

## CRITICAL REVIEW OF THE MODIFIED WINDING FUNCTION THEORY

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**Abstract**—The Modified Winding Function Theory (MWFTh), regarded as a very powerful and general theory, has been extensively used for the last 15 years. This paper performs an in-depth review of the mathematical and physical framework on which the MWFTh is based, showing that it is indeed very well suited to analyse machines with small air gaps of arbitrary shape. However, contrary to what is usually stated in the literature, it is also proved that its general formulae fail when applied to large air gaps. This major finding is deduced from two different approaches, both of which are later reinforced by numerical examples. In spite of that, there is an important industrial field (diagnosis techniques of salient-pole synchronous machines eccentricities) in which good theoretical results have been reported by applying the MWFTh to these large air-gap machines. This issue is addressed and clarified in the paper.

### 1. INTRODUCTION

To the authors' knowledge, the Winding Function Theory (WFTh) can be traced back to [1]. Contrary to the classical d-q model, this theory can take into account all of the winding magnetomotive force (mmf) space harmonics in small air-gap machines. Relying on the WFTh, the authors in [2] presented, in a clear and detailed manner, a coupled circuit model of the squirrel cage induction machine (IM) with no restrictions as to the space distribution of the stator windings and rotor bars, taking into account all of the winding mmf harmonics.

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Following [2], the WFTh has been widely and satisfactorily used in the analysis of faulty IM windings, such as shorting, opening and arbitrary asymmetrical connections of the stator windings, as well as broken rotor bars and cracked end rings (e.g., [3–5]). Needless to say, modern studies based on FEM — e.g., [6–8] — are more accurate, but also more time consuming.

Based on [2], the WFTh was also used in [9] and [10] for the analysis of the static and dynamic eccentricity in IMs. The authors in [9] stated then (page 916) that “as a result of non-uniform air gap, the mutual inductances between stator phases and rotor loops,  $L_{sr}$  are different than the mutual inductances between the rotor loops and stator phases  $L_{rs}$ ”. This is incorrect and was later criticized in [11], where it is indicated that the inequality between  $L_{sr}$  and  $L_{rs}$  is not the result of non-uniform air gap but originates in the way inductances are calculated using the equations in [2], which do not account for air-gap variations. Indeed, when linearity and reciprocity hold, the mutual inductances  $L_{AB}$  and  $L_{BA}$  of two arbitrary circuits  $A$  and  $B$  must always be the same, as proved, e.g., in [12] (notice that, since in [9] the iron permeability was assumed to be infinite, the system is linear).

In [13] an important modification of the old procedure and a new method to calculate inductances, called the Modified Winding Function Approach, was presented. It was essentially aimed at analysing rotor eccentricity effects in asynchronous and synchronous salient-pole machines in order to develop a suitable diagnosis technique, which is very important for improving machine life (e.g., [14]).

Except for the major requirement that all of the machine cross-sections be equivalent (no axial variations; two-dimensional problem), there are no restrictions regarding the rotor shape, air-gap length, windings layout and number of space harmonics, which may be arbitrary. Therefore, being so general and ambitious, there seems to be no reasons to limit the application of the mathematical development in [13] to salient-pole synchronous machines. In fact, the authors in [13] state in the concluding section that “the Modified Winding Function Approach can be used for finding inductances of any electrical machine in cases of healthy and unhealthy conditions”.

Accordingly, the approach in [13] was soon considered an undoubtful achievement. Its importance and great influence, especially in the field of eccentricities studies, cannot be understated, as confirmed by the almost hundred papers which have cited [13] in the last 15 years.

Rotating electrical machines are well known to be very complex systems. The analytical formulation of their behaviour in the case of arbitrary large air gap and, moreover, with arbitrary number of space

harmonics is extremely difficult. Actually, even resorting to partial differential equations, an analytical solution is only possible in very few cases [15]. However, according to [13], such a complexity can be essentially overcome by means of the MWFTh (Modified Winding Function Theory), leading to rather simple formulae in the most general case.

In view of the relevance and potential implications of the MWFTh proposed in [13], one logically expects that the mathematical and physical framework on which it is based should have been thoroughly checked and verified in all detail. Unfortunately, it clearly seems that this has not been done so far.

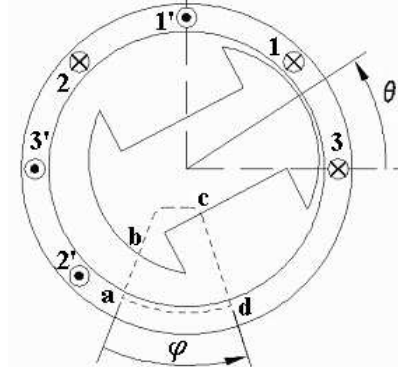
The structure of this paper is as follows: Section 2 briefly reproduces, as presented in the literature, the deduction process that leads to the well-known key quantity called the modified winding function. Then, Section 3 reviews in all detail the mathematical developments in the previous section. From this analytical review it follows that the theory is very well suited to accurately model machines with small air gap of arbitrary shape. However, its general formulae fail when applied to large air gaps. This is also illustrated in this section by simulation results, comparing on several machines the exact air-gap induction distribution with the one implicitly assumed for deducing the MWFTh general formulae in Section 2. Thereafter, Section 4 reviews the inductance calculation method in the MWFTh. It is theoretically shown that, for large air gaps, the validity of the inductances general formulae in [13] can hardly be accepted. This theoretical conclusion is also reinforced by means of a numerical example.

In spite of that, there is an important industrial field (diagnosis of salient-pole synchronous machines eccentricities) in which good theoretical results have been reported by applying the MWFTh to these large air-gap machines. This issue is addressed and thoroughly discussed in Section 5.

