scatterer at or Near a Magnetic Material Interface

If this same scatterer is now placed near (or partially embedded in) the interface $z = 0$ between the half-spaces with permeability μ_1 and μ_2 considered earlier, we must expect in general that the surface current density induced on the scatterer will not be simply related to what is induced when it is in free space. Moreover, the tangential incident field will be most naturally expressed by $\mathbf{H}_t^i$, while the normal incident field is most naturally B_z^i, since these components are the continuous ones when the scatterer is not present. On the other hand, as we have shown in the previous sections, the most natural magnetic dipole moments by means of which to compute the induced field of the scatterer are the normal excess and tangential net dipole moments. It therefore appears most reasonable to define the polarizability of a conducting scatterer in an interface by the equation

$$\mathbf{r}_m = \overleftrightarrow{\alpha}_M \cdot \mathbf{N}_m^i \tag{49}$$

where the generalized incident field vector $\mathbf{N}_m$ is defined by

$$\mathbf{N}_m^i = \begin{bmatrix} H_x^i \\ H_y^i \\ \frac{B_z^i}{\mu_0} \end{bmatrix} \tag{50}$$

and the generalized dipole vector is defined by

$$\mathbf{r}_m = \begin{bmatrix} m_{nx} \\ m_{ny} \\ m_{ez} \end{bmatrix} \tag{51}$$

In general, no simple relation exists between the magnetic polarizability of a scatterer in a homogeneous medium and that of the same scatterer embedded in or near an interface. We can expect that the components of the polarizability dyadic will vary continuously from those for the scatterer located in an infinite medium of permeability μ_1 to those when it is located in an infinite medium of permeability μ_2 as the position of the scatterer varies from $z = +\infty$ to $-\infty$. Little else can be said in general. However, if a certain degree of symmetry exists, some further results in this direction can be obtained, as will be explained in the following subsection. Otherwise, numerical techniques will have to be used, either specialized ones or general-purpose ones such as FDTD or finite-element methods, as we noted for the electric polarizability problem in [5].

5.3. Magnetic Polarizability of a Symmetric Superconducting Scatterer

Consider a superconducting scatterer that has reflection symmetry with respect to the plane $z = 0$. If it is placed in free space, we may consider two cases of excitation by a magnetic static field.

Case 1: If an incident z-directed magnetic field is imposed, given by

$$\mathbf{B}^i = \mathbf{a}_z B_z^i \tag{52}$$

then the total field will have a symmetry about $z = 0$: $B_z(x, y, -z) = B_z(x, y, z)$ and $\mathbf{B}_t(x, y, -z) = -\mathbf{B}_t(x, y, z)$, and similarly for $\mathbf{H}$. In particular,

$$\mathbf{B}(x, y, 0) = \mathbf{a}_z B_z(x, y, 0) \tag{53}$$

i.e., the total field is normal at the plane $z = 0$. The resulting induced dipole moment (either excess or net) of the scatterer will have only a z-component: $\mathbf{m} = \mathbf{a}_z m_z$.

Case 2: If the incident field is oriented in the tangential (xy) plane,

$$\mathbf{H}^i = \mathbf{H}_t^i \tag{54}$$

then the total field obeys the symmetry relations $H_z(x, y, -z) = -H_z(x, y, z)$ and $\mathbf{H}_t(x, y, -z) = \mathbf{H}_t(x, y, z)$, and similarly for $\mathbf{B}$. In particular,

$$\mathbf{H}(x, y, 0) = \mathbf{H}_t(x, y, 0) \tag{55}$$

i.e., the total field is tangential to the plane $z = 0$. The resulting induced magnetic dipole moment of the scatterer will in this case have only transverse components: $\mathbf{m} = \mathbf{m}_t$.

We conclude from these symmetry relations that the free-space magnetic polarizability dyadic for this scatterer has the form

$$\overleftrightarrow{\alpha}_{M0} = \overleftrightarrow{\alpha}^t_{M0} + \mathbf{a}_z \mathbf{a}_z \alpha^{zz}_{M0} \tag{56}$$

where $\overleftrightarrow{\alpha}^t_{M0}$ has only x and y components. These results are in agreement with those of Baum [15]. If the scatterer is placed in an infinite homogeneous magnetic material of permeability μ,

$$\overleftrightarrow{\alpha}_M = \frac{\mu}{\mu_0} \overleftrightarrow{\alpha}^t_{M0} + \frac{\mu_0}{\mu} \mathbf{a}_z \mathbf{a}_z \alpha^{zz}_{M0} \tag{57}$$

Eq. (57) follows from (47)–(51).

