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ONE-STEP LEAPFROG ADI-FDTD METHOD FOR LOSSY MEDIA AND ITS STABILITY ANALYSIS

Jian-Yun Gao[1, *] and Hong-Xing Zheng[2]

[1]Department of Basic Courses, Tianjin Vocational Institute, Tianjin 300410, China

[2]Institute of Antenna and Microwave Techniques, Tianjin University of Technology and Education, Tianjin 300222, China

Abstract—A one-step leapfrog alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method for lossy media is presented. Different from the method provided by others, the proposed method is originated from the conventional ADI-FDTD method instead of considering the leapfrog ADI-FDTD method as a perturbation of the conventional explicit FDTD method. Its unconditional stability is analytically proven through a method that combines the von Neumann method with the Jury criterion. In addition, its unconditional stability and computational efficiency are verified through numerical experiments.

1. INTRODUCTION

Finite-difference time-domain (FDTD) method has been widely developed to simulate various EM problems in the past decades [1]. However, it is constrained by the Courant-Friedrich-Levy (CFL) stability condition. To overcome this problem, the unconditionally sable alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method was developed [2, 3]. But it employs a split time-step scheme where mid time-step computations are required. As a result, the required memory and CPU time are more than those of the conventional FDTD method. Recently, one-step leapfrog ADI-FDTD method was developed from the conventional ADI-FDTD method [4] where no mid time-step computations are needed. Therefore, it has better computational efficiency [5, 6].

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* Corresponding author: Jian-Yun Gao (jianyun_gao@163.com).

On the other hand, both the FDTD and ADI-FDTD methods are widely studied for lossy media [7–10]. In [11], the leapfrog ADI-FDTD method was also developed for lossy media. But the difference equations are derived by considering the leapfrog ADI-FDTD method as a perturbation of the conventional explicit FDTD method instead of from Maxwell's equations or conventional ADI-FDTD method. Moreover, the unconditional stability of the leapfrog ADI-FDTD method for lossy media is only numerically verified.

In this paper, the leapfrog ADI-FDTD method for lossy media is first derived from the conventional ADI-FDTD method. Then its unconditional stability is analytically proved though a method that combines the von Neumann method with the Jury criterion. At last, its accuracy and unconditional stability are studied through numerical experiments.

2. LEAPFROG ADI-FDTD FORMULATION FOR LOSSY MEDIA

The time-dependent Maxwell's curl equations in a linear, lossy, and non-dispersive medium with permittivity $\boldsymbol{\varepsilon}$, permeability $\boldsymbol{\mu}$, and electric conductivity $\boldsymbol{\sigma}$ can be written as

$$\boldsymbol{\varepsilon}\frac{d\mathbf{E}}{dt} = [(\mathbf{A}-\mathbf{B})\mathbf{H}-\boldsymbol{\sigma}\mathbf{E}] \tag{1a}$$

$$\boldsymbol{\mu}\frac{d\mathbf{H}}{dt} = (\mathbf{B}-\mathbf{A})\mathbf{E} \tag{1b}$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 & 0 \end{bmatrix}.$$

Therefore, for sub-step #1 of the ADI-FDTD method:

$$\boldsymbol{\varepsilon}\mathbf{E}^{n+1/2} = \boldsymbol{\varepsilon}\mathbf{E}^{n} + \frac{\Delta t}{2}\left(\mathbf{A}\mathbf{H}^{n+1/2} - \mathbf{B}\mathbf{H}^{n} - \boldsymbol{\sigma}\mathbf{E}^{n+m_1}\right) \tag{2a}$$

$$\boldsymbol{\mu}\mathbf{H}^{n+1/2} = \boldsymbol{\mu}\mathbf{H}^{n} + \frac{\Delta t}{2}\left(\mathbf{B}\mathbf{E}^{n+1/2} - \mathbf{A}\mathbf{E}^{n}\right). \tag{2b}$$