## 2. MODIFIED WINDING FUNCTION THEORY

In this section, the general formulae of the MWFTh are established reproducing exactly the same statements, assumptions and deduction process as in [13]. In the next section, some of these statements, deduction process and resulting equations will be reconsidered.

Figure 1 reproduces Fig. 1 in [13]. The stator reference for the angle  $\varphi$  of the closed path  $abcd$  is taken at an arbitrary point,  $a$ , along the gap. Points  $a$  and  $d$  are located on the stator corresponding to angles 0 and  $\varphi$  respectively and points  $b$  and  $c$  are located on the rotor. Paths  $ab$  and  $cd$  are defined to lie along the lines of flux even



**Figure 1.** Elementary salient-pole machine with eccentric rotor.

though these flux lines cannot be uniquely defined without using flux plots. They will take irregular paths in the air gap but intersect with the stator and rotor at right angles.

Applying Ampere's law to the path  $abcda$  results in

$$\oint_{abcda} \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} \quad (1)$$

where  $S$  is the surface enclosed by the path  $abcda$ . Since all the windings enclosed by the closed path carry the same current  $i$ , (1) reduces to the following:

$$\oint_{abcda} \vec{H} \cdot d\vec{l} = n(\varphi, \theta) \cdot i \quad (2)$$

The function  $n(\varphi, \theta)$  is the turn function and represents the number of turns of the winding enclosed by the path  $abcda$ . In general, for a rotating coil it is assumed to be a function of  $\varphi$  and the rotor position angle  $\theta$  (obviously, for a stationary coil it is only a function of  $\varphi$ ). In terms of mmf drops in a magnetic circuit, (2) can be written as

$$F_{ab} + F_{bc} + F_{cd} + F_{da} = n(\varphi, \theta) \cdot i \quad (3)$$

Since the iron is considered to be infinitely permeable, the mmf drops  $F_{bc}$  and  $F_{da}$  are negligible and (3) reduces to

$$F_{ab}(0, \theta) + F_{cd}(\varphi, \theta) = n(\varphi, \theta) \cdot i \quad (4)$$

The next step is to find an expression for the mmf drop at  $\varphi = 0$ ,  $F_{ab}(0, \theta)$ . To this end one makes use of Gauss's law

$$\oiint_S \vec{B} \cdot d\vec{S} = 0 \quad (5)$$

where  $\vec{B}$  represents magnetic flux density, and the surface integral is carried out over the boundary surface of an arbitrary volume. Taking the surface  $S$  to be a cylindrical volume located just inside the stator inner surface (that is, infinitely close to the stator inner surface), (5) becomes

$$\int_0^{2\pi} \int_0^l \mu_0 H(\varphi, \theta) r dl d\varphi = \mu_0 r l \int_0^{2\pi} H(\varphi, \theta) d\varphi = 0 \quad (6)$$

where  $l$  is the axial stack length of the machine,  $r$  the stator inner radius, and  $\mu_0$  the free space permeability. Since  $B$  does not vary with respect to the axial length, and the mmf is the product of flux radial length times the magnetic field intensity, then

$$\int_0^{2\pi} \frac{F(\varphi, \theta)}{g(\varphi, \theta)} d\varphi = 0 \quad (7)$$

Dividing (4) by the air-gap function  $g(\varphi, \theta)$  and then integrating from 0 to  $2\pi$  yields

$$\int_0^{2\pi} \frac{F_{ab}(0, \theta) + F_{cd}(\varphi, \theta)}{g(\varphi, \theta)} d\varphi = \int_0^{2\pi} \frac{n(\varphi, \theta)}{g(\varphi, \theta)} i \cdot d\varphi \quad (8)$$

Since the second term of the left hand side is zero as found from Gauss's law, (8) reduces to

$$F_{ab}(0, \theta) = \frac{1}{2\pi \langle g^{-1}(\varphi, \theta) \rangle} \int_0^{2\pi} n(\varphi, \theta) g^{-1}(\varphi, \theta) \cdot i \cdot d\varphi \quad (9)$$

where  $\langle g^{-1}(\varphi, \theta) \rangle$  is the average value of the inverse gap function. Substituting (9) into (4) and solving for  $F_{cd}(\varphi, \theta)$  yields

$$\begin{aligned} F_{cd}(\varphi, \theta) &= \left[ n(\varphi, \theta) - \frac{1}{2\pi \langle g^{-1}(\varphi, \theta) \rangle} \int_0^{2\pi} n(\varphi, \theta) g^{-1}(\varphi, \theta) d\varphi \right] i \\ &= M(\varphi, \theta) i \end{aligned} \quad (10)$$

The term  $M(\varphi, \theta)$  is called the modified winding function.

Equations (1) to (10) in this paper replicate Equations (1) to (10) in [13].

### 3. REVIEWING THE DEDUCTION PROCESS OF THE MWFTH GENERAL EQUATIONS

#### 3.1. Analytical Review

We fully agree with Equations (1) to (5). The surface integral (5) can be carried out over the boundary surface of an arbitrary volume. Taking the surface  $S$  to be a cylindrical volume located just inside the stator inner surface, as proposed in [13], and assuming that the flux lines intersect with the stator at right angles, Equation (5) becomes

$$\int_0^{2\pi} \int_0^l \mu_0 H_{Surf}(\varphi, \theta) r \, dl \, d\varphi = \mu_0 r l \int_0^{2\pi} H_{Surf}(\varphi, \theta) d\varphi = 0 \quad (11)$$

where, unlike in (6), we have explicitly stressed that the  $H$  values refer to the surface  $S$ .

From (11) it follows immediately that

$$\int_0^{2\pi} H_{Surf}(\varphi, \theta) d\varphi = 0 \quad (12)$$

By definition, the mmf drop along an air-gap path between stator and rotor of arbitrary shape and length,  $g(\varphi, \theta)$ , is

$$F(\varphi, \theta) = \int_0^{g(\varphi, \theta)} \vec{H}(\varphi, \theta) \cdot d\vec{l} \quad (13)$$

where  $\vec{H}(\varphi, \theta)$  represents, obviously, the  $H$  values along the air-gap path  $g(\varphi, \theta)$ .

