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EXACT CLOSED FORM FORMULA FOR SELF INDUCTANCE OF CONDUCTOR OF RECTANGULAR CROSS SECTION

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Abstract—In this paper, self inductance for a conductor with rectangular cross section is investigated. Using the three-dimensional Fredholm's integral equation of the second kind with weakly singular kernel we obtain an equation for the complex voltage drop in the conductor. Self impedance appearing in the equation is expressed in the form of integral relation for any current density distribution. The imaginary part of this impedance divided by angular frequency is the self inductance of a conductor of any shape and finite length. In the case of direct current (DC), low frequency (LF) or thin strip conductor of rectangular cross section the formulae for the self inductances are given for any length and for length much greater than the other dimensions.

1. INTRODUCTION

The real lumped isolated conductor can be modeled as a connection, in series or in parallel, of a resistance and a self inductance. Self inductance plays an important role not only in power circuits, but also in transmission lines, interconnections in many microwave and digital printed circuit board (PCB) lands and striplines [1–11].

Formulae for self inductances of conductors of rectangular cross-section are the subject of many electrical papers and books. As for DC, power frequency or a thin isolated rectangular conductor with width a, thickness b, and length l as shown in Fig. 1, there are many formulae for its self inductance. The most significant are: Grover's

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given in [2, 3, 12, 13], Bueno and Assis' shown in [10], FastHenry's introduced in [13], Ruehli's presented in [3, 11, 13] as well as Hoer and Love's shown in [3, 13, 14].

In general cases, there are two methods to calculate self inductance: the first one is the calculation of inductance of a current-carrying closed loop and the second is the calculation of induction of a segment of a current loop using the concept of partial inductance [3]. In this paper, a new method for calculating self inductance is presented. The method results in the Fredholm's integral equation. We compare our formulae with several well-known ones given in the literature.

2. INTEGRAL EQUATION

In case of a rectilinear conductor with length l, conductivity σ, cross section S with sinusoidal current input function with angular frequency ω and complex value $\underline{I}$ (Fig. 1), the vector of current density has one component along the Oz axis, that is $\underline{\mathbf{J}}(X) = \mathbf{a}_z \underline{J}(X)$. Then the vector magnetic potential $\underline{\mathbf{A}}(X) = \mathbf{a}_z \underline{A}(X)$ and total electric field $\underline{\mathbf{E}}(X) = \mathbf{a}_z \underline{E}(X)$.

The total electric field may be presented [15–17] in the form of two electric fields sum

$$\underline{E}(X) = \underline{E}_{st}(X) + \underline{E}_{in}(X) \tag{1}$$

where $\underline{E}_{st}(X)$ is the so-called quasi-static source whilst $\underline{E}_{in}(X)$ is an induced electric field.

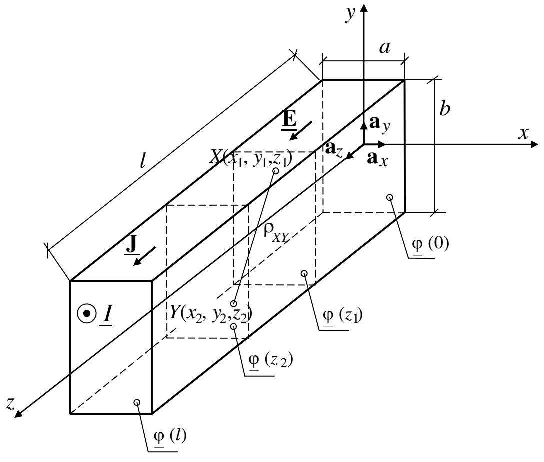

Figure 1. A conductor of rectangular cross section with width a, thickness b, length l, conductivity σ and complex current $\underline{I}$.