5.4. Magnetic Polarizability of a Symmetric Superconducting Scatterer Half Embedded in an Interface

Let this scatterer now be placed symmetrically in the interface $z = 0$ between two different magnetic materials as considered before.

Consider first the case when the incident $\mathbf{H}^i$ field is tangential to the interface and the same as in the free space case. Then the total $\mathbf{H}$ field must also be the same everywhere in space as it was when placed in the homogeneous medium, due to its continuity at $z = 0$. This implies that the excess surface current density $\mathbf{J}_{Se}$ (here equivalent to a free current density) and thus the excess magnetic dipole moment $\mathbf{m}_e$ on the scatterer will also be the same. The net surface current density $\mathbf{J}_{Sn}$ will be either $\frac{\mu_1}{\mu_0}\mathbf{J}_{Se}$ on the top portion of the scatterer in $z > 0$, or $\frac{\mu_2}{\mu_0}\mathbf{J}_{Se}$ on the bottom portion of the scatterer in $z < 0$. Adding the contributions from top and bottom, one can then show that the resulting net magnetic dipole moment for the scatterer embedded symmetrically in the interface can be written in terms of the magnetic dipole moment in free space:

$$\mathbf{m}_n = \frac{1}{2}\left(\frac{\mu_1}{\mu_0} + \frac{\mu_2}{\mu_0}\right)\mathbf{m}_e = \frac{\mu_\parallel}{\mu_0}\mathbf{m}_e \tag{58}$$

On the other hand, if the incident magnetic field is normal to the interface, assume that it is now $\mathbf{B}^i$ that is the same as in the free space case. The total $\mathbf{B}$ field will be unchanged from its value when the scatterer is in free space, because it is normal to and continuous at $z = 0$. The $\mathbf{H}$ field, however, is either μ_0/μ_1 (in the upper half-space) or μ_0/μ_2 (in the lower half-space) larger than its value in free space. Therefore, the excess current density on the superconducting scatterer also increases by those factors. As a result, $\mathbf{m}_n$ is unchanged in this case, while combining the top and bottom contributions to the excess dipole moment gives

$$\mathbf{m}_e = \frac{1}{2}\left(\frac{\mu_0}{\mu_1} + \frac{\mu_0}{\mu_2}\right)\mathbf{m}_n = \frac{\mu_0}{\mu_\perp}\mathbf{m}_n \tag{59}$$

From these results, we conclude that the magnetic polarizabilities for the halfway-embedded symmetric scatterer are expressible in terms of the free-space magnetic polarizabilities as

$$\overleftrightarrow{\alpha}_M^t = \frac{\mu_\parallel}{\mu_0}\overleftrightarrow{\alpha}_{M0}^t \tag{60}$$

$$\alpha_M^{zz} = \frac{\mu_0}{\mu_\perp}\alpha_{M0}^{zz} \tag{61}$$

5.5. Magnetic Polarizability of a Superconducting Sphere Symmetrically Embedded in an Interface

We now apply these results to the case of a superconducting sphere embedded halfway in the interface between the media. This will be accomplished by modifying the well-known solution for a magnetic sphere located in infinite free space subjected to a magnetostatic incident field (see, e.g., [16]) and applying (60)–(61). If the sphere is placed into a uniform incident field $\mathbf{H}^i = \mathbf{a}_z H_0$, the total magnetic field can be written as the negative of the gradient of a scalar magnetic potential as follows:

$$\mathbf{H} = -\nabla\Psi = -\frac{\partial\Psi}{\partial r}\mathbf{a}_r - \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\mathbf{a}_\theta \tag{62}$$

The scalar potential can be written as:

$$\Psi = -H_0 r\cos\theta + \sum_{l=0}^{\infty}\frac{\alpha_l}{r^{l+1}}P_l\left(\cos\theta\right) \tag{63}$$

where a is the radius of the sphere (see Fig. 3).