If the air-gap path  $g(\varphi, \theta)$  lies along a flux line, as proposed in [13], the directions of  $\vec{H}$  and  $d\vec{l}$  coincide at any point and Equation (13) becomes

$$F(\varphi, \theta) \Big|_{\substack{\text{arbitrary air-gap} \\ \text{path along a flux line}}} = \int_0^{g(\varphi, \theta)} H(\varphi, \theta) \cdot dl \quad (14)$$

If, in addition we assume *the particular case* that the length of  $g(\varphi, \theta)$  is negligible (small air gaps), then the  $H$  value along the air-gap path  $g(\varphi, \theta)$  can be considered to be constant and equal to the value at that point of the inner stator surface where the integration

path  $g(\varphi, \theta)$  intersects with this surface. Therefore, in this particular case (14) reduces to

$$F(\varphi, \theta)|_{\substack{\text{small air-gap path} \\ \text{along a flux line}}} = \int_0^{g(\varphi, \theta)} H_{Surf}(\varphi, \theta) dl = H_{Surf}(\varphi, \theta) g(\varphi, \theta) \quad (15)$$

Taking into account (15), Equation (12) becomes, for the particular case of small (negligible) air gap,

$$\int_0^{2\pi} \frac{F(\varphi, \theta)}{g(\varphi, \theta)} d\varphi = 0 \bigg|_{\substack{\text{small air-gap} \\ \text{path along a flux line}}} \quad (16)$$

Equation (16) coincides with (7), except for an essential point: it has been deduced and is valid only for small enough air gaps, not for arbitrary air gaps, as assumed in [13]. Before analyzing in detail in the next section the implications of this fundamental issue, another questionable point will be discussed.

Consider the statement “in the case of salient-pole synchronous machines, the flux lines will intersect with the stator and rotor at right angles” (page 158 in [13], just at the beginning). This will only occur at those particular stator (or rotor) surface points where the linear current density is null, as proved in [16] and shown in Appendix A. It is well known that assuming the classical hypothesis of  $\mu_{Fe}$  to be infinite, the tangential component of the magnetic field intensity  $\vec{H}$  at any point of the stator (or rotor) surface equals the linear current density at this point ([15] page 47, [17] page 244, [18] page 342). Actually, as very well synthesized in [19], page 151 “the tangential component is most important; without it the machine would not work. The radial field alone may be regarded as necessary for the development of armature emf, if relative motion between the armature conductors and the radial flux is arranged but, for the development of mechanical forces on the conductors and the complementary flow of power across the armature conductors/air gap interface, the tangential field is an absolute necessity”. This fact also becomes especially clear in [20] (a paper in which the main rotating electrical machines formulae are derived in a beautiful and rigorous manner resorting to the Poynting vector): without tangential  $H$  field there would be no Poynting vector radial component pointing to the stator inner (or to the rotor outer) surface and therefore, no power flow stator-rotor could take place.

In other words, for Equation (6) to be correct, the total magnetic field intensity,  $H(\varphi, \theta)$  appearing therein should be replaced by its

radial component at the stator surface,  $H_{R, Surf}$ , as follows:

$$\mu_0 r l \int_0^{2\pi} H_{R, Surf}(\varphi, \theta) \cdot d\varphi = 0 \quad (17)$$

On the other hand, if we now choose  $g(\varphi, \theta)$  to be a radial air-gap path we can write

$$F(\varphi, \theta) \Big|_{\substack{\text{arbitrary radial} \\ \text{air-gap path}}} = \int_0^{g(\varphi, \theta)} \vec{H}(\varphi, \theta) \cdot d\vec{l} = \int_0^{g(\varphi, \theta)} H_R(\varphi, \theta) dl \quad (18)$$

This equation can be easily compared with its more complex counterpart (14) where there are both radial and tangential  $H$  components (more details on these two components in Appendix A). If now, in addition to a radial path, we assume again the particular case of small air gap, following just the same procedure as above, we get:

$$\int_0^{2\pi} \frac{F(\varphi, \theta)}{g(\varphi, \theta)} d\varphi = 0 \Big|_{\substack{\text{small air-gap} \\ \text{radial path}}} \quad (19)$$

where  $g(\varphi, \theta)$  represents, as explicitly indicated above, the mathematical expression of the air-gap radial length as a function of the polar coordinate,  $\varphi$  (for a rotor fixed position,  $\theta$ ). This is the so-called air-gap function. In fact, this is the function actually used in the papers on MWFTh dealing with eccentricity problems, which is most appropriate since, for a specified machine with known static, dynamic or mixed eccentricity, this air-gap function can rather easily be modeled by means of different functions (see a comparison in [21]). This is not at all the case for the flux line paths considered in Fig. 1. But, above all, one should be aware that, when applying Gauss's law, the  $H$  value required in (6) is the  $H$  normal component at stator surface, as just explained, so that the correct expression for (6) is actually (17). Thus, the right mathematical process, which should start from (17) leads compulsorily to (19).

Therefore, choosing for  $g(\varphi, \theta)$  the air-gap function, formula (19) coincides with (7), except for a fundamental point: *its theoretical validity has been proved only for the particular case of small enough (negligible) air gaps*, whereas the authors in [13] apply it directly to machines with arbitrary air-gap length in order to establish the general MWFTh formulae, in particular (8) to (10). Therefore, these formulae become very questionable for large air gaps.