The quasi-static source electric field

$$\underline{\mathbf{E}}_{st}(X) = -\mathbf{grad}\underline{\varphi}(z) \tag{2}$$

where $\underline{\varphi}(z)$ is the complex scalar electric potential along a straight conductor of rectangular cross section of length l (Fig. 1). This potential must satisfy Laplace's equation

$$\frac{\mathrm{d}^2\underline{\varphi}(z)}{\mathrm{d}z^2} = 0 \tag{3}$$

the solution of which is [15–17]

$$\underline{\varphi}(z) = \frac{\underline{\varphi}(0) - \underline{\varphi}(l)}{l} z + \underline{\varphi}(0) \tag{4}$$

By introducing a unit voltage drop (in $\mathrm{V} \cdot \mathrm{m}^{-1}$) in a conductor

$$\underline{u} = \frac{\underline{\varphi}(l) - \underline{\varphi}(0)}{l} \tag{5}$$

we obtain from the formula (4)

$$\underline{\varphi}(z) = -\underline{u}\, z + \underline{\varphi}(0) \tag{6}$$

then from it and from the formula (2)

$$\underline{E}_{st}(X) = -\frac{\mathrm{d}\underline{\varphi}(z)}{\mathrm{d}z} = \underline{u} \tag{7}$$

The induced electric field $\underline{E}_{in}(X)$ is determined by $\underline{E}_{in}(X) = -\mathrm{j}\omega\underline{A}(X)$, where $\underline{A}(X)$ is a vector magnetic potential and

$$\underline{A}(X) = \frac{\mu_0}{4\pi} \int\limits_v \frac{\underline{J}(Y)}{\rho_{XY}} \mathrm{d}x_2\, \mathrm{d}y_2\, \mathrm{d}z_2 \tag{8}$$

where: $X = X(x_1, y_1, z_1)$ is the point of observation and $X \in R^3$, $Y = Y(x_2, y_2, z_2)$ is the source point and $Y \in S$, v — conductor's volume, $\rho_{XY} = \sqrt{r_{XY}^2 + (z_2 - z_1)^2}$ is the distance between the point of observation X and the source point Y (Fig. 1), where $r_{XY} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Finally (1) becomes

$$\underline{E}(X) + \mathrm{j}\,\omega\underline{A}(X) = \underline{u} \tag{9}$$

and from it, after substituting $\underline{E}(X) = \frac{\underline{J}(X)}{\sigma}$ and $\underline{A}(X)$ from (8), for $X \in v$ we have

$$\frac{\underline{J}(X)}{\sigma} + \mathrm{j}\omega\frac{\mu_0}{4\pi} \int\limits_v \frac{\underline{J}(Y)}{\rho_{XY}} \mathrm{d}x_2\, \mathrm{d}y_2\, \mathrm{d}z_2 = \underline{u} \tag{10}$$

Equation (10) is a three-dimensional Fredholm's integral equation of a second type with a weak-singular kernel $\frac{1}{\rho_{XY}}$ and it has an unequivocal solution for any (not only constant) right side [16, 17].

3. DEFINITION OF SELF INDUCTANCE

We multiply (10) by the complex conjugate value $\underline{J}^*(X)$ and integrate over the volume v of this conductor. Assuming that $\underline{J}(X)$ does not depend on the variable z and taking into consideration that $\underline{U} = \underline{u}\, l$ is the voltage drop in the conductor (in V) as well as that $I^2 = \underline{I}\,\underline{I}^*$, we may divide both sides of the final equation by $\underline{I}^*$. Then we obtain the equation

$$\underline{U} = \underline{Z}\,\underline{I} = (R + \mathrm{j}\,\omega L)\,\underline{I} \tag{11}$$

where the self-inductance of the conductor

$$L = \frac{1}{\omega}\,\mathrm{Im}\,\underline{Z} = \frac{\mu_0}{4\pi I^2}\int\limits_v\int\limits_v \frac{\underline{J}(Y)\,\underline{J}^*(X)}{\rho_{XY}}\,\mathrm{d}v\,\mathrm{d}v \tag{12}$$

One can see from (12) that the self inductance of the conductor depends on the distribution of the current density in this conductor. The formula (12) specifies the self-inductance of a solid conductor as a parameter standing next to jω in (11) which specifies the voltage drop and from the point of theory of circuits it is called [16, 17] self inductance. It may not be associated with a closed loop (according to the classical view of self-inductance of a closed circuit) but it should be merely considered as a quantity helpful in calculating self inductances of real closed electrical circuits. The above formula can be also deduced from the magnetic energy of a loop carrying current $\underline{I}$ [5].