For sub-step #2:

$$\boldsymbol{\varepsilon}\mathbf{E}^{n+1} = \boldsymbol{\varepsilon}\mathbf{E}^{n+1/2} + \frac{\Delta t}{2}\left(\mathbf{A}\mathbf{H}^{n+1/2} - \mathbf{B}\mathbf{H}^{n+1} - \boldsymbol{\sigma}\mathbf{E}^{n+m_2}\right) \quad \text{(3a)}$$

$$\boldsymbol{\mu}\mathbf{H}^{n+1} = \boldsymbol{\mu}\mathbf{H}^{n+1/2} + \frac{\Delta t}{2}\left(\mathbf{B}\mathbf{E}^{n+1/2} - \mathbf{A}\mathbf{E}^{n+1}\right). \quad \text{(3b)}$$

where m_1 and m_2 in (2a) and (3a) are the time indices within one time step. Following the same derivative process as given in [4], by substituting (2b) into (2a), we have

$$\boldsymbol{\varepsilon}\mathbf{E}^{n+\frac{1}{2}} = \boldsymbol{\varepsilon}\mathbf{E}^{n} + \frac{\Delta t}{2}\left(\mathbf{A}\mathbf{H}^{n} + \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{B}\mathbf{E}^{n+\frac{1}{2}} - \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{A}\mathbf{E}^{n} - \mathbf{B}\mathbf{H}^{n}\right) - \frac{\Delta t}{2}\boldsymbol{\sigma}\mathbf{E}^{n+m_1}. \quad (4)$$

Replacing n with $n-1$ in (3a) and (3b), and then substituting (3b) back into (3a), we have

$$\boldsymbol{\varepsilon}\mathbf{E}^{n} = \boldsymbol{\varepsilon}\mathbf{E}^{n-\frac{1}{2}} + \frac{\Delta t}{2}\left(\mathbf{A}\mathbf{H}^{n} - \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{B}\mathbf{E}^{n-\frac{1}{2}} + \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{A}\mathbf{E}^{n} - \mathbf{B}\mathbf{H}^{n}\right) - \frac{\Delta t}{2}\boldsymbol{\sigma}\mathbf{E}^{n+m_2-1}. \quad (5)$$

Add (4) and (5) on their both sides, the leapfrog ADI-FDTD equation for electric field $\mathbf{E}$ is obtained as

$$\left(\boldsymbol{\varepsilon} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{B}\right)\mathbf{E}^{n+\frac{1}{2}} = \left(\boldsymbol{\varepsilon} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{B}\right)\mathbf{E}^{n-\frac{1}{2}} + \Delta t\left(\mathbf{A}\mathbf{H}^{n} - \mathbf{B}\mathbf{H}^{n}\right) - \frac{\Delta t}{2}\left(\boldsymbol{\sigma}\mathbf{E}^{n+m_1} + \boldsymbol{\sigma}\mathbf{E}^{n+m_2-1}\right). \quad (6)$$

With similar procedure, the equation for the magnetic field $\mathbf{H}$ can be obtained as follows.
Rewrite (2a) as

$$\boldsymbol{\varepsilon}\mathbf{E}^{n} = \boldsymbol{\varepsilon}\mathbf{E}^{n+1/2} - \frac{\Delta t}{2}\left(\mathbf{A}\mathbf{H}^{n+1/2} - \mathbf{B}\mathbf{H}^{n} - \boldsymbol{\sigma}\mathbf{E}^{n+m_1}\right). \quad (7)$$

Substitute (7) into (2b), we have

$$\boldsymbol{\mu}\mathbf{H}^{n+\frac{1}{2}} = \boldsymbol{\mu}\mathbf{H}^{n} + \frac{\Delta t}{2}\left(\mathbf{B}\mathbf{E}^{n+1/2} + \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{A}\mathbf{H}^{n+\frac{1}{2}} - \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{B}\mathbf{H}^{n} - \mathbf{A}\mathbf{E}^{n+1/2}\right) - \frac{\Delta t}{4}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\boldsymbol{\sigma}\mathbf{E}^{n+m_1}. \quad (8)$$

Then substitute (3a) into (3b), we have

$$\boldsymbol{\mu}\mathbf{H}^{n+1} = \boldsymbol{\mu}\mathbf{H}^{n+1/2} + \frac{\Delta t}{2}\left(\mathbf{B}\mathbf{E}^{n+1/2} - \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{A}\mathbf{H}^{n+\frac{1}{2}} + \frac{\Delta t}{2}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{B}\mathbf{H}^{n} - \mathbf{A}\mathbf{E}^{n+1/2}\right) + \frac{\Delta t}{4}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\boldsymbol{\sigma}\mathbf{E}^{n+m_2}. \quad (9)$$