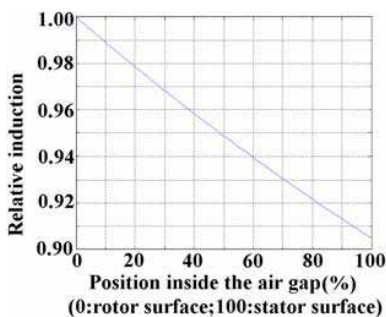


### 3.2. Numerical Verification for Uniform Large Air Gaps

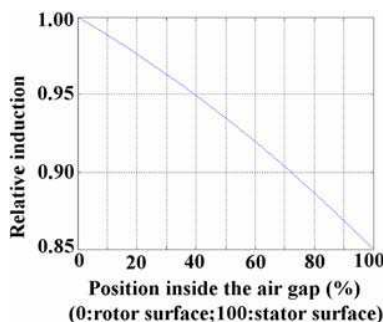
Equation (7) is a key equation to establish the MWFTh general formulae. However, as just proved, *its deduction requires that the  $H$ -value along  $g(\varphi, \theta)$  remains practically constant*. Obviously, this is true for small air gaps, but it is very dubious that this also holds for large air gaps. This issue is analysed in detail in this section.

In a seminal paper [16], Doherty and Nickle, starting directly from Maxwell's equations, and assuming the classical hypothesis of infinite iron permeability, deduce and solve a system of two partial differential equations in order to determine with accuracy the induction at any point inside a cylindrical uniform air gap of arbitrary length produced by an arbitrary current sheet located at the armature surface (see Appendix A of this paper for details). We have programmed these equations in Matlab and obtained the different  $B$  values along the path  $g(\varphi, \theta)$  for very different conditions (variations in air-gap length, mmf harmonics order, machine pole number, winding structure, etc.).

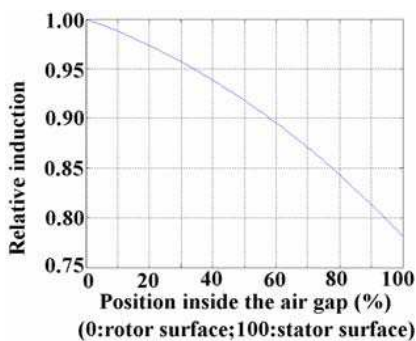
Figures 2 to 11 represent air-gap induction values versus air-gap position at a fixed polar coordinate referred to the induction value at rotor surface. The values have been calculated by applying (A5) and (A6) in Appendix A to a constant air-gap machine with inner stator radius 10 times as large as the air-gap length. Notice that this is not a too much small value for the required verifications in this section. On the contrary, the above ratio is actually greater than that found in salient-pole synchronous machines for the region outside the poles



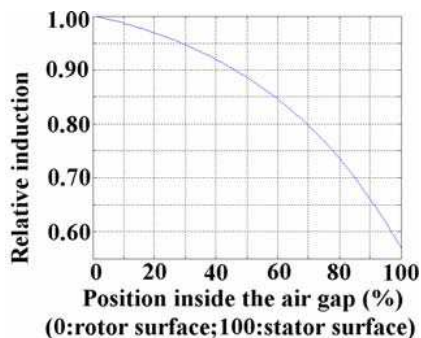
**Figure 2.** Induction versus position inside the air gap for a fixed polar coordinate. Values referred to the induction at rotor surface. Only fundamental mmf wave considered.  $p = 1$ ,  $q = 2$ .



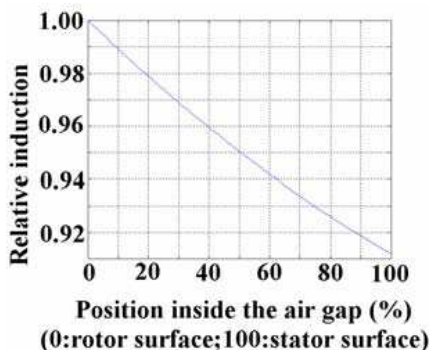
**Figure 3.** Same as Fig. 2, with mmf harmonics 1, 5, 7, 11.



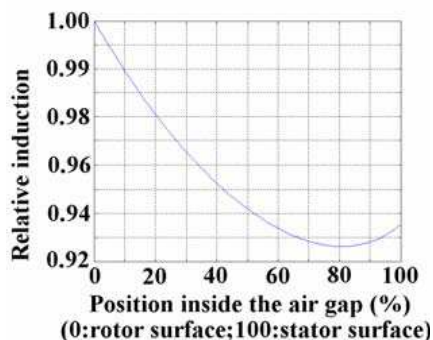
**Figure 4.** Same as Fig. 2 with all mmf harmonics (1, 5, 7, 11, 13, 17).



**Figure 5.** Same as Fig. 4 with  $p = 2$ .



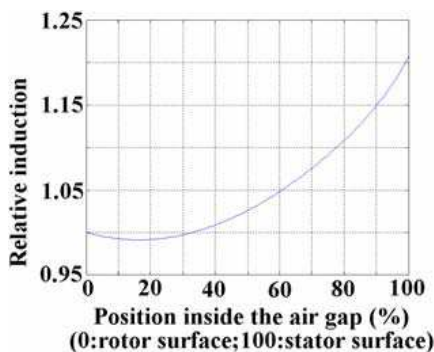
**Figure 6.** Same as Fig. 4 with  $q = 3$ .



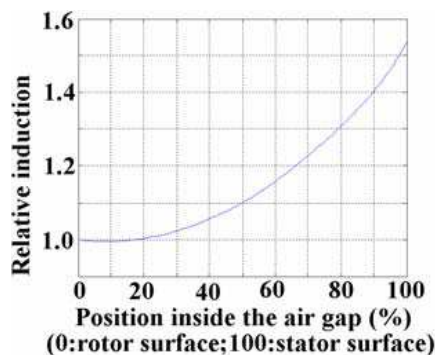
**Figure 7.** Same as Fig. 6 with  $p = 3$ .