If a conductor has a constant cross-sectional area S along its length and in the case of DC, low frequency or for a thin strip conductor we can assume that the current density is uniform and given as $\underline{J}(X) = \underline{I}/S$ and then, from (12), we obtain the self inductance of a straight conductor

$$L = \frac{\mu_0}{4\pi S^2}\int\limits_v\int\limits_v \frac{1}{\rho_{XY}}\,\mathrm{d}v\,\mathrm{d}v \tag{13}$$

4. SELF INDUCTANCE OF A CONDUCTOR OF RECTANGULAR CROSS SECTION

The self inductance of a rectangular conductor of dimensions $a \times b \times l$ shown in Fig. 1 is given by the formula

$$L = \frac{\mu_0}{4\pi}\frac{1}{a^2 b^2}F \tag{14}$$

where

$$F = \int\limits_0^l\int\limits_0^l\int\limits_0^b\int\limits_0^b\int\limits_0^a\int\limits_0^a \frac{1}{\rho_{XY}}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z_1\,\mathrm{d}z_2 \tag{15}$$

is a sixtuple definite integral of an integrable function ρ_{XY}^{-1} of six variables $(x_1, x_2, y_1, y_2, z_1, z_2)$. In general cases this integral is very difficult to calculate. But in our case we can put $x = x_2 - x_1$, $y = y_2 - y_1$, $z = z_2 - z_1$ and first calculate a sixtuple indefinite integral

$$F(x, y, z) = \iint \iint \iint \frac{1}{\rho_{XY}(x, y, z)} \, \mathrm{d}x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}z \qquad (16)$$

of a function

$$\rho_{XY}^{-1}(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}} \qquad (17)$$

of three variables (x, y, z) — twice with respect to x, twice with respect to y and twice with respect to z. After each double integration we omit terms which depend only on one or two variables — they are constants with respect to the considered variable.

If two variables, for example x_1 and x_2, can be replaced with only one variable $x = x_2 - x_1$ then a double definite integral can be calculated from following formula

$$\begin{aligned} H &= \int_{s_3}^{s_4} \int_{s_1}^{s_2} f(x_2 - x_1, y, z) \mathrm{d}x_2 \, \mathrm{d}x_1 \\ &= H(s_4 - s_1) - H(s_4 - s_2) + H(s_3 - s_2) - H(s_3 - s_1) \\ &= [H(x)] \begin{matrix} {\scriptstyle s_4 - s_1, s_3 - s_2} \\ (x) \\ {\scriptstyle s_4 - s_2, s_3 - s_1} \end{matrix} = [H(x)] \begin{matrix} {\scriptstyle p_1,\ p_3} \\ (x) \\ {\scriptstyle p_2, p_4} \end{matrix} = \sum_{i=1}^{i=4} (-1)^{i+1} H(p_k) \end{aligned} \qquad (18)$$

where

$$H(x, y, z) = \iint f(x, y, z) \mathrm{d}x \, \mathrm{d}x \qquad (19)$$

is an indefinite integral of $f(x, y, z)$. So in (16), we can also omit terms proportional to one variable like $H(x, y, z) = xg(y, z)$. Finally, after a lengthy integration, (16) yields an expression for the sixtuple indefinite integral

$$\begin{aligned} F(x, y, z) &= \iint \iint \iint \frac{1}{\rho_{XY}(x, y, z)} \, \mathrm{d}x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}z \\ &= \iint \iint \iint \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, \mathrm{d}x \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}z \end{aligned}$$