Applying the boundary condition

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=a^+} = 0 \tag{64}$$

to (63) determines the coefficients α_l in the expansion of the potential, all of which vanish except for $l = 1$. We obtain:

$$\Psi = -H_0\cos\theta\left(r + \frac{a^3}{2r^2}\right) \tag{65}$$

Applying Eq. (62) to Eq. (65), the magnetic field can be written as

$$\mathbf{H} = H_0\left[\mathbf{a}_r\cos\theta\left(1 - \frac{a^3}{r^3}\right) - \mathbf{a}_\theta\sin\theta\left(1 + \frac{a^3}{2r^3}\right)\right] \tag{66}$$

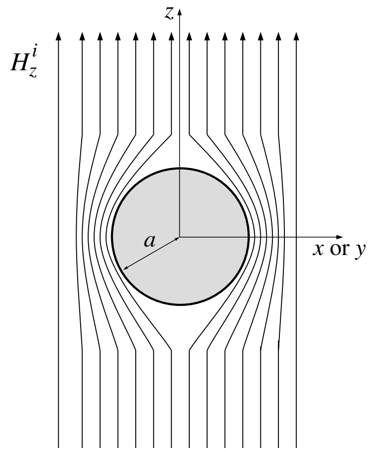

Figure 3. Superconducting sphere in a uniform incident magnetic field.

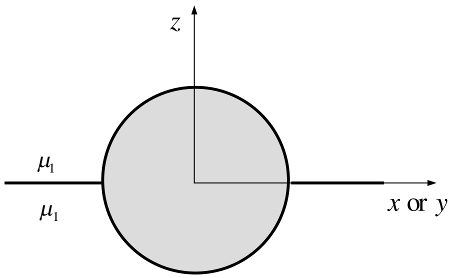

Figure 4. Superconducting sphere in the interface between magnetic materials.

The magnetic dipole moment of the superconducting sphere can be calculated from:

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r} \times \mathbf{J}_S \, dS = \frac{1}{2}\int \mathbf{r} \times (\mathbf{a}_r \times \mathbf{H}) \, dS = -\frac{a}{2}\int \mathbf{a}_\theta H_\theta \, dS \quad (67)$$

where $\mathbf{J}_S$ is the surface current on the sphere. Substituting (66) into (67),

$$\mathbf{m} = \frac{3a^3}{4} H_0 \iint \mathbf{a}_\theta \sin^2\theta d\theta d\phi - \frac{3\pi a^3}{2} H_0 \mathbf{a}_z \int \sin^3\theta d\theta = -2\pi a^3 H_0 \mathbf{a}_z \quad (68)$$

From (68) and (45), the zz component of the polarizability of the sphere is

$$\alpha_{M0}^{zz} = -2\pi a^3 \quad (69)$$

and from the symmetry of a sphere we have

$$\alpha_{M0}^{xx} = \alpha_{M0}^{yy} = -2\pi a^3 \quad (70)$$

as well.

From (49)–(51) and (60)–(61), the polarizability of a sphere embedded halfway in the interface, as seen in Fig. 4 can now be written as:

$$\overleftrightarrow{\alpha}_M = \left(-2\pi a^3\right) \begin{bmatrix} \frac{\mu_\parallel}{\mu_0} & 0 & 0 \\ 0 & \frac{\mu_\parallel}{\mu_0} & 0 \\ 0 & 0 & \frac{\mu_0}{\mu_\perp} \end{bmatrix} \quad (71)$$

6. CONCLUSION

In this paper, we have considered the effects of a material interface on the fields produced by a magnetic dipole partially embedded in that interface. We believe that our approach provides an unambiguous way of accounting for the scattering by small particles at an interface, and will enable the proper treatment of arrays of such particles at an interface. Specifically this work together with that of [5] will permit an extension of our previous work in [1–4] to obtain GSTCs for a metafilm embedded in an interface. Although our study largely concerns the macroscopic field, we found that it is dependent in a fundamental way on the microscopic details of the positioning of the scatterer in the interface. It is thus important to consider this effect carefully when modeling small particles at an interface.

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