Add (8) and (9) on their both sides, the leapfrog ADI-FDTD equation for magnetic field **H** is obtained as

$$\left(\boldsymbol{\mu} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{B}\right)\mathbf{H}^{n+1} = \left(\boldsymbol{\mu} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{B}\right)\mathbf{H}^{n} + \Delta t\left(\mathbf{B}\mathbf{E}^{n+\frac{1}{2}} - \mathbf{A}\mathbf{E}^{n+\frac{1}{2}}\right) + \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\varepsilon}^{-1}(\boldsymbol{\sigma}\mathbf{E}^{n+m_2} - \boldsymbol{\sigma}\mathbf{E}^{n+m_1}). \quad (10)$$

It is noted that (6) and (10) are the leapfrog ADI-FDTD equations for lossy media which is one-step and arbitrary order in space. Like in [9], various values for the ADI-FDTD method can be chosen for the values of m_1 and m_2, e.g., $m_1 = 0$, $m_2 = 0.5$ for the backward-backward method; $m_1 = 0$, $m_2 = 1$ for the backward-forward method; $m_1 = 0.5$, $m_2 = 0.5$ for the forward-backward method; and $m_1 = 0.5$, $m_2 = 1$ for the forward-forward method. We can see that only for $m_1 = 0.5$, $m_2 = 0.5$ the cross term in (10) can be eliminated. Then we can obtain the leapfrog ADI-FDTD method for lossy media as

$$\left(\boldsymbol{\varepsilon} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{B} + \frac{\Delta t\boldsymbol{\sigma}}{2}\right)\mathbf{E}^{n+\frac{1}{2}} = \left(\boldsymbol{\varepsilon} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\mu}^{-1}\mathbf{B} - \frac{\Delta t\boldsymbol{\sigma}}{2}\right)\mathbf{E}^{n-\frac{1}{2}} + \Delta t(\mathbf{A}\mathbf{H}^{n} - \mathbf{B}\mathbf{H}^{n}) \quad (11a)$$

$$\left(\boldsymbol{\mu} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{B}\right)\mathbf{H}^{n+1} = \left(\boldsymbol{\mu} - \frac{\Delta t^2}{4}\mathbf{A}\boldsymbol{\varepsilon}^{-1}\mathbf{B}\right)\mathbf{H}^{n} + \Delta t\left(\mathbf{B}\mathbf{E}^{n+\frac{1}{2}} - \mathbf{A}\mathbf{E}^{n+\frac{1}{2}}\right). \quad (11b)$$

For clarity, the update equations for E_x and H_x are as

$$\left(1 - \frac{\Delta t^2}{4\varepsilon_x}\frac{\partial}{\partial y}\mu_z^{-1}\frac{\partial}{\partial y} + \frac{\Delta t\sigma_x}{2\varepsilon_x}\right)E_x^{n+\frac{1}{2}} = \left(1 - \frac{\Delta t^2}{4\varepsilon_x}\frac{\partial}{\partial y}\mu_z^{-1}\frac{\partial}{\partial y} - \frac{\Delta t\sigma_x}{2\varepsilon_x}\right)E_x^{n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon_x}\left(\frac{\partial}{\partial y}H_z^n - \frac{\partial}{\partial z}H_y^n\right) \quad (12a)$$

$$\left(1 - \frac{\Delta t^2}{4\mu_x}\frac{\partial}{\partial y}\varepsilon_z^{-1}\frac{\partial}{\partial y}\right)H_x^{n+1} = \left(1 - \frac{\Delta t^2}{4\mu_x}\frac{\partial}{\partial y}\varepsilon_z^{-1}\frac{\partial}{\partial y}\right)H_x^{n} + \frac{\Delta t}{\mu_x}\left(\frac{\partial}{\partial z}E_y^{n+\frac{1}{2}} - \frac{\partial}{\partial y}E_z^{n+\frac{1}{2}}\right). \quad (12b)$$