(see Fig. 1), which is just the region where the claims for the MWFTh formulae general validity are to be checked. Obviously, this general validity, if only true, must hold not only for the complex case of the irregular interpolator space of a salient pole machine with eccentricity (see the MWFTh application to path  $cd$  in Fig. 1) but also for the much simpler case of a uniform air gap of similar length. This last case, contrary to the one with irregular air gaps, can be tackled in an exact analytical manner, and will be considered below.

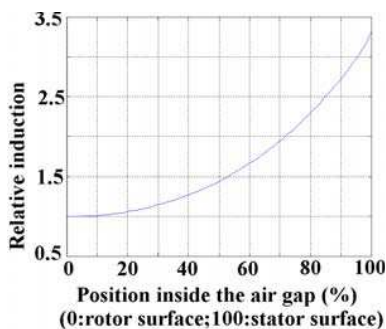
The air-gap length in Figs. 2 to 11 is displayed in percentage (0 means rotor surface and 100 stator surface). The mmf space harmonics considered are 1 (fundamental wave), 5, 7, 11, 13 and 17. For more details, refer to Appendix A.



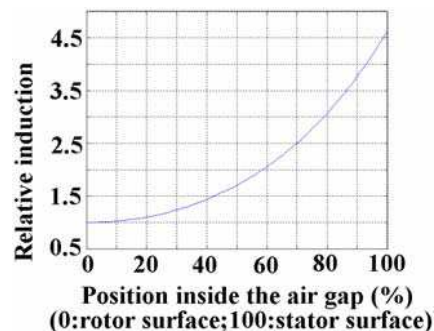
**Figure 8.** Same as Fig. 6 with  $p = 7$ .



**Figure 9.** Same as Fig. 6 with  $p = 10$ .



**Figure 10.** Same as Fig. 6 with  $p = 18$ .



**Figure 11.** Air-gap induction produced exclusively by the 11 mmf harmonic in a machine with  $p = 2$ .

Figures 2 to 4 refer to the most favourable case of a bipolar machine,  $p = 1$  (the air gap is as small as possible compared to the pole pitch). Even in this case, if  $q$  (slots per pole and per phase) is equal to 2, there are important differences between the  $B$  values, especially when all the space harmonics are taken into account. Of course, for  $p = 2$ , the situation is worse, as shown in Fig. 5.

With  $q \geq 3$ , and  $p = 1$ ,  $p = 2$  (which are the cases often found in the literature in experimental measurements) or  $p = 3$ , the differences between  $B$  values are not too much important (see Figs. 6 and 7). However, as predicted in [16], this is no longer true as the pole number increases (notice that some hydroalternators may have more than 100 poles). This is shown in Figs. 8 to 10. For  $p = 7$ , the errors are already

greater than 20%. For  $p = 10$ , they go over 50%. Moreover, even in the case of  $p = 1$  or  $p = 2$ , if we are only interested in the induction produced by a given mmf harmonic, the differences in the  $B$  values are again very important, as shown in Fig. 11.

In summary, as explained at the beginning of this section, proving the validity of (7) in the general case requires that, not only for small but also for large and irregular air gaps (see, e.g., eccentric interpolar region in Fig. 1), the  $H$  value along  $g(\varphi, \theta)$  remains practically constant. However, this does not hold even in the very much simpler case of large uniform air gaps, as shown in Figs. 2–11.

#### 4. REVIEWING THE INDUCTANCE CALCULATION PROCEDURE IN THE MWFTH

In order to calculate the winding inductances, the authors in [13] start from the statement that the permeance  $P$  of the air-gap flux path is given by Equation (16) in [13], which is repeated here:

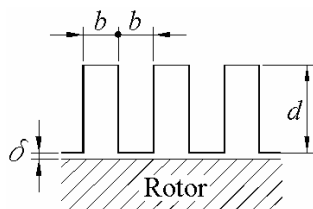
$$P = \frac{\mu A}{l} \quad (20)$$

where  $\mu$  is the permeability,  $A$  is the cross sectional area and  $l$  is the length of the magnetic path. That is, air-gap permeance is proportional to the inverse of the air-gap length and is independent of the order of the applied mmf wave. In this regard, after determining with accuracy by means of partial differential equations the permeance of an air gap of uniform length, Doherty and Nickle summarize their work as follows ([16], page 938): “The equations show that, when the air gap is small compared to the pole pitch of the harmonic wave, the permeance is simply equal to the reciprocal of the airgap length, as ordinary assumed. When the air gap is large<sup>†</sup>, the permeance is a function of the order of the space harmonic of mmf” This conclusion is repeated in page 937, right column, where they write “it is clear that the permeance of a given machine is different for every harmonic mmf”. Also in the same paper [16], in the discussion section, J. Douglas already underlined that “It is not generally appreciated that the permeance of the air gap. . . is different for different orders of the armature harmonics of mmf.” (page 944, right column). The same statements can be found, e.g., in chapter 4 of [15].

In summary, for big air-gap enlargements or expansions (and, of course, also for uniform, but large air gaps) Equation (20) above is

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<sup>†</sup> Notice, from another different perspective, that for vey large air gaps (more precisely: when air gap length and pole pitch of the harmonic wave are of similar order), the errors in determining the air gap  $B$  values become completely unacceptable, as shown, especially, in Fig. 11.



**Figure 12.** One-side slotted machine with rectangular slots.

not valid. Just as the key Equation (7) in Section 2 is restricted to small air gaps, Equation (20), chosen in [13] as starting point for general inductance calculations, is restricted to small air gaps as well. Therefore, general formulae (23) and (24) in [13] for mutual and magnetizing self inductances, which rely on (20), have not general validity and may lead (at least in certain cases) to remarkable inaccuracies in inductance calculations, as shown in the following example.