$$= \frac{1}{72}\left\{\begin{array}{l} \frac{6}{5}\left(x^4+y^4+z^4-3x^2y^2-3x^2z^2-3y^2z^2\right)\sqrt{x^2+y^2+z^2} \\ -12xyz\left(z^2\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right. \\ \left.+y^2\tan^{-1}\frac{xz}{y\sqrt{x^2+y^2+z^2}}+x^2\tan^{-1}\frac{yz}{x\sqrt{x^2+y^2+z^2}}\right) \\ -3x(y^4-6y^2z^2+z^4)\ln\left(x+\sqrt{x^2+y^2+z^2}\right) \\ -3y(x^4-6x^2z^2+z^4)\ln\left(y+\sqrt{x^2+y^2+z^2}\right) \\ -3z(x^4-6x^2y^2+y^4)\ln\left(z+\sqrt{x^2+y^2+z^2}\right) \end{array}\right\} \quad (20)$$

and the self inductance of the conductor of rectangular cross section is given by following formula

$$\begin{aligned} L &= \frac{\mu_0}{4\pi}\frac{1}{a^2b^2}\left[\left[\left[F(x,y,z)\right]\begin{matrix}a,-a\\(x)\\0,0\end{matrix}\right]\begin{matrix}b,-b\\(y)\\0,0\end{matrix}\right]\begin{matrix}l,-l\\(z)\\0,0\end{matrix} \\ &= \frac{\mu_0}{4\pi}\frac{1}{a^2b^2}\sum_{i=1}^{i=4}\sum_{j=1}^{j=4}\sum_{k=1}^{k=4}(-1)^{i+j+k+1}F(p_i,q_j,r_k) \end{aligned} \quad (21)$$

where $p_1 = a$, $p_3 = -a$, $q_1 = b$, $q_3 = -b$, $r_1 = l$, $r_3 = -l$ and $p_2 = p_4 = q_2 = q_4 = r_2 = r_4 = 0$.

On the basis of (21), we have an analytical formula for the self inductance of the straight conductor of rectangular cross section

$$\begin{aligned} L = &\frac{\mu_0}{120\pi a^2b^2} \\ &\left\{\begin{array}{l} 4\left(a^5+b^5+l^5\right)-4\left(a^4-3a^2b^2+b^4\right)\sqrt{a^2+b^2} \\ -4\left(a^4-3a^2l^2+l^4\right)\sqrt{a^2+l^2}-4\left(b^4-3b^2l^2+l^4\right)\sqrt{b^2+l^2} \\ +4(a^4+b^4+l^4-3a^2b^2-3a^2l^2-3b^2l^2)\sqrt{a^2+b^2+l^2} \\ -40abl\left[a^2\tan^{-1}\frac{bl}{a\sqrt{a^2+b^2+l^2}}+b^2\tan^{-1}\frac{al}{b\sqrt{a^2+b^2+l^2}}+l^2\tan^{-1}\frac{ab}{l\sqrt{a^2+b^2+l^2}}\right] \\ +30abl\left[ab\ln\frac{l+\sqrt{a^2+b^2+l^2}}{-l+\sqrt{a^2+b^2+l^2}}+al\ln\frac{b+\sqrt{a^2+b^2+l^2}}{-b+\sqrt{a^2+b^2+l^2}}+bl\ln\frac{a+\sqrt{a^2+b^2+l^2}}{-a+\sqrt{a^2+b^2+l^2}}\right] \end{array}\right\} \\ +&\frac{\mu_0}{120\pi a^2b^2} \\ &\left\{\begin{array}{l} -5a\left[b^4\ln\frac{\left(-a+\sqrt{a^2+b^2}\right)\left(a+\sqrt{a^2+b^2+l^2}\right)}{\left(a+\sqrt{a^2+b^2}\right)\left(-a+\sqrt{a^2+b^2+l^2}\right)}+l^4\ln\frac{\left(-a+\sqrt{a^2+l^2}\right)\left(a+\sqrt{a^2+b^2+l^2}\right)}{\left(a+\sqrt{a^2+l^2}\right)\left(-a+\sqrt{a^2+b^2+l^2}\right)}\right] \\ -5b\left[a^4\ln\frac{\left(-b+\sqrt{a^2+b^2}\right)\left(b+\sqrt{a^2+b^2+l^2}\right)}{\left(b+\sqrt{a^2+b^2}\right)\left(-b+\sqrt{a^2+b^2+l^2}\right)}+l^4\ln\frac{\left(-b+\sqrt{b^2+l^2}\right)\left(b+\sqrt{a^2+b^2+l^2}\right)}{\left(b+\sqrt{b^2+l^2}\right)\left(-b+\sqrt{a^2+b^2+l^2}\right)}\right] \\ -5l\left[a^4\ln\frac{\left(-l+\sqrt{a^2+l^2}\right)\left(l+\sqrt{a^2+b^2+l^2}\right)}{\left(l+\sqrt{a^2+l^2}\right)\left(-l+\sqrt{a^2+b^2+l^2}\right)}+b^4\ln\frac{\left(-l+\sqrt{b^2+l^2}\right)\left(l+\sqrt{a^2+b^2+l^2}\right)}{\left(l+\sqrt{b^2+l^2}\right)\left(-l+\sqrt{a^2+b^2+l^2}\right)}\right] \end{array}\right\} \end{aligned} \quad (22)$$