Other field equations can be obtained with cyclic permutations of $\{x, y, z\}$. It should be noted again that in (12a) and (12b), the difference approximations are only used for the time. Therefore, the proposed method is in arbitrary order in space, and the spatial derivatives can be replaced with their finite-difference counterparts of a chosen order in actual computations. For example, if we chose the center second-order difference in space domain in an isotropic homogeneous media characterized by permittivity ε, permeability μ, and electric conductivity σ, (12a) and (12b) can be written as

$$\left(1+\frac{\Delta t^2}{2\varepsilon\mu\Delta y^2}+\frac{\Delta t\sigma}{2\varepsilon}\right)E_x\Big|_{i+1/2,j,k}^{n+\frac{1}{2}}-\frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\left(E_x\Big|_{i+1/2,j+1,k}^{n+\frac{1}{2}}+E_x\Big|_{i+1/2,j-1,k}^{n+\frac{1}{2}}\right)$$
$$=\left(1+\frac{\Delta t^2}{2\varepsilon\mu\Delta y^2}-\frac{\Delta t\sigma}{2\varepsilon}\right)E_x\Big|_{i+1/2,j,k}^{n-\frac{1}{2}}$$
$$-\frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\left(E_x\Big|_{i+1/2,j+1,k}^{n-\frac{1}{2}}+E_x\Big|_{i+1/2,j-1,k}^{n-\frac{1}{2}}\right)$$
$$+\frac{\Delta t}{\varepsilon}\left(\frac{H_z\big|_{i+1/2,j+1/2,k}^{n}-H_z\big|_{i+1/2,j-1/2,k}^{n}}{\Delta y}\right.$$
$$\left.-\frac{H_y\big|_{i+1/2,j,k+1/2}^{n}-H_y\big|_{i+1/2,j,k-1/2}^{n}}{\Delta z}\right) \tag{13a}$$

$$\left(1+\frac{\Delta t^2}{2\varepsilon\mu\Delta y^2}\right)H_x\Big|_{i,j+1/2,k+1/2}^{n+1}$$
$$-\frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\left(H_x\Big|_{i,j+3/2,k+1/2}^{n+1}+H_x\Big|_{i,j-1/2,k+1/2}^{n+1}\right)$$
$$=\left(1+\frac{\Delta t^2}{2\varepsilon\mu\Delta y^2}\right)H_x\Big|_{i,j+1/2,k+1/2}^{n}$$
$$-\frac{\Delta t^2}{4\varepsilon\mu\Delta y^2}\left(H_x\Big|_{i,j+3/2,k+1/2}^{n}+H_x\Big|_{i,j-1/2,k+1/2}^{n}\right)$$
$$+\frac{\Delta t}{\varepsilon}\left(\frac{E_y\big|_{i,j+1/2,k+1}^{n+1/2}-E_y\big|_{i,j+1/2,k}^{n+1/2}}{\Delta z}-\frac{E_z\big|_{i,j+1,k+1/2}^{n+1/2}-E_z\big|_{i,j,k+1/2}^{n+1/2}}{\Delta y}\right). \tag{13b}$$

where i, j and k are the spatial indices in the x-, y- and z-directions, respectively. Other field equations can be obtained with cyclic permutations of $\{x,\ y,z\}$.

3. STABILITY ANALYSIS OF THE LEAPFROG ADI-FDTD METHOD FOR LOSSY MEDIA

To analyze the stability of the leapfrog ADI-FDTD method for lossy media, the von Neumann method combining with the Jury criterion is employed [12]. The von Neumann method mainly consists of considering a Fourier series expansion of the field components at the mesh nodes at a given time as

$$f^n(i,j,k) = f_0 Z^n \exp\left[J(i\Delta x k_x + j\Delta y k_y + k\Delta z k_z)\right] \tag{14}$$

where J is the imaginary unit, f_0 the complex amplitude, Z the amplification factor, and k_x, k_y and k_z are the wavenumbers in the x-, y- and z-directions. To determine the value of $|Z|$, field solutions of the form (14) are substituted into the difference equations of the leapfrog ADI-FDTD method for lossy media. This leads to a characteristic polynomial in Z as

$$S(Z) = \sum_{i=0}^{N} a_i Z^i. \tag{15}$$

The condition for stability can be written as $|Z_i| \leq 1.0$ where Z_i are the roots of $S(Z) = 0$. A sufficient condition to ensure this is the Jury criterion which only uses the coefficients of $S(Z)$. To do so, a Jury table should be given by

Table 1. Jury table.