In Fig. 12 there are only slots in the stator. Assume air-gap length,  $\delta = 2.54$  mm, air-gap radius,  $r = 422.656$  mm, axial stack length  $l = 273.05$  mm and 48 slots (these data coincide with the ones of the machine in appendix of [13]). Slot depth,  $d = 100$  mm. We want to determine the magnetizing self-inductance of a diametral coil placed on the rotor.

For the machine with no slots the classical formula gives,

$$L = \frac{\mu_0}{2\delta} l \pi r = 0.089 \text{ mH}$$

Simulation with the Finite Element Method (FEM) gives 0.0894 mH. Applying the MWFTh (Equation (24) in [13]) we get

$$\begin{aligned} L_{MWF} &= \mu_0 r l \int_0^{2\pi} n(\varphi) M(\varphi) g^{-1}(\varphi) d\varphi \\ &= \mu_0 r l \int_0^{2\pi} \left(\frac{1}{2}\right) \frac{1}{\delta} d\varphi = L = 0.0897 \text{ mH} \end{aligned}$$

So, for small air gaps, the three methods provide the same result. However, for the machine with slots, the MWFTh gives, no matter the

rotor position,

$$\begin{aligned} L_{MWF} &= \mu_0 r l \int_0^{2\pi} n(\varphi) M(\varphi) g^{-1}(\varphi) d\varphi \\ &= \left(\frac{1}{2}\right)^2 \mu_0 r l \int_0^{\pi} \left(\frac{1}{\delta} + \frac{1}{\delta + d}\right) d\varphi = 0.0460 \text{ mH} \end{aligned}$$

whereas the FEM simulations provide 0.0582 mH. This last value is in close agreement with the one obtained (second method) using Carter's factor ( $k_c = 1.5212$  for Fig. 12), namely

$$L_{\text{Carter}} = \frac{\mu_0}{2\delta} l \pi r \frac{1}{k_c} = 0.0590 \text{ mH}$$

In other words, the actual coil self inductance is about 28% greater than the incorrect value provided by the MWFTh. This discrepancy can also be obtained very fast by a third alternative way, namely applying the air-gap length inverse law, which underlies the MWFTh inductance calculation. According to this law, the permeance of the slotted area in Fig. 12 would be about 40 times smaller than the permeance of the non-slotted area. Thus, the total permeance would be  $0.512P_{\text{smooth}}$  where  $P_{\text{smooth}}$  means the permeance of the air gap with no slots. However, since Carter's factor is 1.5212, the actual permeance value is, in fact,  $0.657P_{\text{smooth}}$ , that is, 28% greater than the value given by the air-gap length inverse law. Had other machine parameters been chosen in Fig. 12, the errors could have exceeded 50%.

This example shows, by three different ways, that the claims in the concluding section of [13] ("in fact, the MWFTh can be used for finding inductances of any electrical machine in cases of healthy and unhealthy conditions") can hardly be accepted.

## 5. ON THE APPLICATION OF THE MWFTH TO SALIENT-POLE SYNCHRONOUS MACHINES ECCENTRICITY STUDIES

In spite of equations, figures and statements in Sections 3 and 4, it could be objected that there is an important industrial field (diagnosis techniques of salient-pole synchronous machines eccentricities) in which very good theoretical results are reported by applying the MWFTh to these machines, where very large air gaps are present. This seems to be, at least at first sight, in contradiction with the findings discussed in this work. Actually, this is not so for two main reasons.

To begin with, when resorting to the MWFTh for condition monitoring, diagnosis of rotor eccentricities is actually based not on direct inductance measurements but on detecting some specific stator current harmonics due to eccentricity. Yet the coincidence of theoretical and experimental frequencies of some current harmonics does not actually prove that the inductances have been correctly computed.

Regarding this point, in the conclusions of [11], it was already explicitly underlined that although “incorrect inductances had been used in [10] in the analysis of the dynamic eccentricity” (the mutual inductances  $L_{AB}$  and  $L_{BA}$  of two arbitrary circuits  $A$  and  $B$  in a linear system must always be equal, as already said) the extra current harmonic however was the same in both papers.

An analogous conclusion related to a different practical case can be found in [15], where the air-gap permeance of a one-side slotted machine, like the one in Fig. 12, is given by the expression:

$$P(\varphi) = P_0 (1 + b_1 \cos N\varphi + b_2 \cos 2N\varphi + \dots) \quad (21)$$

The author underlines that the coefficients  $b_1$ ,  $b_2$ , etc. in (21) are different for different mmf harmonics, which is in total agreement with [16], since, as also indicated in [15], slots expansions cannot at all be considered “negligible air gap”. Nevertheless the author in [15] also emphasizes that, as can be mathematically checked, the *frequencies* of the stator current harmonics due to machine slotting are correctly given by (21), even assuming (as often erroneously done) that the coefficients  $b_1$ ,  $b_2$ , etc. are constant, that is, independent of the mmf harmonic order ([15], page 102 for more details). In other words, in spite of the fact that inductances calculated from (21) with constant coefficients are not correct, the theoretical frequencies (but not at all the amplitudes) of current harmonics are in agreement with the measurements.

The second reason referred to above is in close connection with the first one. Indeed, the extra harmonics coming from rotor eccentricity are due to modifications or changes in the magnetic energy distribution of the eccentric machine with respect to the healthy one. Yet, these magnetic energy changes are located, by far, in the small air gap under the poles. Therefore, for the purpose of eccentricity diagnosis based on the detection of the frequencies of some extra significant harmonics, the large air gap outside the poles can be neglected in the machine equations. In fact, this is just the assumption explicitly chosen as starting point for eccentricity analysis, for instance in [22], where the authors write (page 1551) that in their equations “this quantity (the permeance) is thus neglected for the points far from the poles shoes air-gap, and taken into account *only* for the air-gap between the salient-poles and the stator”. This assumption is also shown in a clear manner

in Figs. 3 and 4 of [22]. Obviously, applying the MWFTh in such cases will not show its limitations, simply because in fact only small air gaps are being (and are needed for our purposes to be) considered.