5. SELF INDUCTANCE OF A THIN TAPE

The self inductance of a thin tape of width a, thickness $b \approx 0$ and length l is given by formula

$$L = \frac{\mu_0}{4\pi}\frac{1}{a^2}F \tag{23}$$

where

$$F = \int_0^l \int_0^l \int_0^a \int_0^a \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}z_1 \mathrm{d}z_2}{\sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2}} \tag{24}$$

is a quadruple definite integral of four variables (x_1, x_2, z_1, z_2). Now we can put $x = x_2 - x_1$ and $z = z_2 - z_1$ and first calculate a quadruple indefinite integral

$$F(x, z) = \iint \iint \frac{\mathrm{d}x\,\mathrm{d}x\ \mathrm{d}z\,\mathrm{d}z}{\sqrt{x^2 + z^2}} \tag{25}$$

twice with respect to x and twice with respect to z. Finally, after a lengthy integration, (25) yields an expression for quadruple indefinite integral

$$\begin{aligned} F(x, z) &= \frac{1}{2}\left[-\frac{3}{2}z\left(x^2 + z^2\right) - \frac{1}{3}\left(x^2 + z^2\right)^{3/2} \right. \\ &\left. + xz^2 \ln\left(x + \sqrt{x^2 + z^2}\right) + x^2 z \ln\left(z + \sqrt{x^2 + z^2}\right) \right] \end{aligned} \tag{26}$$

So the self inductance of the thin tape is given by the following formula

$$L = \frac{\mu_0}{4\pi}\frac{1}{a^2}\left[[F(x, z)] \begin{matrix} a,-a \\ (x) \\ 0,0 \end{matrix} \right] \begin{matrix} l,-l \\ (z) \\ 0,0 \end{matrix} = \frac{\mu_0}{4\pi}\frac{1}{a^2} \sum_{i=1}^{i=4} \sum_{k=1}^{k=4} (-1)^{i+k} F(p_i, r_k) \tag{27}$$

On the basis of (27) we have an analytical formula for the self inductance of the thin tape

$$\begin{aligned} L &= \frac{\mu_0}{6\pi a^2} \left[3a^2 l \ln \frac{l + \sqrt{l^2 + a^2}}{a} - \left(l^2 + a^2\right)^{3/2} \right. \\ &\left. + 3al^2 \ln \frac{a + \sqrt{l^2 + a^2}}{l} + l^3 + a^3 \right] \end{aligned} \tag{28}$$

It is exactly the Hoer's formula for the thin tape given in [14].