1	a_N	a_{N-1}	a_{N-2}	$\ldots$	a_2	a_1	a_0
2	a_0	a_1	a_2	$\ldots$	a_{N-2}	a_{N-1}	a_N
3	b_{N-1}	b_{N-2}	b_{N-3}	$\ldots$	b_1	b_0	0
4	b_0	b_1	b_2	$\ldots$	b_{N-2}	b_{N-1}	0
5	c_{N-2}	c_{N-3}	c_{N-4}	$\ldots$	c_0	0	0
6	c_0	c_1	c_2	$\ldots$	c_{N-2}	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2N-3$	d_2	d_1	d_0	0	0	0	0

In Table 1, the entries

$$b_j = \begin{vmatrix} a_N & a_{N-1-j} \\ a_0 & a_{j+1} \end{vmatrix}, \quad c_k = \begin{vmatrix} b_{N-1} & b_{N-2-k} \\ b_0 & b_{k+1} \end{vmatrix}, \quad \ldots \tag{16}$$

where $j = 0, 1, \ldots, N-1$ and $k = 0, 1, \ldots, N-2$.

Jury criterion is that if the conditions below are satisfied,

$$\begin{aligned}&S(1)>0, \quad (-1)^N S(-1)>0, \quad a_N>|a_0|, \quad b_{N-1}>|b_0|, \\ &c_{N-2}>|c_0|, \quad \ldots, \quad d_2>|d_0|. \end{aligned} \tag{17}$$

then all the roots of $S(Z)=0$ are less or equal to unity in magnitude.

Without generosity, for the leapfrog ADI-FDTD method for lossy media, by substituting (14) into the second-order differential leapfrog ADI-FDTD Equation (13), we have

$$\begin{bmatrix} Z^{-\frac{1}{2}}-Z^{\frac{1}{2}}-\frac{Q_L}{Q_y}\left(Z^{-\frac{1}{2}}+Z^{\frac{1}{2}}\right) & 0 & 0 & 0 & \frac{2JW_z}{\varepsilon Q_y} & -\frac{2JW_y}{\varepsilon Q_y} \\ 0 & Z^{-\frac{1}{2}}-Z^{\frac{1}{2}}-\frac{Q_L}{Q_z}\left(Z^{-\frac{1}{2}}+Z^{\frac{1}{2}}\right) & 0 & -\frac{2JW_z}{\varepsilon Q_z} & 0 & \frac{2JW_x}{\varepsilon Q_z} \\ 0 & 0 & Z^{-\frac{1}{2}}-Z^{\frac{1}{2}}-\frac{Q_L}{Q_x}\left(Z^{-\frac{1}{2}}+Z^{\frac{1}{2}}\right) & \frac{2JW_y}{\varepsilon Q_x} & -\frac{2JW_x}{\varepsilon Q_x} & 0 \\ 0 & -\frac{2JW_z}{\mu Q_y} & \frac{2JW_y}{\mu Q_y} & Z^{-\frac{1}{2}}-Z^{\frac{1}{2}} & 0 & 0 \\ \frac{2JW_z}{\mu Q_z} & 0 & -\frac{2JW_x}{\mu Q_z} & 0 & Z^{-\frac{1}{2}}-Z^{\frac{1}{2}} & 0 \\ -\frac{2JW_y}{\mu Q_x} & \frac{2JW_x}{\mu Q_x} & 0 & 0 & 0 & Z^{-\frac{1}{2}}-Z^{\frac{1}{2}} \end{bmatrix}$$

$$\begin{bmatrix} E_{x0} \\ E_{y0} \\ E_{z0} \\ H_{x0} \\ H_{y0} \\ H_{z0} \end{bmatrix} = 0 \tag{18a}$$

where

$$W_\alpha = \Delta t/\Delta\alpha \sin(k_\alpha \Delta\alpha/2) \tag{18b}$$

$$Q_\alpha = 1+W_\alpha^2/(\varepsilon\mu), \quad \alpha=x,y,z \tag{18c}$$

$$Q_L = \Delta t\sigma/(2\varepsilon) \tag{18d}$$

For a nontrivial solution of (18), the determinant of coefficient matrix should be zero. Therefore, with the help of the software Mathematica