The above reasoning becomes reinforced by the refined approach followed in [23]. Its authors apply the MWFTh equations only to the small air gap above the pole shoes (see Equation (24) in page 414 of [23]), adding thereafter, in the same page, that the contribution of the slots and the area outside the poles (saliency) has to be taken into account (if only necessary) by means of special techniques. This procedure constitutes, in fact, a confirmation (although implicit and rather hidden) of the MWFTh limitations. Indeed, should the MWFTh actually be valid for arbitrary air gaps, then it could be directly applied to the whole machine and there would be no need at all to resort to special techniques for dealing with the large air-gap areas (saliency and slots). In summary, references [22, 23], and other similar ones, do not contradict the conclusions in the present paper, for none of them applies the MWFTh to the machine large air-gap areas.

Finally, it is worth emphasizing that the mathematical proofs in Sections 3 and 4 of this paper, on the MWFTh restricted validity, are fully independent of the contents of this section. In other words, they would keep on being true even if the objection above had not been addressed and rebutted in the paper.

## 6. CONCLUSIONS

Rotating electrical machines are well known to be very complex systems. The analytical formulation of their behaviour in the case of arbitrary large air gaps and, moreover, with arbitrary number of space harmonics is extremely difficult. Actually, even resorting to partial differential equations, an analytical solution is only possible in very few cases. However, this complexity is stated to be easily overcome by means of the MWFTh, which leads to rather simple formulae in the most general case.

The core of this paper is the in-depth review of the mathematical and physical framework underlying the MWFTh presented in [13], a theory regarded as very general (able to cope with different and very complex problems in large air gap machines), reproduced in several books and cited by numerous publications in the last 15 years, particularly in the field of fault diagnosis techniques.

In this sense, the central thesis of this paper is as follows: the MWFTh is indeed very well suited to analyse machines with small air gaps of arbitrary shape; however we do question its general validity on the ground of the following main reasons:



a) The key Equation (7) upon which the MWFTh general equations rely, does not hold for large air gaps. As proved mathematically, it must be replaced by Equation (19) which is only valid for negligible air gaps. This fact has been also checked by numerous simulations, solving the exact two partial differential equations system that gives the magnetic field of a large cylindrical air gap.

b) The inductance or, more precisely, the permeance calculation method in [13] is based on the so called air-gap length inverse law. Yet, it has long been known [16] that this law does not hold for large air gaps.

c) Starting from the assumption that there are no tangential field components on the stator and rotor surfaces is unacceptable, especially for a theory which claims to be as general as the MWFTh. For it has been also well known for a long time that, without tangential components, the machines would simply not work.

Finally, it is worth mentioning that this paper, as explained in detail in Section 5, does not question at all the, in our opinion, correct results, and the diagnosis techniques for salient-pole machines eccentricities proposed in publications like [22, 23]. Notice that none of them applies the MWFTh to the machine large air-gap areas.

## APPENDIX A.

In pages 935–937 of [16] Doherty and Nickle, resorting to Maxwell's equations, undertake the problem of determining with accuracy the flux density distribution at any point inside a cylindrical uniform air gap of arbitrary length produced by an arbitrary current sheet located at the armature surface, assuming the classical hypothesis of infinite iron permeability. To this end, they start from the two following differential equations (Equations (1b) and (2b) in page 936 of [16]):

$$\frac{\partial B_\lambda}{\partial \lambda} + B_R + R \frac{\partial B_R}{\partial R} = 0 \quad (\text{A1})$$

$$\frac{\partial B_R}{\partial \lambda} - R \frac{\partial B_\lambda}{\partial R} - B_\lambda = 0 \quad (\text{A2})$$

where

$R$  = radius to any point in the air-gap space

$\lambda$  = mechanical angular displacement on armature periphery

$B_\lambda$  = induction tangential component at any point in the air gap

$B_R$  = induction radial component at any point in the air gap

Differentiating (A1) and (A2) with respect to  $R$  and  $\lambda$  and following the mathematical manipulations indicated in [16], we get the two following second order differential equations with separated variables (Equations (9b) and (10b) in [16]):

$$\frac{1}{R^2} \frac{\partial^2 B_R}{\partial \lambda^2} + \frac{\partial^2 B_R}{\partial R^2} + \frac{3}{R} \frac{\partial B_R}{\partial R} + \frac{B_R}{R^2} = 0 \quad (\text{A3})$$

$$\frac{1}{R^2} \frac{\partial^2 B_\lambda}{\partial \lambda^2} + \frac{\partial^2 B_\lambda}{\partial R^2} + \frac{3}{R} \frac{\partial B_\lambda}{\partial R} + \frac{B_\lambda}{R^2} = 0 \quad (\text{A4})$$

Doherty and Nickle solved the above system in the case of a sinusoidal current sheet (or a sinusoidal mmf) of arbitrary amplitude and pole number placed at the stator surface, obtaining the tangential and radial components of the induction at any air-gap point. Their values are given by expression (24b) and (27b) in page 937 of [16], which are repeated here using today's more usual symbols for some quantities:

$$B_\lambda = \frac{n p F_n R_a^{pn}}{(R_a^{2pn} - R_f^{2pn})} \left( R^{pn-1} - R_f^{2pn} R^{-pn-1} \right) \sin(pn\lambda) \quad (\text{A5})$$

$$B_R = \frac{n p F_n R_a^{pn}}{(R_a^{2pn} - R_f^{2pn})} \left( R^{pn-1} + R_f^{2pn} R^{-pn-1} \right) \cos(pn\lambda) \quad (\text{A6})$$

where  $n$  is the order of the sinusoidal mmf (or of the sinusoidal current sheet),  $R_a$  the radius of inner armature surface, and  $R_f$  the radius of rotor surface (more details in [16]).