6. CASE OF LENGTH OF CONDUCTOR MUCH GREATER THAN THE OTHER DIMENSIONS

The double definite integral is

$$f(x,y)=\int_0^l\int_0^l \frac{1}{\rho_{XY}}\,\mathrm{d}z_1\,\mathrm{d}z_2=2l\left(\ln\frac{l+\sqrt{l^2+r_{XY}^2}}{r_{XY}}-\frac{\sqrt{l^2+r_{XY}^2}}{l}+\frac{r_{XY}}{l}\right) \tag{29}$$

If $l \gg r_{XY}$ the function $f(x,y)$ becomes

$$f(x,y)=2l\left(\ln\frac{2l}{r_{XY}}-1\right) \tag{30}$$

and the self inductance of the rectangular conductor is expressed by the formula

$$L=\frac{\mu_0 l}{2\pi}\left[\ln(2l)-1+G\right] \tag{31}$$

where

$$G=-\frac{1}{2a^2b^2}\int_0^b\int_0^b\int_0^a\int_0^a \ln\left[(x_2-x_1)^2+(y_2-y_1)^2\right]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2 \tag{32}$$

After calculating quadruple indefinite integral

$$\begin{aligned} G(x,y) &= -\frac{1}{2a^2b^2}\iint\iint \ln\left[x^2+y^2\right]\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}y \\ &= \frac{1}{288a^2b^2}\left\{150x^2y^2-6\left[8xy^3\tan^{-1}\frac{x}{y}+8x^3y\tan^{-1}\frac{y}{x}\right.\right. \\ &\quad \left.\left.-\left(x^4-6x^2y^2+y^4\right)\ln\left(x^2+y^2\right)\right]\right\} \end{aligned} \tag{33}$$

we determine the self inductance of the long conductor of rectangular cross section

$$\begin{aligned} L &= \frac{\mu_0 l}{2\pi}\left\{\ln(2l)-1+\left[\left[G(x,y)\right]\begin{matrix}a,-a\\(x)\\0,\ 0\end{matrix}\right]\begin{matrix}b,-b\\(y)\\0,0\end{matrix}\right\} \\ &= \frac{\mu_0 l}{2\pi}\left\{\ln(2l)-1+\sum_{i=1}^{i=4}\sum_{j=1}^{j=4}(-1)^{i+j}\,G(p_i,q_j)\right\} \end{aligned} \tag{34}$$

On the basis of (34) we have the analytical formulae for the self inductance of the straight long conductor of rectangular cross section

$$L = \frac{\mu_0 l}{2\pi}\left\{\ln\frac{2l}{a} + \frac{13}{12} - \frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{a}{b}\tan^{-1}\frac{b}{a}\right]\right.$$
$$\left. + \frac{1}{12}\left[\left(\frac{a}{b}\right)^2 - 6 + \left(\frac{b}{a}\right)^2\right]\ln\left[1 + \left(\frac{a}{b}\right)^2\right] + \frac{1}{6}\left[6 - \left(\frac{a}{b}\right)^2\right]\ln\frac{a}{b}\right\} \quad (35)$$

or

$$L = \frac{\mu_0 l}{2\pi}\left\{\ln\frac{2l}{b} + \frac{13}{12} - \frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{a}{b}\tan^{-1}\frac{b}{a}\right]\right.$$
$$\left. + \frac{1}{12}\left[\left(\frac{a}{b}\right)^2 - 6 + \left(\frac{b}{a}\right)^2\right]\ln\left[1 + \left(\frac{b}{a}\right)^2\right] + \frac{1}{6}\left[6 - \left(\frac{b}{a}\right)^2\right]\ln\frac{b}{a}\right\} \quad (36)$$

as well as

$$L = \frac{\mu_0 l}{2\pi}\left\{\ln\frac{2l}{a+b} + \frac{13}{12} - \frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{a}{b}\tan^{-1}\frac{b}{a}\right] + \frac{1}{2}\ln\left[1 + \frac{a}{b}\frac{2}{1 + \left(\frac{a}{b}\right)^2}\right]\right.$$
$$\left. + \frac{1}{12}\left[\left(\frac{a}{b}\right)^2\ln\left[1 + \left(\frac{b}{a}\right)^2\right] + \left(\frac{b}{a}\right)^2\ln\left[1 + \left(\frac{a}{b}\right)^2\right]\right]\right\} \quad (37)$$