8.0, we have

$$\begin{aligned} S(Z) &= S_1(Z)\cdot S_2(Z)\cdot S_3(Z)/\left(\varepsilon\mu Q_x^2 Q_y^2 Q_z^2 Z^3\right) \\ &= [-1+Z]\cdot\left[4Z\left(Q_y W_x^2 + Q_Z W_y^2\right) + Q_x\left((-1+Z)^2\varepsilon\mu Q_y Q_z + 4ZW_z^2\right)\right] \\ &\cdot\left[\begin{array}{l}(1+Z)Q_L\left((-1+Z)^2\varepsilon\mu Q_x Q_z + 4ZW_x^2\right) + (-1+Z) \\ \cdot\left(4Z\left(Q_y W_x^2 + Q_z W_y^2\right) + Q_x\left((-1+Z)^2\varepsilon\mu Q_y Q_z + 4ZW_z^2\right)\right)\end{array}\right] \\ &/(\varepsilon\mu Q_x^2 Q_y^2 Q_z^2 Z^3). \end{aligned} \tag{19}$$

Here, for simplicity, we divide $S(Z)$ into three different polynomials $S_1(Z)$, $S_2(Z)$, and $S_3(Z)$.

It is obvious that the root for $S_1(Z) = 0$ is unity.

For $S_2(Z)$, by applying Jury criterion, we have

$$S_2(1) = 4\left(Q_y W_x^2 + Q_z W_y^2 + Q_x W_z^2\right) > 0 \tag{20a}$$

$$S_2(-1) = 4\varepsilon\mu + 4W_x^2 W_y^2 W_z^2/\left(\varepsilon^2\mu^2\right) > 0 \tag{20b}$$

$$a_2 = a_0 = \varepsilon\mu Q_x Q_y Q_z > 0. \tag{20c}$$

Note that (20b) is obtained by substitution (18c) into the right hand of $S_2(-1)$. Equation (20c) is the critical condition of the Jury criterion. This means that both the roots of $S_2(Z) = 0$ is unity in magnitude.

For $S_3(Z)$, Jury criterion is applied again, we have

$$S_3(1) = 8Q_L W_x^2 > 0 \tag{21a}$$

$$(-1)^3 S_3(-1) = 8\varepsilon\mu + 8W_x^2 W_y^2 W_z^2/\left(\varepsilon^2\mu^2\right) > 0 \tag{21b}$$

$$a_3 - |a_0| = \begin{cases} 2\varepsilon\mu Q_x Q_y Q_z > 0, & a_0 \geq 0 \\ 2\varepsilon\mu Q_L Q_x Q_z > 0, & a_0 < 0 \end{cases}, \text{ i.e., } a_3 > |a_0| \tag{21c}$$

$$b_2 - |b_0| = \begin{cases} 8\varepsilon\mu Q_L Q_x Q_z\left(Q_z W_y^2 + Q_x W_z^2\right) > 0, \ a_1 a_3 - a_0 a_2 \geq 0 \\ 8\varepsilon\mu Q_L Q_x Q_z(\varepsilon\mu + W_x^2 + W_x^2 W_y^2/(\varepsilon\mu) \\ +W_x^2 W_y^2 W_z^2/\left(\varepsilon^2\mu^2\right)) > 0, \ a_1 a_3 - a_0 a_2 \leq 0 \end{cases}$$

$$\text{i.e., } b_2 > |b_0|. \tag{21d}$$

Note again that (21b) is obtained by substitution (18c) into the right hand of $S_3(-1)$. So the magnitudes of the three roots of $S_3(Z) = 0$ are less than unity.

As a result, all the six roots of $S(Z) = 0$ are not lager than unity. Therefore, the proposed leapfrog ADI-FDTD method for lossy media is unconditionally stable.

4. NUMERICAL RESULTS AND DISCUSSION

To prove the unconditional stability and accuracy of the proposed leapfrog ADI-FDTD method, a lossy material-filled PEC cavity with Yee grid cells of $50 \times 30 \times 9$ is calculated as in Fig. 1.

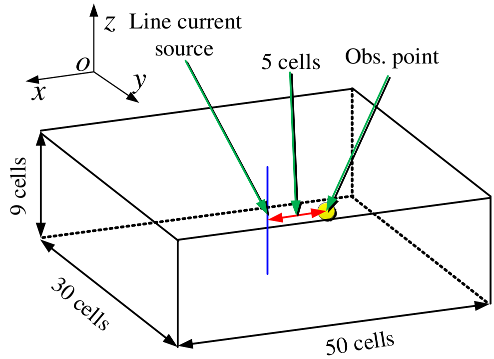

Figure 1. Lossy material-filled PEC cavity.