Obviously, in actual machines, the current sheet wave produced by the stator winding is not sinusoidal. In this case we simply proceed to its Fourier expansion, apply (A5) and (A6) to each sinusoidal harmonic wave and add up in due manner the individual contributions.

Notice that according to (A5) and (A6) there are both radial and tangential induction components at the stator surface, as underlined after establishing (16) in this paper. Notice too that at all points of the radial straight line defined by  $\lambda = 0$  there is only radial component, no matter the order  $n$  of the mmf harmonic considered. In other words, this particular radial straight line is at the same time a flux line. We have chosen this straight line in the calculations and figures in Section 3 (simulations with other radial straight lines provide similar results).

## REFERENCES

1. Schmitz, N. L. and D. W. Nowotny, *Introductory Electromechanics*, The Ronald Press Company, New York, 1965.

2. Luo, X., Y. Liao, H. A. Toliyat, A. El-Antably, and T. A. Lipo, "Multiple coupled circuit modeling of induction machines," *IEEE Trans. Ind. Applicat.*, Vol. 31, No. 2, 311–318, 1995.
3. Toliyat, H. A. and T. A. Lipo, "Transient analysis of cage induction machines under stator, rotor bar and end ring faults," *IEEE Transactions on Energy Conversion*, Vol. 10, No. 2, 241–247, 1995.
4. Milimonfared, J., H. M. Kelk, A. Der Minassians, S. Nandi, and H. A. Toliyat, "A novel approach for broken rotor bar detection in cage induction motors," *IEEE Trans. Ind. Applicat.*, Vol. 35, 1000–1006, 1999.
5. Joksimovic, M. G. and J. Penman, "The detection of inter-turn short circuits in the s stator windings of operating motors," *IEEE Trans. Ind. Electron.*, Vol. 47, 1078–1084, 2000.
6. Faiz, J. and B. M. Ebrahimi, "Mixed fault diagnosis in three-phase squirrel-cage induction motor using analysis of air-gap magnetic field," *Progress In Electromagnetics Research*, Vol. 64, 239–255, 2006.
7. Faiz, J., B. M. Ebrahimi, and M. B. B. Sharifian, "Time stepping finite element analysis of broken bars fault in a three-phase squirrel-cage induction motor," *Progress In Electromagnetics Research*, Vol. 68, 53–70, 2007.
8. Vaseghi B., N. Takorabet, and F. Meibody-Tabar, "Transient finite element analysis of induction machines with stator winding turn fault," *Progress In Electromagnetics Research*, Vol. 95, 1–18, 2009.
9. Toliyat, H. A., M. Areffen, and A. Parlos, "A method for dynamic simulation of air-gap eccentricity in induction machines," *IEEE Trans. Ind. Applicat.*, Vol. 32, 910–918, 1996.
10. Joksimovic, M. G., D. M. Durovic, J. Penman, and N. Arthur, "Dynamic simulation of dynamic eccentricity in induction machines-winding function approach," *IEEE Transactions on Energy Conversion*, Vol. 15, 143–148, 2000.
11. Faiz, J. and I. Tabatbaei, "Extension of winding function theory for non-uniform air gap in electric machinery," *IEEE Transactions on Magnetics*, Vol. 38, No. 6, 2002.
12. Ballantine, S., "Reciprocity in electromagnetic, mechanical, acoustical, and interconnected systems," *Proc. of the Institute of Radio Engineers*, Vol. 17, No. 6, 927–951, 1929.
13. Al-Nuim, N. A. and H. A. Toliyat, "A novel method for modeling dynamic air-gap eccentricity in synchronous machines based on

- modified winding function theory,” *IEEE Transactions on Energy Conversion*, Vol. 13, 156–162, 1998.
14. Faiz, J. and B. M. Ebrahimi, “Static eccentricity fault diagnosis in an acceleration no-load three-phase saturated squirrel-cage induction motor,” *Progress In Electromagnetics Research B*, Vol. 10, 35–54, 2008.
  15. Müller, G., *Elektrische Maschinen. Betriebsverhalten Elektrischer Maschinen*, VEB Verlag Technik, Berlín, 1990.
  16. Doherty, R. E. and C. A. Nickle, “Synchronous Machines. I. An extension of Blondel’s two-reaction theory. II. Steady state power-angle characteristics,” *AIEE Trans.*, Vol. 45, 912–947, 1926.
  17. Seely, S., *Electromechanical Energy Conversion*, Mc. Graw Hill, New York, 1962.
  18. Ivanov-Smolensky, A., *Machines Electriques*, MIR, Moscu, 1983.
  19. Gray, C. B., *Electrical Machines and Drive Systems*, Longman Scientific and Technical, Essex, 1991.
  20. Palit, B. B., “Einheitliche Untersuchung der elektrischen Maschinen mit Hilfe des Poynting-Vektors und des elektromagnetischen Energieflusses im Luftspaltraum,” *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 31, 384–412, 1980.
  21. Faiz, J., I. Tabatabaei, and H. A. Toliyat, “An evaluation of inductances of a squirrel-cage induction motor under mixed eccentric conditions,” *IEEE Transactions on Energy Conversion*, Vol. 18, No. 2, 252–258, 2003.
  22. Tabatabaei, I., J. Faiz, H. Lesani, and M. T. Nabavi-Razavi, “Modeling and simulation of a salient pole synchronous generator with dynamic eccentricity using modified winding function approach,” *IEEE Transactions on Magnetics*, Vol. 40, No. 3, 2004.
  23. Akbari, H., H. Meshgin-Kelk, and J. Milimonfared, “Extension of winding function theory for radial and axial nonuniform air-gap in salient pole synchronous machines,” *Progress In Electromagnetics Research*, Vol. 114, 407–428, 2011.