7. COMPUTATIONAL RESULTS

For fast digital computations the following normalizations are introduced: $u = l/a$ and $w = b/a$. From (22) the per-unit-length self inductance of the straight conductor of rectangular cross section is then

$$\frac{L}{l} = \frac{\mu_0}{\pi}\left\{\begin{array}{l} \frac{1}{30}\left[\begin{array}{l} \left(\frac{1}{uw^2} + \frac{w^3}{u} + \frac{u^4}{w^2}\right) - \left(\frac{1}{uw^2} - 3\frac{1}{u} + \frac{w^2}{u}\right)A_2 \\ - \left(\frac{1}{uw^2} - 3\frac{u}{w^2} + \frac{u^3}{w^2}\right)A_1 - \left(\frac{w^2}{u} - 3u + \frac{u^3}{w^2}\right)A_3 \\ + \left(\frac{1}{uw^2} + \frac{w^2}{u} + \frac{u^3}{w^2} - 3\frac{1}{u} - 3\frac{u}{w^2} - 3u\right)A_4 \end{array}\right] \\ -\frac{1}{3}\left(\frac{1}{w}\tan^{-1}\frac{uw}{A_4} + w\tan^{-1}\frac{u}{wA_4} + \frac{u^2}{w}\tan^{-1}\frac{w}{uA_4}\right) \\ +\frac{1}{4}\left(\ln\frac{u+A_4}{-u+A_4} + \frac{u}{w}\ln\frac{w+A_4}{-w+A_4} + u\ln\frac{1+A_4}{-1+A_4}\right) \\ -\frac{1}{24}\left[\begin{array}{l} \frac{w^2}{u}\ln\frac{(-1+A_2)(1+A_4)}{(1+A_2)(-1+A_4)} + \frac{u^3}{w^2}\ln\frac{(-1+A_1)(1+A_4)}{(1+A_1)(-1+A_4)} \\ + \frac{1}{uw}\ln\frac{(-w+A_2)(w+A_4)}{(w+A_2)(-w+A_4)} + \frac{u^3}{w}\ln\frac{(-w+A_3)(w+A_4)}{(w+A_3)(-w+A_4)} \\ + \frac{1}{w^2}\ln\frac{(-u+A_1)(u+A_4)}{(u+A_1)(-u+A_4)} + w^2\ln\frac{(-u+A_3)(u+A_4)}{(u+A_3)(-u+A_4)} \end{array}\right] \end{array}\right\} \quad (38)$$

where $A_1 = \sqrt{1 + u^2}$, $A_2 = \sqrt{1 + w^2}$, $A_3 = \sqrt{u^2 + w^2}$ and $A_4 = \sqrt{1 + u^2 + w^2}$.

From (28) the per-unit-length self inductance of the thin tape is given by following formula

$$\frac{L}{l}=\frac{\mu_0}{6\pi}\left[3\ln\left(u+\sqrt{1+u^2}\right)-\frac{1}{u}\left(1+u^2\right)^{3/2}+3u\ln\frac{1+\sqrt{1+u^2}}{u}+u^2+\frac{1}{u}\right] \quad (39)$$