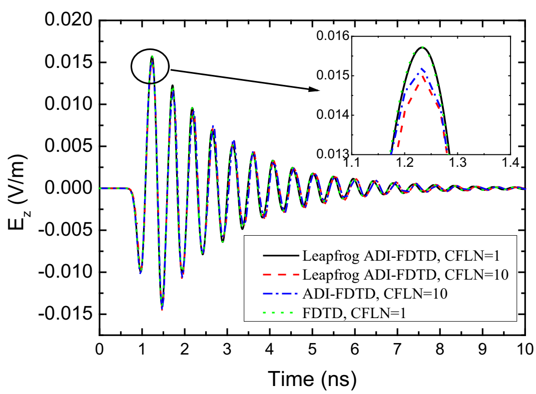

Figure 2. Recorded E_z computed with the leapfrog ADI-FDTD method of CFLN = 1 and CFLN = 10, ADI-FDTD method of CFLN = 10, and FDTD method of CFLN = 1.

A line current source centred at the domain from bottom to top along the z-direction was used to excite the EM field. The current source is given by

$$J_z = \exp\left[-\frac{4\pi(t-t_0)^2}{\tau^2}\right] \quad (22)$$

where $\Delta x = \Delta y = \Delta z = \Delta = 2\,\text{mm}$, $\Delta t = \Delta/\sqrt{3}c \cdot \text{CFLN}$, $\tau = 150\Delta t/\text{CFLN}$, and $t_0 = 2\tau$, CFLN being the CFL number. We set the observation point five cells away from the centre of the computational domain along the x-direction.

Figure 2 shows E_z component versus time at the observation point obtained with (i) leapfrog ADI-FDTD method with CFLN = 1 and CFLN = 10, (ii) conventional ADI-FDTD with CFLN = 10, (iii) FDTD with CFLN = 1. We choose the permittivity ε and permeability μ as the values in vacuum. The electric conductivity σ is chosen as 0.01 S/m. It can be seen that the result obtained by CFLN = 1 for the leapfrog ADI-FDTD method agrees well with that obtained by FDTD; the errors are increasing slightly with the increasing of CFLN from 1 to 10; for CFLN = 10, the leapfrog ADI-FDTD and conventional ADI-FDTD result have almost the same errors. To check the stability, many more experiments were carried out with different electric conductivities and CFLNs. We use 100 million iterations for each simulation. The electric conductivities are changed from 0 to 10^6 S/m and CFLN from 1 to 10^3. It is found that the recorded E_z always decreases as the time progresses. Therefore, the stability of the proposed method is numerically verified.

Table 2 shows the comparisons of the CPU time and memory used by the conventional FDTD with CFLN = 1, conventional ADI-FDTD with CFLN = 1 and 10, proposed leapfrog ADI-FDTD with CFLN = 1 and 10. The simulation platform is Lenovo PC with Intel Dual Core i3 M330 of 2.13 GHz and RAM of 4 GB; the source code was written in Fortran language. It can be seen that the proposed method used the same memory as that of FDTD. This is because, unlike the conventional ADI-FDTD method, the leapfrog ADI-FDTD method has only one-step and with no mid-time EM fields needed to be stored. As for the CPU time consumed, the proposed method with CFLN = 10 uses the least.

Table 2. Comparison of the CPU time and memory used by the conventional FDTD method, conventional ADI-FDTD method, and proposed leapfrog ADI-FDTD method.

Method	CFLN	Number of steps	CPU Time (s)	Memory (MB)
leapfrog ADI-FDTD	1	2000	25	3.7
	10	200	2.6	3.7
ADI-FDTD	10	200	3.3	4.1
FDTD	1	2000	9.4	3.7

5. CONCLUSION

A one-step leapfrog ADI-FDTD method for lossy media has been presented in this paper. This method is derived from the conventional ADI-FDTD method. Its unconditional stability has been analytically proven though a method that combines the von Neumann method with the Jury criterion. In addition, its unconditional stability and accuracy are verified through numerical experiments.

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