From (35), (36) and (37) the per-unit-length self inductance of the straight long conductor of rectangular cross section is given by following formulae

$$\frac{L}{l}=\frac{\mu_0}{2\pi}\left[\ln 2u+\frac{13}{12}-\frac{2}{3}\left(w\tan^{-1}\frac{1}{w}+\frac{1}{w}\tan^{-1}w\right)\right.$$
$$\left.+\frac{1}{12}\left(\frac{1}{w^2}-6+w^2\right)\ln\left(1+\frac{1}{w^2}\right)+\frac{1}{6}\left(6-\frac{1}{w^2}\right)\ln\frac{1}{w}\right] \quad (40)$$

or

$$\frac{L}{l}=\frac{\mu_0}{2\pi}\left[\ln\frac{2u}{w}+\frac{13}{12}-\frac{2}{3}\left(w\tan^{-1}\frac{1}{w}+\frac{1}{w}\tan^{-1}w\right)\right.$$
$$\left.+\frac{1}{12}\left(\frac{1}{w^2}-6+w^2\right)\ln\left(1+w^2\right)+\frac{1}{6}\left(6-w^2\right)\ln w\right] \quad (41)$$

as well as

$$\frac{L}{l}=\frac{\mu_0}{2\pi}\left\{\ln\frac{2u}{1+w}+\frac{13}{12}-\frac{2}{3}\left(w\tan^{-1}\frac{1}{w}+\frac{1}{w}\tan^{-1}w\right)\right.$$
$$\left.+\frac{1}{2}\ln\left(1+\frac{2w}{1+w^2}\right)+\frac{1}{12}\left[\frac{1}{w^2}\ln\left(1+w^2\right)+w^2\ln\left(1+\frac{1}{w^2}\right)\right]\right\} \quad (42)$$

For given rations $u=l/a$ and $w=b/a$ the per-unit-length self inductance is independent of the values of width a and thickness b.

Table 1. Per-unit-length self inductance of a rectangular conductor for normalized length for DC or low frequency.

$w=b/a=0.1$								
$u=l/a$	Grover $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$	Bueno $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$	FastHenry $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$	Ruehli $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$	Hoer* $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$	Eq. (38) $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$	Eq. (39) $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$	Eq. (40) $\frac{L}{l}\left(\frac{\text{nH}}{\text{m}}\right)$
0.01	negative	negative	6.863510	6.863510	6.863510	6.863510	11.60329	negative
0.10	negative	negative	57.04253	57.04253	57.04253	57.04253	70.57298	negative
1.00	170.3974	219.1475	278.7982	278.7982	278.7982	278.7982	297.3209	219.1475
10.0	675.1674	679.6645	686.3510	686.3510	686.3510	686.3510	705.7298	679.6645
100	1140.109	1140.1816	1140.857	1140.857	1140.857	1140.857	1160.329	1140.181
1000	1601.069	1600.698	1600.766	1600.764	1600.346	1600.441	1620.247	1600.698

*According to Eq. (20) from [14].

For the chosen ratio $w = b/a$ and different normalized lengths of the straight conductor of rectangular cross section the calculations of its per-unit-length self inductance have been made according to all previous, shown above, formulae — Table 1.

8. CONCLUSIONS

In this paper, we have presented a new method for calculating self inductance of an isolated conductor of rectangular cross section. We have defined the self inductance of conductor of any shape and finite length given by sixtuple definite integral. In the case of DC or low frequency we have given general formulae for self inductance of conductors of rectangular cross section of any dimensions including thin tapes and "very long" conductors. By computations we have shown that our formulae give the same results as FastHenry's, Ruehli's and all Hoer's formulae (21) and (22) for all dimensions of a conductor. But computational results show that our formulae are numerically more stable and accurate for all dimensions of a conductor, particularly for long on-chip interconnections, than the others. In addition we have also obtained analytical forms of all formulae, which are more useful than general ones. Of course they give the same results as the general formulae.

Our formulae are analytically simple and can also replace the traditional working ones or tables.

These formulae can be used in the methods of numerical calculation of AC self inductance of a rectangular conductor. Then the cross section of the conductor is divided into rectangular subbars (elementary bars) in which the current is assumed to be uniformly distributed over the cross section of each subbar.

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